THE ASTROPHYSICAL JOURNAL, 342:635–638, 1989 July 15

EXACT VACUUM SOLUTION TO CONFORMAL WEYL GRAVITY AND GALACTIC ROTATION CURVES

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Received 1988 November 7; accepted 1988 December 21

ABSTRACT

We present the complete, exact exterior solution for a static, spherically symmetric source in locally conformal invariant Weyl gravity. The solution includes the familiar exterior Schwarzschild solution as a special case and contains an extra gravitational potential term which grows linearly with distance. Our obtained solution provides a potential explanation for observed galactic rotation curves without the need for dark matter. Our solution also has some interesting implications for cosmology.

Subject headings: cosmology — dark matter — galaxies: internal motions — gravitation

I. INTRODUCTION

There is by now a broad consensus in the community that the correct theory of gravity (at least at the classical level) is that based on the Einstein equations, i.e., the equations derived by the variation of the Einstein-Hilbert action $I_{\mathrm{EH}} = -(1/16\pi G) \int d^4x(-g)^{1/2}R^{\alpha}{}_{\alpha}$. This consensus has been reached on account of the fact that the standard tests of General Relativity have established that, in the vicinity of the Sun, the geometry which fixes the geodesics is that given by the experimentally tested part of the Schwarzschild line element. The fact that particles should indeed move on geodesics is actually a rather general property of curved space time, resulting from the fact that the energy momentum tensor is covariantly conserved, i.e., that $T^{\mu\nu}{}_{;\nu} = 0$, with this latter property itself being true by virtue of the Bianchi identities. A well known (see, e.g., Eddington 1922, pp. 140–144) but not always appreciated fact is that $T^{\mu\nu}$ will be covariantly conserved for any action which is a coordinate scalar, since for any such action there would always be an appropriate set of Bianchi identities. Thus there is therefore an inherent freedom in the choice of the action for the gravitational field. What makes the choice of the Einstein action unique is the additional (simplifying) requirement that the resulting equations of motion be no higher than second order in the metric. Other choices for the action scalar can then yield field equations different from those of Einstein which could still in principle comply with the standard tests of General Relativity.[2] In this note we present (§ II) the field equations resulting from a highly restrictive prescription for choosing the action, one *based on an invariance principle rather than the demand of second-order equations of motion* and provide their complete, exact, vacuum solution for a spherically symmetric geometry (i.e., the analog of the Schwarzschild solution for this theory). Finally, in § III the solution is discussed and contact with observations is made, suggesting a possible explanation for the observed galactic rotation curves without the need to invoke the existence of dark matter.

II. ACTION AND FIELD EQUATIONS

In analogy with the principle of local gauge invariance which severely restricts the structure of possible Lorentz invariant actions in flat spacetime theories, we impose the quite analogous principle of local conformal invariance as the requisite principle to restrict the choice of action for the gravitational field in curved spacetime. This principle requires the action to remain invariant under any and all local stretchings $g_{\mu\nu}(x) \to \Omega^2(x)g_{\mu\nu}(x)$ of the geometry. The action

$$I_W = -\alpha \int d^4x(-g)^{1/2}C_{\lambda\mu\nu\kappa}C^{\lambda\mu\nu\kappa} = -2\alpha \int d^4x(-g)^{1/2}[R_{\mu\kappa}R^{\mu\kappa} - (1/3)(R^{\alpha}{}_{\alpha})^2], \tag{1}$$

where $C_{\lambda\mu\nu\kappa}$ is the conformal Weyl tensor (Weyl 1918) and α is a purely dimensionless coefficient, is completely locally conformal invariant and represents the *unique* four-dimensional coordinate scalar with the requisite properties.[3] Recently one of us (Mannheim 1988) has reopened the question of the Weyl alternative to Einstein gravity by pointing out that in the Weyl theory it is possible to have a nontrivial de Sitter geometry as a solution even in the absence of any cosmological constant term, with such a term vanishing identically precisely because of the conformal invariance of the theory. Thus, such a theory naturally alleviates the discrepancy between theoretical considerations (which suggest that the cosmological constant should be of the order of the Planck energy density) and observation (which suggests that it is at least 10^{120} times smaller). Under the same principle, the familiar Einstein-Hilbert action with its intrinsic Newton's constant scale is also absent. By then bypassing the Einstein action completely, we differ from other approaches to higher order gravity which try either to incorporate the Einstein term in the higher order action *ab initio* or induce it in some appropriate limit. Moreover, since we have a strictly conformally invariant theory, particle masses can only arise through dynamics (i.e., via the spontaneous breaking of the symmetry of the action), a situation which is now anyway thought

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[2] A specific example is given by the field equations resulting from variation of the action $I = \int d^4x(-g)^{1/2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, which as shown by Eddington (1922) do admit the Schwarzschild line element as a solution (this is not their most general solution though) and hence could pass the standard tests of General Relativity.

[3] We hereafter refer to the theory based on the action I_W of eq. (1) as Weyl gravity; this should not be confused with Weyl geometry, a non-Riemannian geometry in which $g^{\mu\nu}{}_{;\alpha}$ is equal to $b_\alpha g^{\mu\nu}$ rather than zero. Our geometry is indeed Riemannian, i.e., b_α is zero.

to occur in modern theories of elementary particles, and is thus also suggestive of a point of view which is compatible with Weyl gravity.

The action of equation (1) leads to the following gravitational field equations (DeWitt 1964)

$$(-g)^{-1/2} g_{\mu\alpha} g_{\nu\beta} \frac{\delta I_W}{\delta g_{\alpha\beta}} = -2\alpha[W^{(2)}_{\mu\nu} - \tfrac{1}{3} W^{(1)}_{\mu\nu}] = -2\alpha W_{\mu\nu} = -\tfrac{1}{2} T_{\mu\nu} , \tag{2}$$

in the presence of a matter energy momentum tensor $T_{\mu\nu}$, with $W^{(1)}_{\mu\nu}$ and $W^{(2)}_{\mu\nu}$ being given by

$$W^{(1)}_{\mu\nu} = 2g_{\mu\nu}(R^{\alpha}{}_{\alpha})^{;\beta}{}_{;\beta} - 2(R^{\alpha}{}_{\alpha})_{;\mu;\nu} - 2R^{\alpha}{}_{\alpha} R_{\mu\nu} + \tfrac{1}{2} g_{\mu\nu}(R^{\alpha}{}_{\alpha})^2 ;$$

$$W^{(2)}_{\mu\nu} = \tfrac{1}{2} g_{\mu\nu}(R^{\alpha}{}_{\alpha})^{;\beta}{}_{;\beta} + R_{\mu\nu}{}^{;\beta}{}_{;\beta} - R_{\mu}{}^{\beta}{}_{;\nu;\beta} - R_{\nu}{}^{\beta}{}_{;\mu;\beta} - 2R_{\mu\beta} R_{\nu}{}^{\beta} + \tfrac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} . \tag{3}$$

Thus whenever $R_{\mu\nu}$ is zero it follows that $W_{\mu\nu}$ vanishes too, so any vacuum solution of the Einstein equations would automatically satisfy those of Weyl gravity too. However, the vacuum vanishing of $W_{\mu\nu}$ may be realized other than by having $R_{\mu\nu}$ vanish and hence it is not immediately clear in general as to whether $W_{\mu\nu}$ may indeed replace the more familiar $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{\alpha}{}_{\alpha}$ in the gravitational field equations. Standing in the way of answering this question is the fact that, as can be seen from equation (3), the set of fourth-order (in general) equations $W_{\mu\nu} = 0$ is somewhat intractable to solve. Therefore, in order to gain insight as to how relevant Weyl gravity may in fact be in the real world we reexamine the problem of the general line element associated with $W_{\mu\nu}$ in a static, spherically symmetric geometry, and in this paper we present the complete and exact vacuum solution that we have found for it.

In trying to solve vacuum Weyl gravity we note first that the general static, spherically symmetric line element

$$ds^2 = -b(\rho)dt^2 + a(\rho)d\rho^2 + \rho^2 d\Omega \tag{4}$$

may be rewritten as

$$ds^2 = \frac{p^2(r)}{r^2} [-B(r)dt^2 + A(r)dr^2 + r^2 d\Omega] \tag{5}$$

under the general coordinate transformation

$$\rho = p(r) , \qquad B(r) = \frac{r^2 b(r)}{p^2(r)} , \qquad A(r) = \frac{r^2 a(r) p'^2(r)}{p^2(r)} \tag{6}$$

with the function $p(r)$ being so far arbitrary. Choosing $p(r)$ according to

$$-\frac{1}{p(r)} = \int \frac{dr}{r^2[a(r)b(r)]^{1/2}} \tag{7}$$

then yields for the line element

$$ds^2 = \frac{p^2(r)}{r^2} \left[-B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega \right] . \tag{8}$$

The general line element is thus conformal to a standard line element in which $A(r)^{-1} = B(r)$. However, since the Weyl theory is itself conformal, the $W_{\mu\nu}(\rho)$ tensor associated with equation (4) transforms as $W_{\mu\nu}(x) \to \Omega^{-2}(x) W_{\mu\nu}(x)$ under a conformal transformation on the geometry, and $W_{\mu\nu}(\rho)$ is thus conformal to the $W_{\mu\nu}(r)$ tensor associated with the line element

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega . \tag{9}$$

Hence it may be determined (up to an arbitrary overall r-dependent conformal factor) using the line element of equation (9). Moreover, in the vacuum $W_{\mu\nu}(\rho)$ is to vanish, and thus so must $W_{\mu\nu}(r)$, implying that for the vacuum all observable information is contained in $W_{\mu\nu}(r)$ as calculated from equation (9).

To extract out $W_{\mu\nu}(r)$ from equation (3) using the line element of equation (9) is still quite formidable since it involves quite complicated covariant differentiations. Rather than do this we note instead that for any action $I = \int d^4x(-g)^{1/2} L$ with Lagrangian density L and any static line element

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + C(r)d\theta^2 + D(r, \theta)d\phi^2 \tag{10}$$

we may calculate

$$g^{1/2} W^{rr} = \frac{\delta I}{\delta A} = \frac{\partial}{\partial A}(g^{1/2} L) - \frac{\partial}{\partial r}\left(g^{1/2} \frac{\partial L}{\partial A'}\right) + \frac{\partial^2}{\partial r^2}\left(g^{1/2} \frac{\partial L}{\partial A''}\right) \tag{11}$$

(and likewise for $\delta I/\delta B$, $\delta I/\delta C$ and $\delta I/\delta D$) by ordinary differentiation. For the Weyl action of equation (1) and the line element of equation (10) this yields in general [on setting $C(r) = r^2$ and $D(r, \theta) = r^2 \sin^2\theta$]

$$\begin{aligned} 48A^5B^4r^4W^{rr} = {} & [8A^2B^2B'B''' - 4A^2B^2B''^2 - B''(12A^2BB'^2 + 8AA'B^2B') + 7A^2B'^4 + 6AA'BB'^3 + B^2B'^2(7A'^2 - 4AA'')]r^4 \\ & + [-16A^2B^3B''' + B''(48A^2B^2B' + 16AA'B^3) - 20A^2BB'^3 - 16AA'B^2B'^2 + B^3B'(16AA'' - 28A'^2)]r^3 \\ & + [-32A^2B^3B'' - 4A^2B^2B'^2 + 8AA'B^3B' + B^4(28A'^2 - 16AA'')]r^2 + 32A^2B^3B'r + B^4(16A^4 - 16A^2) . \end{aligned} \tag{12}$$

We do not need to give the expressions for W^{00} and $W^{\theta\theta}$ here (in a static geometry W^{0r} is identically zero) since they can be determined indirectly via the Bianchi and trace identities that $W_{\mu\nu}$ satisfies, viz.,

$$\left(\frac{\partial}{\partial r}+\frac{2}{r}+\frac{A'}{A}+\frac{B'}{2B}\right)W^{rr}+\frac{B'}{2A}W^{00}-\frac{2r}{A}W^{\theta\theta}=0\ ; \qquad -BW^{00}+AW^{rr}+2r^2W^{\theta\theta}=0 \tag{13}$$

(thus incidentally providing us with an independent check on our use of equation [11] in deriving them.) In a conformal theory then all information is contained in W^{rr} which itself only involves derivatives up to third order.

Restricting now to the line element of interest given in equation (9) yields for W^{rr} the expression

$$B^{-1}W^{rr}=\frac{1}{6}B'B'''-\frac{1}{12}B''^2-\frac{1}{3r}(BB'''-B'B'')-\frac{1}{3r^2}(BB''+B'^2)+\frac{2}{3r^3}BB'-\frac{B^2}{3r^4}+\frac{1}{3r^4}. \tag{14}$$

The substitution $B(r)=r^2f(r)$ enables us to rewrite equation (14) as

$$B^{-1}W^{rr}=\frac{1}{6}r^4\left[f'f'''-\frac{1}{2}f''^2+\frac{2}{r^8}+\frac{4}{r}f'f''+\frac{4}{r^2}f'^2\right]. \tag{15}$$

The further substitution $f'(r)=y^2(r)r^{-4}$ then yields

$$B^{-1}W^{rr}=\frac{1}{3r^4}(1+y^3y'')\ , \tag{16}$$

an extraordinarily simple expression.

The vacuum equation $W^{rr}=0$ associated with equation (16) is itself readily solved. First and second integrals are given by

$$y'^2-\frac{1}{y^2}=\gamma\ , \tag{17}$$

$$\gamma r=(1+\gamma y^2)^{1/2}+3\beta\gamma-1\ , \tag{18}$$

in terms of conveniently defined integration constants γ and $3\beta-1/\gamma$, so that a third integration with associated constant k yields finally for the exterior line element

$$A(r)^{-1}=B(r)=1-\frac{\beta(2-3\beta\gamma)}{r}-3\beta\gamma+\gamma r-kr^2 \tag{19}$$

a solution which is exact without any approximation at all.

III. CONCLUSIONS AND DISCUSSION

With regard to our solution we note first that when the parameter γ is set equal to zero the line element associated with equation (19) becomes precisely the familiar Schwarzschild solution in a de Sitter background whose scalar curvature is given by $R^{\alpha}{}_{\alpha}=-12k$. [In the Einstein theory the metric with $\gamma=0$ and $k\neq 0$ would only be obtained in the presence of a cosmological term. In the Weyl theory, as already noted by Mannheim (1988) the de Sitter solution is a vacuum ($T_{\mu\nu}=0$) solution, and thus involves no cosmological term at all]. Consequently the parameter γ measures all departures of the Weyl theory from that of Einstein with a cosmological term, and for small enough γ the Weyl theory would appear to enjoy the experimental successes of Einstein relativity. Theoretically, just like the parameters β and k, the parameter γ should be related to the interior dynamics of the static source of interest or the properties of the background geometry (or both) and should hence be expressible in terms of these properties if the exterior and interior solutions could be matched. Unfortunately, we have not yet been able to solve the associated interior problem in order to perform this matching, partly because $W_{\mu\nu}$ is still somewhat unwieldy when it is nonzero, but mainly because in a conformal theory we need an explicit dynamical model for how the gravitating system is to get its mass in the first place in order to actually solve the interior Weyl gravity problem. Fortunately though, as we have seen, it is not necessary to solve this difficult dynamical problem in order to extract out the general form of the exterior solution in terms of phenomenological system dependent parameters β and γ.

In order to confront our theory with observations, we now note that the form of the line element of equation (19) indicates the presence of a Newtonian $1/r$ term which should dominate at small distances with the γr term being the more dominant one at larger distances.[4] Further, since k is the cosmological scalar curvature, the kr^2 term should become important at cosmological distances. Unfortunately, at this point nothing more can currently be said theoretically about the relative importance of these terms. One can, however, set limits from the compatibility of observations with dynamics based solely on a geometry given by $A(r)^{-1}=B(r)=1-2MG/r$. As is well known, there exists a celebrated case where observations are apparently in conflict with such dynamics; this involves the discrepancy between the observations of galactic rotation curves and expectations based on the standard laws of gravity using the observed distribution of luminous matter (Rubin, Ford, and Thonnard 1978). In our theory, however, we note that if the γr term is comparable in magnitude on a typical galactic scale (say 10 kpc for definitiveness) to the Newtonian $1/r$ term with typical galactic coefficient $10^{11}\,M_{\odot}G$, it would lead to a potential which grows with distance for $r>10$ kpc, which is approximately constant for $r\sim 10$ kpc, and which becomes Newtonian for $r<10$ kpc. This potential would lead to

[4] In passing we note that our potential is reminiscent of the confining potentials currently being considered in bound state models of heavy quark-antiquark systems.

rotational velocities that are approximately constant in the intermediate (10 kpc) regime and that eventually grow like $r^{1/2}$ at large enough distances. This picture accords well qualitatively with observed galactic rotation curves which are either more or less flat (Rubin *et al.* 1978) or even growing with distance (Blitz 1979) on the kiloparsec scales of interest. As to an actual value for the parameter γ for a typical galaxy, the value of γ for which the $1/r$ and r terms are comparable is $\gamma \approx 10^{-28}$ cm^{-1}, (which intriguingly is roughly the value of the inverse Hubble length) with the dimensionless quantity $\beta\gamma$ then taking the value $\beta\gamma \approx 10^{-12}$. For this value of γ a typical expected flat rotational velocity would be of order 10^2 km s^{-1}, which is in general order of magnitude accord with experimental findings.[5]

One should further note that when $\beta = 0$, or for that matter when r is sufficiently large so that all the β-dependent terms in equation (19) can be ignored, the resulting metric, under the coordinate transformation

$$\rho = \frac{4r}{2(1 + \gamma r - kr^2)^{1/2} + 2 + \gamma r} \quad \text{and} \quad \tau = \int R(t)dt \tag{20}$$

can be brought to the form

$$ds^2 = \frac{1}{R^2(\tau)} \frac{[1 - \rho^2(\gamma^2/16 + k/4)]^2}{[(1 - \gamma\rho/4)^2 + k\rho^2/4]^2} \left\{ -d\tau^2 + \frac{R^2(\tau)}{[1 - \rho^2(\gamma^2/16 + k/4)]^2} (d\rho^2 + \rho^2 d\Omega) \right\} . \tag{21}$$

This metric is conformal to a RW metric with arbitrary scale factor $R(\tau)$ and three-space curvature $K = -k - \gamma^2/4$ (it may therefore be quite natural in Weyl gravity that the value of γ is correlated with the inverse Hubble length). On the other hand, for small enough r the form of the metric (neglecting the $3\beta\gamma$ term which is of order 10^{-12} for a galaxy in our solution) becomes that of the exterior Schwarzschild solution. Our solution is therefore able to interpolate between those of Robertson-Walker and Schwarzschild in a continuous and smooth manner by exploiting the conformal structure of the theory. As far as we know, this is the first time that such a type of solution has been obtained. In view of the above arguments the effect of the γr term in galactic dynamics could be due to the influence of the background geometry on the stellar trajectories (Mach's principle?) at the outskirts of a galaxy.

It is also of interest to calculate the Weyl tensor associated with the space of the metric given in equation (19). The relevant components of the Weyl tensor are all found to be proportional to the quantity $r^2B'' - 2rB' + 2(B - 1)$. Substituting the expression for $B(r)$, the above quantity becomes proportional to $\beta(2 - 3\beta\gamma + \gamma r)/r$. One observes that when $\beta = 0$ the solution is conformally flat, in agreement with the arguments given above. One should also note that when $\beta \neq 0$ the contribution to the Weyl tensor arising from the coefficient of the $1/r$ term vanishes as $r \to \infty$. The nonvanishing contribution comes from the constant term $-3\beta\gamma$. Our solution appears to represent a massive body embedded in a conformally flat spherically symmetric space. The conformal flatness of the background space is broken by the mere presence of the massive body. As indicated by the evaluation of the Weyl tensor, the breaking of conformal flatness is manifest even at infinity where one would otherwise expect the gravitational effects of the source to be negligible. The theory hence allows the breaking of the conformal symmetry of its action *in its solution*, thus suggesting the possibility of a gravitational origin for inertial mass.

While there have been several attempts made in the literature to account for the galactic rotation curves without the introduction of any so far unobserved dark matter by ad hoc modifications of either Newton's second law (Milgrom 1983; Bekenstein and Milgrom 1984) or the Newtonian law of gravity (Sanders 1984), the present approach differs from them in that it was not at all introduced for the purpose of resolving a specific problem in astrophysics; rather our proposed solution is simply a byproduct of a theory of gravity which is itself based on a principle, namely that of local conformal invariance of the world geometry. Given its potential for addressing successfully one of the long-standing problems in astrophysics, and given its remarkable economy of assumptions, we believe it to be a potentially viable theory of gravity which merits further consideration and study.

We would like to thank B. Schaefer for useful discussions regarding possible irregularities in the orbits of the planets Neptune and Pluto. One of us (P. D. M.) wishes to acknowledge the National Research Council of the National Academy of Sciences for financial support for this work and also wishes to thank S. Holt, R. Ramaty and F. Stecker for the warm hospitality of the Laboratory for High Energy Astrophysics at the Goddard Space Flight Center where this work was performed. The work of P. D. M. has also been supported in part by the US Department of Energy under grant No. DE-AC02-79ER10336.A.

[5] Given the fact that our potential grows with distance it would at first appear that its effects would be even more important outside of galaxies than in them. However, far enough away from a given galaxy, a test particle would feel the effects of the linear terms due to the contribution of the other galaxies. If the galaxies are distributed homogeneously on the largest scales then all of their linear terms will merge into a uniform contribution to the general cosmological background.

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