Some Background on the Weyl Tensor

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September 21, 2024

Abstract

Of all the known mathematical symmetries in modern physics, *conformal invariance* is probably the least one encountered by students. In quantum field theory it is better known as *gauge invariance* or *phase invariance*, where it is perhaps the most fundamental symmetry underlying all modern quantum physics. But in general relativity it is a backwater symmetry, due primarily to our ignorance as to whether or not it is even needed in gravitational work. It is usually just ignored, as Einstein's 1915 gravity theory works just fine, even though it is not conformally invariant.

In this paper we explore how conformal invariance can be brought into relativistic gravitational physics. At the most fundamental level, this effort requires the introduction of the *Weyl tensor*. The motivation for this is solely due to the possibility that the problem of *dark matter*, believed to be the explanation for the anomalous behavior of galactics stars, lensing and clustering, might be explained instead by modifying Einsteinian gravity to include conformal invariance. In this regard, the Weyl tensor (or its various forms) is indispensible.

1. Introduction

It is assumed that the reader is already familiar with dark matter and its assumed responsibility for flat stellar rotation curves, galactic lensing and clustering. But despite nearly half a century of dedicated and costly experimental research, not a single dark matter particle has been verifiably detected. Although the search for dark matter continues with larger and more elaborate experimental equipment, some researchers have given up hope that it will ever be detected, with not a few believing that dark matter simply does not exist. This situation has motivated the possibility that Einsteinian gravity might be modified to provide an alternate and perhaps more correct explanation for dark matter's effects.

In 1918, the noted German mathematical physicist Hermann Weyl sought to generalize Einstein's 1915 theory in the hope that a suitable modification of that theory might also incorporate electromagnetism as a purely geometrical construct. Weyl found a way to introduce the electromagnetic four-potential A_μ into the Levi-Civita (or Christoffel) connection term

$$\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} = \frac{1}{2} g^{\lambda\beta} \left(g_{\mu\beta|\nu} + g_{\beta\nu|\mu} - g_{\mu\nu|\beta} \right)$$

(where the single subscripted bar stands for ordinary partial differentiation). He then wrote that the revised connection should now read

$$\Gamma^\lambda_{\mu\nu} = \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} + \frac{1}{2} \delta^\lambda_\mu A_\nu + \frac{1}{2} \delta^\lambda_\nu A_\mu - \frac{1}{2} g_{\mu\nu} g^{\lambda\beta} A_\beta$$

Weyl's connection term was invariant with respect to the local transformation

$$g_{\mu\nu}(x) \to e^{\pi(x)} g_{\mu\nu}(x)$$

where $g_{\mu\nu}$ is the metric tensor and π is an arbitrary function of space and time, while the potential A_μ transforms accordingly. Such a transformation is now known as a *Weyl transformation*, and quantities invariant to this transformation are called *conformally invariant*. All of this is explained in great detail in Reference 3.

Despite initial admiration by the physics community, Einstein spotted a fatal flaw in Weyl's theory, and Weyl subsequently abandoned it. However, he was still convinced that gravitation should be a conformally invariant theory, and he sought a geometrical quantity that displayed that invariance that involved only the Riemann curvature tensor $R_{\mu\nu\alpha\beta}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R. Weyl was able to derive a unique 4-rank tensor with the required property, now known as the Weyl tensor which, in n dimensions, is

$$C^{\lambda}_{\ \nu\alpha\beta} = R^{\lambda}_{\ \nu\alpha\beta} + \frac{1}{n-2}\left(\delta^{\lambda}_{\beta}R_{\nu\alpha} - \delta^{\lambda}_{\alpha}R_{\nu\beta} + g_{\nu\alpha}R^{\lambda}_{\beta} - g_{\nu\beta}R^{\lambda}_{\alpha}\right) + \frac{1}{(n-1)(n-2)}\left(\delta^{\lambda}_{\alpha}\, g_{\beta\nu} - \delta^{\lambda}_{\beta}\, g_{\alpha\nu}\right)$$

The derivation is straightforward but cumbersome, and can be found in Reference 4. (The writer can't help noting that the calculation is greatly simplified by adopting a locally flat coordinate system, where the metric tensor is considered a constant.) Note that any contraction of components in the Weyl tensor vanishes due to the nature of the terms in the tensor itself. Consequently, quantities like $C^{\lambda}_{\ \nu\lambda\beta} = 0$.

2. The Weyl Action

Unfortunately, the Weyl tensor cannot be used as an action Lagrangian because it is not a Lorentz scalar. To construct such a scalar, Weyl was forced to consider the tensor's square, $C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$. This quantity provides a suitable invariant, and the associated action

$$\int \sqrt{-g}\, C_{\mu\nu\alpha\beta}\, C^{\mu\nu\alpha\beta}\, d^4x \tag{2.1}$$

is fully conformally invariant (note that this action is valid only for the case $n=4$). By the laborious process of multiplying $C_{\mu\nu\alpha\beta}$ by its fully contravariant form $C^{\mu\nu\alpha\beta}$, the integral reduces to

$$\int \sqrt{-g}\, C_{\mu\nu\alpha\beta}\, C^{\mu\nu\alpha\beta}\, d^4x = \int \sqrt{-g}\left(R_{\mu\nu\alpha\beta}\, R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right) d^4x \tag{2.2}$$

At this point it is instructive to note that the labor involved in calculating (2.2) can be avoided by considering the conformal variations of the individual quantities $R_{\mu\nu\alpha\beta}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R (this is also shown in Reference 3). By writing

$$\int \sqrt{-g}\, C_{\mu\nu\alpha\beta}\, C^{\mu\nu\alpha\beta}\, d^4x = \int \sqrt{-g}\left(R_{\mu\nu\alpha\beta}\, R^{\mu\nu\alpha\beta} + AR_{\mu\nu}R^{\mu\nu} + BR^2\right) d^4x \tag{2.3}$$

(where A and B are constants) and taking the conformal variation of each term in the integrand, it is straightforward to see that this reduces to (2.2) provided that the simple condition

$$A = -3B - 1 \tag{2.4}$$

holds. It is a simple matter to show that for $A=-2$ we have $B=1/3$, and we recover (2.2). This is sometimes known as the Gauss-Bonnet case.

However, any hope we have of deriving the equations of motion from (2.2) is greatly complicated by the presence of the Riemann curvature term. But in 1938 Cornelius Lanczos (Reference 1) demonstrated that there is a divergence term buried in the equation that allows for the elimination of the curvature term while retaining the conformal invariance of the action.

While he did not elaborate on this idea, there is an extremely simple way of accomplishing this. Let us assume any other value for the constant A; for example, let us assume $A = 23.178$. Then from (2.4) we have $B=-8.059333....$ Then the action becomes

$$\int \sqrt{-g}\left(R_{\mu\nu\alpha\beta}\, R^{\mu\nu\alpha\beta} + 23.178\, R_{\mu\nu}R^{\mu\nu} - 8.059333...R^2\right) d^4x$$

Remarkably, this action is conformally invariant! But let us now subtract the integrand in (2.2) from this. The Riemann curvature terms cancel, and we're left with

$$\int \sqrt{-g}\left[(23.178+2)R_{\mu\nu}R^{\mu\nu} + (-8.059333...-1/3)R^2\right] d^4x$$

or

$$25.178 \int \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) d^4x$$

Ignoring the irrelevant leading constant, we have

$$S = \int \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) d^4x \tag{2.5}$$

This quantity is recognized as the "official" action for conformal gravity. The reason that (2.4) works is not because there is a divergence buried in the action, but because the all-important Bianchi identity

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right)_{||\nu} = 0$$

is hiding in the action, where the double subscripted bar stands for covariant differentiation.

3. Problems

An annoying problem with (2.5) is that it is of fourth order with regard to the metric tensor and its derivatives. This introduces ghost terms into the theory, particularly when attempts are made to join it with quantum field theory, despite the fact that the Weyl action (being dimensionless) is renormalizable.

Another problem, and one that seems to be ignored in the research, is that being a square (the bare integrand has the dimension L^{-4}) and of fourth order greatly complicates the task of finding a suitable energy-momentum tensor to match it with. All traditional forms of the energy-momentum tensor are of dimension L^{-2}, and it seems meaningless to try matching the Weyl action with the square of this tensor.

Furthermore, although the Weyl action omits the Riemann curvature tensor, variation of the action with respect to the metric tensor $g^{\mu\nu}$ is still very difficult. Mannheim and Kazanas managed to find an exact vacuum solution using a Schwarzschild-like metric, where $ds^2 = e^\nu c^2 dt^2 - e^\lambda dr^2 - r^2 d\Omega^2$ and with

$$e^\nu = e^{-\lambda} = 1 - \frac{(2 - 3\beta\gamma)}{r} - 3\beta\gamma + \gamma r + kr^2 \tag{3.1}$$

where β, γ, k being constants. This solution has direct application to the dark matter problem (especially in view of the last two terms), but the the authors' subsequent inclusion of a second-order energy-momentum tensor seems contrived.

Finally, despite its formal elegance and apparent applicability it is still not known if conformal invariance plays an important role in gravitation.

Nevertheless, fourth-order modified gravity theories like Weyl's have experienced a resurgence of interest in recent years, particularly because all efforts to date to detect dark matter particles have failed.

References

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