Selflimitation & Holism in Mathematics

The example of Hermann Weyl and World War I

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Luminy, 26 January 2007
Inspirations received

- Erhard Scholz, personal communication, and several of his articles about Weyl.
- ETH Archives Zürich and personal communications by Michael Hagner.
- Solomon Feferman: *In the Light of Logic*. OUP 1998. Esp. the logical analysis of H. Weyl’s *Das Kontinuum* from 1918.
Anne Harrington’s argument (for the life sciences)

Starting from Kant’s Third Critique and Goethe’s *Farbenlehre*, continued in the early 19th century by the philosophers’ quest for completed systems in the face of political fragmentation of Germany, holistic ideas within the sciences in Germany increasingly turned into a revolt against the machine image of life towards the end of the 19th century.

An important point of Harrington’s book is that holistic currents in the sciences respond (by way of metaphors) to political agenda.

Harrington's case studies of holistic thinkers, often outside the scientific mainstream, but strongly marking the history of their disciplines:

✶ Jakob v. Uexküll, *Umwelt*; also Hans Driesch.
✶ Constantin von Monakow, pan-psychism through brain research.
✶ Kurt Goldstein, WWI brain damages call for holistic approach to the human brain and to therapy.
But what does “holism” mean in mathematics?

Gerolamo Cardano (1501-1576) had a holistic approach to nature, but this does not transpire in his mathematical texts.

All through the nineteenth century it seems problematic to attribute “holistic” tendencies to mathematicians (like Gauss or Klein), or to use it to better understand ongoing debates (like those between Cantor - Dedekind - Kronecker). The category of modernity, however, often seems adequate, as Mehrtens has shown.

Claim defended and illustrated in this talk:

The attribute “holistic” is adequate (and then more precise than “counter-modern”) to explain certain mathematicians’ attitudes at certain times between 1905 and 1945.
Hermann Weyl (1885-1955)
After the end of the theory of magnitudes in 1872 and Cantor’s set theory, various problems arise, among them:

1905 Jules Antoine Richard’s [1862-1956] Paradox:
The set of real numbers definable by finitely many words is countable.

1908 Emile Borel: Les “Paradoxes” de la théorie des ensembles, *Annales de l'ENS*, criticizes the idea that every countable set can effectively be enumerated.

April 1909 Henri Poincaré [1854-1912] gives 6 Wolfskehl lectures in Göttingen, published shortly afterwards in German. Weyl is in the audience.
The fifth lecture: *Über transfinite Zahlen*, resolves Richard’s paradox by distinguishing predicative from impredicative definitions.
In letzter Zeit habe ich viel über die Grundlagen der Mengenlehre nachgedacht und komme da zu Ansichten, die von den Zermelo’schen ziemlich stark abweichen und sich im gewissen Sinne dem hier vielverspotteten Borel-Poincaré’schen Standpunkt nähern.

I have recently thought about the foundations of set theory and were led to views which diverge rather strongly from Zermelo’s, coming close in a certain sense to the point of view of Borel and Poincaré which is generously derided around here [= in Göttingen].
Hermann Weyl:
Über die Definition der mathematischen Grundbegriffe
Habilitationssvortrag Göttingen 1910

Darf man sagen, was nach dem bisher Ausgeführten naheliegt, Mathematik sei die Wissenschaft vom \( \varepsilon \) und denjenigen Beziehungen, die sich auf Grund dieses Begriffes mittels der erwähnten Prinzipien definieren lassen? Vielleicht wird durch eine solche Erklärung die Mathematik ihrem logischen Gehalt nach in der Tat zutreffend bestimmt. Trotzdem erblicke ich den eigentlichen Wert und die eigentliche Bedeutung des so zustande kommenden Begriffssystems einer logisierten Mathematik doch darin, daß sich ihre Begriffe auch, ohne daß dabei die Wahrheit der auf sie bezüglichen Sätze Schaden leidet, anschauungsmäßig deuten lassen, und ich glaube, der menschliche Geist kann auf keinem anderen Wege als durch Verarbeitung der gegebenen Wirklichkeit zu den mathematischen Begriffen aufsteigen.

May we say - as is suggested by what we have developed - that mathematics is the science of \( \varepsilon \) and of those relations which can be defined from this notion via the principles discussed? Maybe such an explanation does actually determine mathematics correctly as for its logical substance. However, I see the proper value and the meaning proper of the system of notions of logicised mathematics thus constructed in that its notions may also be interpreted intuitionwise without affecting the truth of the statements about them. And I believe that the human spirit has no other way to ascend to mathematical notions but by digesting the given reality.
Die “anschauliche Evidenz” enthebt uns aber, daran heute kein Zweifel mehr sein, keineswegs der Notwendigkeit, für eben diese Wahrheiten Beweise zu erbringen, die letzten Endes auf die Axiome der Arithmetik gestützt sind.


The “intuitive evidence” does not, however, allow us - there can be no doubt about this today - to disregard the necessity of supplying proofs for these truths, which in the last resort have to be based on the axioms of arithmetic.

It cannot be denied: the discovery of the generality of such notions as “function,” “curve,” etc. on the one hand - a generality extending far beyond all our intuitions - and the desire of logical rigour on the other - in spite of it being tremendously productive, and in fact necessary for our science - have also brought about unhealthy phenomena. Part of the mathematical production, which strives to trace these notions to their ultimate fine points and ... distortions or to grasp them in their widest contours, has lost - volatised in the void or trickling away in side corridors - the connection with the living stream of science.
Also the idea of the Riemannian surface requires, if we want to do justice to the rigorous demands of modernity, plenty of abstract and subtle concepts and reflections. But we just have to sharpen our view a little bit to realize that this whole finely woven web (in which the beginner may get tangled up) is not what really counts: it is only the net by which we lift up the idea proper, which by its very nature is simple and grand and divine, from the ΤΟΠΟΣ ΑΤΟΠΟΣ, as Plato says - like a pearl from the sea - to the surface of our world of reason.

To grasp the core, however, which this tissue of delicate and meticulous concepts enwraps; to grasp what accounts for the life, the true substance, the inner value of the theory - for this a book (and even a teacher) can only provide scanty indications; here everyone has to wrestle himself afresh to gain understanding.

The holistic conception remains an individual task to be mastered beyond and in spite of the modern, distorting presentation. This dilemma is only mentioned in the preface.
Hermann Weyl through WWI

- 1913 Fall, marries Helene Joseph in Göttingen; appointed professor ETH Zürich.
- 1915 May 11 German military service, soon in a garrison near Saarbrücken; article in *Jahresbericht D.M.-V.* on Riemann surfaces written “without literature.”
- 1916 May 3 after intervention of the Swiss authorities, released by the German army (only *dauernd garnisonsdienstfähig*); back to Zürich.
- Starting 1916 reads J.G. Fichte and Meister Eckart with Fritz Medicus.
- Courses at ETH: W 1916/17 Mathematical theory of the electromagnetic field; S 1917 *Raum, Zeit, Materie*; W 1917/18 Logical foundations of mathematics.
- Two books published in 1918 in his new fields of work:
  - *Das Kontinuum - Kritische Untersuchungen über die Grundlagen der Analysis.* Veit & Co Leipzig.
Swiss Mathematical Society Meeting Zürich 1917

Hilbert

The Weyl couple
DAS KONTINUUM

Kritische Untersuchungen
über die Grundlagen der Analysis

von

Dr. Hermann Weyl
Professor der Mathematik a. d. Eidgen.
Technischen Hochschule Zürich

Leipzig
Verlag von Veit & Comp.
1918
From the preface ...

The goal of this treatise is not to drape the “solid rock” on which the house of analysis is founded with a wooden scaffolding in the sense of formalism, and to then pretend to the reader, and finally even to oneself, that this is the real foundation. Here, rather we defend the opinion that an essential part of the said house is grounded on sand. I believe I can replace this faltering ground by props of reliable stability; but they do not sustain everything one generally takes today to be certain; the rest I sacrifice, since I do not see any other possibility.

At any rate I want to be understood not only by the chairs but also by all students who have become acquainted with the “rigorous” foundations taught today.
The circulus vitiosus to which we point and which is concealed behind the foggy nature of the usual notions of set and function is not at all an easily emended formal blunder in the build-up of analysis. The realization of its fundamental importance is something which can just not be delivered to the reader in so many words. But the more distinctly one pictures for oneself the logical web of analysis, the more deeply and completely the glance of consciousness penetrates it, the more clearly it will appear that in today’s style of foundation each and every cell, so to say, of this formidable organism is infested by the poison of contradiction; and that thorough control is necessary to remedy the situation.
Eine reelle Zahl ist eine Menge rationaler Zahlen, die einer bestimmten Eigenschaft rationaler Zahlen korrespondiert. Eine Menge reeller Zahlen entspricht also einer Eigenschaft $A$ von Eigenschaften rationaler Zahlen. Die obere Grenze dieser Menge reeller Zahlen ist selbst die Menge derjenigen rationalen Zahlen $x$, welche eine gewisse Eigenschaft $E$ besitzen, nämlich die folgende: daß es eine Eigenschaft der Art $A$ gibt, welche der Zahl $x$ zukommt. Eine solche Erklärung ... ist evident sinnlos; der Begriff “Eigenschaft rationaler Zahlen” ist nicht umfangsdefinit.

A real number is a set of rational numbers, which corresponds to a certain property of rational numbers. A set of real numbers thus corresponds to a property $A$ of properties of rational numbers. The upper bound of this set of real numbers is itself the set of of those rational numbers $x$ which satisfy a certain property $E$, to wit the following: that there be a property of type $A$ satisfied by the number $x$. Such an explanation is .... evidently nonsense; the notion “property of rational numbers” is not extensionally definite.

*Das Kontinuum* therefore shows the way to an analysis built from rational numbers exclusively via predicative definitions; it is strictly poorer than the usual analysis: e.g., arbitrary bounded subsets of the continuum do no longer have an upper bound in general.
This predicative analysis provides maximum security, but falls very much short of giving what we would like to have for applications to the real world: ... from page 37 ....

Fassen wir den Mengenbegriff in dem präzisen Sinne, wie ich es hier befürwortet habe, so gewinnt die Behauptung, daß jedem Punkte einer Geraden (nach Wahl eines Anfangspunktes und einer Einheitsstrecke) als Maßzahl eine reelle Zahl [...] entspricht und umgekehrt, einen schwerwiegenden Inhalt. Sie stellt eine merkwürdige Verknüpfung her zwischen dem in der Raumschauung Gegebenen und dem auf logisch-begrifflichem Wege Konstruierten. Offenbar aber fällt diese Aussage gänzlich aus dem Rahmen dessen heraus, was uns die Anschauung irgendwie über das Kontinuum lehrt und lehren kann; es handelt sich da nicht mehr um eine morphologische Beschreibung des in der Anschauung sich Darbietenden (das vor allem keine Menge diskreter Elemente, sondern ein fließendes Ganzes ist), vielmehr werden der unmittelbar gegebenen, ihrem Wesen nach inexakten Wirklichkeit exakte Wesen substriuirt.

If we make precise the notion of set in the way here proposed then the claim that to every point on the line ... correspond a real number, and vice versa, acquires a profound content. It establishes a peculiar link between what is given in our intuition of space and what is construed in a logical-conceptual manner. But this claim obviously leaves entirely the scope of what intuition teaches us or may teach us about the continuum; it is no longer a morphological description of what intuition offers us (and which above all is not a set of discrete elements but a flowing totality); ....
The inadequacy of Weyl’s poor continuum becomes glaringly obvious when he tentatively tries to apply it to *time*. At the end of this attempt he writes on page 68:

Well, I think everything we ask here is evidently nonsense: to all these questions, our intuition of time gives us no answer (from which we would expect conceptual clarification of the nature of its flow); just as someone gives no answer to questions that were obviously addressed to him only because of a mix-up, and that he therefore cannot understand.

......

It is the great merit of Bergson’s philosophy to have underscored this profound alienation of the world of mathematical concepts from the immediately experienced continuity of the phenomenon of time (“*la durée*”).

Besides Bergson, Husserl is mentioned, receives a copy of the book, and writes to Weyl.
The security of sound foundations is thus obtained in *Das Kontinuum* at the high price of violating our intuition of space and time. This *Kontinuum* is all but holistically satisfying for the mathematician-physicist Hermann Weyl. But since he sees no way out of this dilemma during the war, he does sketch the principles of physical applications of the poor continuum. First, to determine a point in homogeneous space-time, one has to refer to a coordinate system:

Das Koordinatensystem ist das unvermeidliche Residuum der Ich-Vernichtung in jener geometrisch-physikalischen Welt, welche die Vernunft aus dem Gegebenen unter der Norm der “Objektivität” herausätscht – letztes dürftiges Wahrzeichen noch in dieser objektiven Sphäre dafür, daß Dasein nur gegeben ist und gegeben sein kann als intentionaler Inhalt der Bewußtseinserlebnisse eines reinen, sinngebenden Ich.

The coordinate system is the inevitable residue of the annihilation of the ego in that geometrical-physical world which reason carves out of what is given under the norm of “objectivity” - the last meagre symbol even in this objective sphere for the fact that existence is only given and can only be given as intentional content of the conscious experience of a pure, sense-creating ego.

*German idealism as a last consolation.*

(page 72)
Weyl’s holistic turn after the war

For the traditional analysis the continuum appeared as a set of its points; it was treated as a special case of the basic logical relationship between element and set. Who would not have noticed already that the equally basic relationship between the whole and its parts had found no place at all in mathematics? But the fact that it has parts is the fundamental property of the continuum; and thus Brouwer’s theory (in accordance with intuition upon which today’s “atomism” infringes so badly) takes this relation as the foundation of the mathematical treatment of the continuum.

Note the catchwords of holism, plus “intuition.”

(1921)
A new type of unison between humanity and formal mathematics suddenly seemed possible; from a 1920 petition of ETH students to prevent Weyl from leaving Zürich:


... Our conviction that Herr Prof. Weyl is irreplaceable has its source in the following reasons: We admire in him the ingenious creator of new cultural values which consist in that the exact sciences come into fruitful interaction with life itself. It is this exceedingly fortunate fusion of the man and the scholar in Herr Prof. Weyl which inspires in each one of us a sense of liberation ... and seems to us to guarantee most surely that whole men will emerge from the eighth section.
Existential states of affairs are empty inventions of logicians. “2 is an even number”: this is a real judgment expressing a state of affairs; “there is an even number” is merely a judgment abstract (Urteilsabstrakt). If knowledge is a precious treasure, then the judgment abstract is a piece of paper indicating the presence of a treasure, yet without revealing at which place. ...

1921:
Mathematics is, as Brouwer occasionally puts it, more of an activity than a doctrine. ...... Brouwer’s view ties together the highest intuitive clarity with freedom. It must have the effect of a deliverance from a nightmare for whoever has maintained any sense for intuitively given facts in the abstract formalism of mathematics.

1925:
The ice cover has burst into floes, and now the element of flux was soon altogether master over the solid. L.E.J. Brouwer sketches a rigorous mathematical theory in which ... the continuum is not conceived as a rigid Being but as medium of free becoming. With this we also regain our freedom as concerns number sequences and sets of numbers. We no longer try to gain a yes or no answer ... by stretching the sequences on the Procrustean bed of construction principles. ...... With Brouwer, mathematics gains the highest intuitive clarity ......
We have seen 3 phases of Weyl’s thinking:

- Holistic nostalgia in prefaces or endnotes
- Selflimitation for security in wartime
- Holism within the foundations of analysis
- Weyl leaves this paradise after a few years (for reasons that are open to debate....)

Phases 2 and 3 are openly counter-modern, but in very different ways. Phase 2 provides a perfect metaphorical reaction to the menace of war.
Epilogue written after talk and discussion.

The critical discussion of the talk suggested on the one hand to incorporate at least some aspects of Weyl's successive editions of “Raum, Zeit, Materie” into the picture presented. On the other hand, clear criteria allowing one to recognize “holistic” approaches, and their detailed verification in the case of Weyl’s intuitionist writings were found lacking in my presentation. I accept this criticism; the last part of the presentation, about Weyl’s adaption of Brouwer, is far too short and sketchy. Yet, I would like to stress that, in my opinion, what makes a writer “holistic” is not so much a matter of his endorsing certain philosophical creeds, but of his attitude [“his whole attitude”, so to say] with respect to the ambient discourse and to the mainstream of his science. That was the reason why I started from Harrington’s case studies and tried to present my case by way of analogy. For the same reason, the question, brought up in the discussion, whether Riemann was holistic, can also not be discussed independently of the question of what holism may have meant in the middle of the 19th century, i.e., before Bergson etc.