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The Quantum Theory of the Electron. Part II.

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In a previous paper by the author* it is shown that the general theory of quantum mechanics together with relativity require the wave equation for an electron moving in an arbitrary electromagnetic field of potentials, A_0, A_1, A_2, A_3 to be of the form

$$\mathrm{F}\psi \equiv \left[p_0 + \frac{e}{c}\mathrm{A}_0 + \alpha_1\left(p_1 + \frac{e}{c}\mathrm{A}_1\right) + \alpha_2\left(p_2 + \frac{e}{c}\mathrm{A}_2\right) + \alpha_3\left(p_3 + \frac{e}{c}\mathrm{A}_3\right) + \alpha_4 mc\right]\psi = 0. \qquad (1)$$

The α's are new dynamical variables which it is necessary to introduce in order to satisfy the conditions of the problem. They may be regarded as describing some internal motion of the electron, which for most purposes may be taken to be the spin of the electron postulated in previous theories. We shall call them the spin variables.

The α's must satisfy the conditions

$$\alpha_\mu{}^2 = 1, \qquad \alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 0. \qquad (\mu \neq \nu.)$$

They may conveniently be expressed in terms of six variables ρ_1, ρ_2, ρ_3, σ_1, σ_2, σ_3 that satisfy

$$\left.\begin{aligned} &\rho_r{}^2 = 1, \quad \sigma_r{}^2 = 1, \quad \rho_r\sigma_s = \sigma_s\rho_r, \quad (r, s = 1, 2, 3) \\ &\rho_1\rho_2 = i\rho_3 = -\rho_2\rho_1, \quad \sigma_1\sigma_2 = i\sigma_3 = -\sigma_2\sigma_1 \end{aligned}\right\}, \qquad (2)$$

and

together with the relations obtained from these by cyclic permutation of the suffixes, by means of the equations

$$\alpha_1 = \rho_1\sigma_1, \quad \alpha_2 = \rho_1\sigma_2, \quad \alpha_3 = \rho_1\sigma_3, \quad \alpha_4 = \rho_3.$$

The variables σ_1, σ_2, σ_3 now form the three components of a vector, which corresponds (apart from a constant factor) to the spin angular momentum vector that appears in Pauli's theory of the spinning electron. The ρ's and σ's vary with the time, like other dynamical variables. Their equations of motion, written in the Poisson Bracket notation [ ], are

$$\dot{\rho}_r = c\,[\rho_r, \mathrm{F}], \qquad \dot{\sigma}_r = c\,[\sigma_r, \mathrm{F}].$$

* 'Roy. Soc. Proc.,' A, vol. 117, p. 610 (1928). This is referred to later by *loc. cit.*

It should be observed that these equations of motion are consistent with the conditions (2), so that if the conditions are satisfied initially they always remain satisfied. For example, we have

$$ih/c \cdot \dot{\sigma}_1 = \sigma_1 F - F\sigma_1 = 2i\rho_1\sigma_3 (p_2 + e/c \cdot A_2) - 2i\rho_1\sigma_2 (p_3 + e/c \cdot A_3).$$

Thus $\dot{\sigma}_1$ anticommutes with σ_1, so that

$$d\sigma_1^2/dt = \dot{\sigma}_1\sigma_1 + \sigma_1\dot{\sigma}_1 = 0.$$

The ρ's and σ's, and therefore also any function of them, can be represented by matrices with four rows and columns. A possible representation, in which ρ_3 and σ_3 are diagonal matrices, is given in (*loc. cit.*) § 2. Such a representation can apply only to a single instant of time, since the ρ's and σ's vary with the time. To get a scheme of representation which holds for all times, so that the equations of motion are valid in it, we should have to have only constants of the motion as diagonal matrices. It is, however, quite correct for the purpose of solving the wave equation (1) to take a matrix representation for the ρ's and σ's which holds only for a single instant of time (as was done in *loc. cit.*), since the wave function is then the transformation function connecting the ρ's, σ's and x's at this particular time with a set of variables that are constants of the motion, as is required for the general interpretation of quantum mechanics.

Before we proceed with the theory of atoms with single electrons that was begun in *loc. cit.*, the proof will be given of the conservation theorem, which states that the change in the probability of the electron being in a given volume during a given time is equal to the probability of its having crossed the boundary. This proof is supplementary to the work of *loc. cit.* § 3, and is necessary before one can infer that the theory will give consistent results that are invariant under a Lorentz transformation.

§ 1. *The Conservation Theorem.*

We shall first make a slight generalisation of the usual interpretation of wave mechanics to apply to cases when the Hamiltonian is not Hermitian. Let the wave equation, written in certain variables q, be

$$(H - W)\psi = 0. \qquad \text{(i)}$$

Consider also the equation

$$(\tilde{H} - \tilde{W})\phi = 0$$

or

$$(\tilde{H} + W)\phi = 0, \qquad \text{(ii)}$$

where the symbol $\tilde{a}$ denotes the matrix obtained from the matrix a by transposing rows and columns. If ψ_m, ϕ_n are suitably normalised solutions of (i) and (ii) respectively, referring to the states m and n, we take $\phi_n\psi_m$ to be the corresponding matrix element of the probability of the q's having specified values. If H is Hermitian, $\tilde{\mathrm{H}}$ is the conjugate imaginary of H (obtained by writing $-i$ for i) and the solutions of (ii) are just the conjugate imaginaries to the solutions of (i), so that in this case our probability $\phi_n\psi_m$ becomes the usual one $\bar{\psi}_n\psi_m$. In the general case it is necessary to use the transposed Hamiltonian instead of the conjugate imaginary Hamiltonian in (ii) in order to secure that if ϕ_n, ψ_m are initially orthogonal or mutually normalised $\left(i.e., \int\phi_n\psi_m\,dq = 1\right)$, they always remain orthogonal or mutually normalised respectively.

Our wave equation for an electron in an electromagnetic field is

$$[p_0 + e'\mathrm{A}_0 + \rho_1(\boldsymbol{\sigma}, \mathbf{p} + e'\mathbf{A}) + \rho_3 mc]\,\psi = 0 \qquad (3)$$

where $e' = e/c$. The Hamiltonian here will be Hermitian if a matrix scheme for the spin variables is chosen in which they are Hermitian. However, if one now applies a Lorentz transformation to this wave equation and divides out by the coefficient of the new p_0, the resulting new Hamiltonian will not, in general, be Hermitian, although, as shown in *loc. cit.*, § 3, it may be brought back to its original Hermitian form by a canonical transformation of the matrix scheme for the spin variables. In the following work we require to have the same matrix representation of the spin variables for all frames of reference, so we cannot assume our Hamiltonian is Hermitian, and must use the above generalised interpretation.

The equation obtained by transposing rows and columns in the operator of (3) is

$$[-p_0 + e'\mathrm{A}_0 + \tilde{\rho}_1(\tilde{\boldsymbol{\sigma}}, -\mathbf{p} + e'\mathbf{A}) + \tilde{\rho}_3 mc]\,\phi = 0. \qquad (4)$$

The probability per unit volume of the electron being in the neighbourhood of any point is given, according to the above assumption, by $\phi\psi$, where this product must now be understood to mean the sum of the products of each of the four components of ϕ (referring respectively to the four rows or columns of the matrices ρ, σ) into the corresponding component of ψ. We have to prove that this probability is the time component of a 4-vector, and that the divergence of this 4-vector vanishes.

From (3)

$$[\rho_3(p_0 + e'A_0) + \rho_1\rho_3(\sigma, \mathbf{p} + e'\mathbf{A}) + mc]\rho_3\psi = 0$$

or

$$[\gamma_0(p_0 + e'A_0) + \Sigma_{r=1,2,3}\gamma_r(p_r + e'A_r) + mc]\chi = 0, \tag{5}$$

where

$$\gamma_0 = \rho_3, \qquad \gamma_r = \rho_1\rho_3\sigma_r, \qquad \chi = \rho_3\psi.$$

Equation (5) is symmetrical between the four dimensions of space and time, and shows that γ_0, $-\gamma_1$, $-\gamma_2$, $-\gamma_3$ are the contravariant components of a 4-vector. If we multiply (4) by $\tilde{\rho}_3$ on the left-hand side, we get

$$[\tilde{\gamma}_0(-p_0 + e'A_0) + \Sigma_r\tilde{\gamma}_r(-p_r + e'A_r) + mc]\phi = 0, \tag{6}$$

since

$$\tilde{\gamma}_0 = \rho_3. \qquad \tilde{\gamma}_r = \tilde{\sigma}_r\tilde{\rho}_3\tilde{\rho}_1 = \tilde{\rho}_3\tilde{\sigma}_1\tilde{\rho}_r.$$

The operator in this equation is just the transposed operator of (5). The probability per unit volume of the electron being in any place is now given by

$$\phi\psi = \phi\rho_3\chi = \phi\gamma_0\chi, \tag{7}$$

where $\phi\alpha\chi$ denotes the sum of the products of each component of ϕ into the corresponding component of $\alpha\chi$, α being any function of the spin variables, represented by a matrix with four rows and columns. [Note that quite generally $\phi\alpha\chi = \chi\tilde{\alpha}\phi$.] Expression (7) is the time component of a 4-vector, whose spacial components, namely,

$$-\phi\gamma_1\chi, \qquad -\phi\gamma_2\chi, \qquad -\phi\gamma_3\chi,$$

must give $1/c$ times the probability per unit time of the electron crossing unit area perpendicular to each of the three axes respectively.

We must now show that the divergence of this 4-vector vanishes, *i.e.*, that

$$\frac{1}{c}\frac{\partial}{\partial t}(\phi\gamma_0\chi) - \Sigma_r\frac{\partial}{\partial x_r}(\phi\gamma_r\chi) = 0. \tag{8}$$

Multiplying (5) by ϕ and (6) by χ and subtracting, we get

$$\phi[\gamma_0 p_0 + \Sigma_r\gamma_r p_r]\chi + \chi[\tilde{\gamma}_0 p_0 + \Sigma_r\tilde{\gamma}_r p_r]\phi = 0,$$

which gives

$$\phi\left[\gamma_0\frac{\partial}{c\partial t} - \Sigma_r\gamma_r\frac{\partial}{\partial x_r}\right]\chi + \chi\left[\tilde{\gamma}_0\frac{\partial}{c\partial t} - \Sigma_r\tilde{\gamma}_r\frac{\partial}{\partial x_r}\right]\phi = 0,$$

or

$$\phi\left[\gamma_0\frac{\partial}{c\partial t} - \Sigma_r\gamma_r\frac{\partial}{\partial x_r}\right]\chi + \frac{1}{c}\frac{\partial\phi}{\partial t}\gamma_0\chi - \Sigma_r\frac{\partial\phi}{\partial x_r}\gamma_r\chi = 0.$$

This gives immediately the conservation equation (8) as the γ's are here constant matrices.

§ 2. *The Selection Principle.*

In *loc. cit.* the quantum number j was introduced, which determines the magnitude of the resultant angular momentum for an electron moving in a central field of force. j can take both positive and negative integral values. Again, the magnetic quantum number $u = M_3/h$, say, that determines the component of the total angular momentum in some specified direction, was shown to take half odd integral values from $-|j| + \frac{1}{2}$ to $|j| - \frac{1}{2}$. The state $j = 0$ is thus excluded, and the weight of any state j is $2|j|$. The equation obtained to determine the energy levels, *i.e.*, equation (25) or (26), involves j only through the combination $j(j+1)$ except in the last term, which represents the spin correction. Thus two values of j which give the same value for $j(j+1)$ form a spin doublet, so that $j = j'$ and $j = -(j'+1)$ form a spin doublet when $j' > 0$. The connection between j-values and the usual notation for alkali spectra is therefore given by the following scheme :—

$$\begin{array}{cccccc} j = -1 & & 1 \quad -2 & & 2 \quad -3 & & 3 \quad -4 \quad \ldots \\ \text{S} & & \text{P} & & \text{D} & & \text{F} \end{array}$$

There is no azimuthal quantum number k in the present theory, an orbit for an electron in an atom being defined by three quantum numbers n, j, u only. One might on this account expect the selection rules, the relative intensities of the lines of a multiplet, etc., in the usual derivation of which k plays an important part, to be different in the present theory, but it will be found that they do just happen to be the same.

We shall first determine the selection rule for j. We use the following two theorems :—

(i) If a dynamical variable X anticommutes with j, its matrix elements all refer to transitions of the type $j \to -j$.

(ii) If a dynamical variable Y satisfies

$$[[Y, jh], jh] = -Y, \tag{9}$$

its matrix elements all refer to transitions of the type $j \to j \pm 1$.

To prove (i) we observe that the condition $jX + Xj = 0$ gives

$$j' \cdot X(j'j'') + X(j'j'') \cdot j'' = 0$$

or

$$(j' + j'') \cdot X(j'j'') = 0.$$

Hence $X(j'j'') = 0$ unless $j'' = -j'$.

A proof of (ii) involving angle variables has been given in a previous paper.* A simple proof analogous to the foregoing proof of (i) is as follows. Equation (9) gives

$$\mathrm{Y}j^2 - 2j\,\mathrm{Y}j + j^2\mathrm{Y} = \mathrm{Y}$$

or

$$\mathrm{Y}(j'j'')\,.\,j''^2 - 2j'\,.\,\mathrm{Y}(j'j'')\,.\,j'' + j'^2\,.\,\mathrm{Y}(j'j'') = \mathrm{Y}(j'j'').$$

Hence $\mathrm{Y}(j'j'') = 0$ except when

$$j''^2 - 2j'j'' + j'^2 = 1,$$

i.e., when

$$j'' = j' \pm 1.$$

We shall now evaluate $[[x_3, jh], jh]$. The definition of j is

$$jh = \rho_3\{(\boldsymbol{\sigma}, \mathbf{m}) + h\}.$$

Hence

$$\begin{aligned}[x_3, jh] &= \rho_3\{\sigma_1[x_3, m_1] + \sigma_2[x_3, m_2]\} \\ &= \rho_3(\sigma_1 x_2 - \sigma_2 x_1),\end{aligned} \tag{10}$$

so that

$$[[x_3, jh], jh] = [\sigma_1 x_2 - \sigma_2 x_1, (\boldsymbol{\sigma}, \mathbf{m})].$$

Now

$$ih\,[\sigma_1, (\boldsymbol{\sigma}, \mathbf{m})] = \sigma_1(\boldsymbol{\sigma}, \mathbf{m}) - (\boldsymbol{\sigma}, \mathbf{m})\,\sigma_1 = 2i(\sigma_3 m_2 - \sigma_2 m_3)$$

or

$$\tfrac{1}{2}h\,[\sigma_1, (\boldsymbol{\sigma}, \mathbf{m})] = \sigma_3 m_2 - \sigma_2 m_3,$$

and similarly

$$\tfrac{1}{2}h\,[\sigma_2, (\boldsymbol{\sigma}, \mathbf{m})] = \sigma_1 m_3 - \sigma_3 m_1.$$

Hence

$$\begin{aligned}\tfrac{1}{2}h\,[[x_3, jh], jh] &= (\sigma_3 m_2 - \sigma_2 m_3)\,x_2 + \tfrac{1}{2}h\,\sigma_1(\sigma_3 x_1 - \sigma_1 x_3) \\ &\quad - (\sigma_1 m_3 - \sigma_3 m_1)\,x_1 - \tfrac{1}{2}h\,\sigma_2(\sigma_2 x_3 - \sigma_3 x_2) \\ &= \sigma_3(\mathbf{m}, \mathbf{x}) - m_3(\boldsymbol{\sigma}, \mathbf{x}) + \tfrac{1}{2}h\{-\sigma_3(\boldsymbol{\sigma}, \mathbf{x}) - x_3\} \\ &= -\mathrm{M}_3(\boldsymbol{\sigma}, \mathbf{x}) - \tfrac{1}{2}hx_3,\end{aligned}$$

so that

$$[[x_3, jh], jh] = -2u(\boldsymbol{\sigma}, \mathbf{x}) - x_3.$$

Thus x_3 does not quite satisfy the condition that Y satisfies in (9), owing to the extra term $-2u(\boldsymbol{\sigma}, \mathbf{x})$ This extra term, however, anticommutes with j. If we now form the expression $x_3 - cu(\boldsymbol{\sigma}, \mathbf{x})$, where c is some quantity that commutes with j, we can choose c so as to make this expression satisfy completely the condition that Y satisfies in (9). We have, in fact,

$$\begin{aligned}[[x_3 - cu(\boldsymbol{\sigma}, \mathbf{x}), jh], jh] &= -2u(\boldsymbol{\sigma}, \mathbf{x}) - x_3 + cu\,.\,4j^2(\boldsymbol{\sigma}, \mathbf{x}) \\ &= -\{x_3 - cu(\boldsymbol{\sigma}, \mathbf{x})\}\end{aligned}$$

* 'Roy. Soc. Proc.,' A, vol. 111, p. 281 (1926), § 3.

if c is chosen such that

$$-2+4j^2c=c,$$

i.e., if

$$c=1/2\,(j^2-\tfrac{1}{4}).$$

Hence x_3 can be expressed as the sum of two terms, namely,

$$\frac{u}{2\,(j^2-\frac{1}{4})}\,(\sigma,\,\mathbf{x}) \qquad \text{and} \qquad x_3-\frac{u}{2\,(j^2-\frac{1}{4})}\,(\sigma,\,\mathbf{x}),$$

of which the first anticommutes with j, and therefore contains only matrix elements referring to transitions of the type $j\rightarrow -j$, while the second satisfies the condition that Y satisfies in (9), and therefore contains only matrix elements referring to transitions of the type $j\rightarrow j\pm 1$. A similar result holds for x_1 and x_2. Hence the selection rule for j is

$$j\rightarrow -j \qquad \text{or} \qquad j\rightarrow j\pm 1.$$

Thus from states with $j=2$ transitions can take place to states with $j=1$, -2 or 3. Comparing this selection rule with the above scheme connecting j-values with the S, P, D notation, we see that it is exactly equivalent to the two selection rules for j and k of the usual theory, and is therefore in agreement with experiment.

§ 3. *The Relative Intensities of the Lines of a Multiplet.*

The relative intensities of the various components into which a line is split up in a weak magnetic field must be the same on the present theory as on previous theories, as they depend only on the Vertauschungs relations connecting the co-ordinates x_r with the components of total angular momentum M_r, which are taken over unchanged into the present theory. It will therefore be sufficient, for determining the relative intensities of the lines of a multiplet, to consider only one Zeeman component of each line, say, the component for which $\Delta u=0$, *i.e.*, the component that comes from x_3.

We shall determine the matrix elements of x_3, when expressed as a matrix in a scheme in which r, j, u and ρ_3 are diagonal. x_3 is diagonal in (*i.e.*, commutes with) all of these variables except j. The part of x_3 referring to transitions $j\rightarrow -j$ we found to be

$$\frac{u}{2(j^2-\frac{1}{4})}\,(\sigma,\mathbf{x})=\frac{u}{2(j^2-\frac{1}{4})}\,\varepsilon\,\rho_1\,r, \tag{11}$$

using the ε introduced in *loc. cit.* § 6. $\varepsilon\rho_1$ anticommutes with j, so that it can

contain only matrix elements of the type $\varepsilon\rho_1(j, -j)$, and from the condition $(\varepsilon\rho_1)^2 = 1$ we must have

$$|\varepsilon\rho_1(j, -j)| = 1.$$

Hence

$$|x_3(j, -j)| = \frac{u}{2(j^2 - \frac{1}{4})} r\,|\varepsilon\rho_1(j, -j)| = \frac{u}{2(j^2 - \frac{1}{4})} r. \tag{12}$$

Again, we have from (10)

$$\{x_3 - i[x_3, jh]\}\{x_3 + i[x_3, jh]\} = \{x_3 - i\rho_3(\sigma_1 x_2 - \sigma_2 x_1)\}\{x_3 + i\rho_3(\sigma_1 x_2 - \sigma_2 x_1)\}$$
$$= x_3^2 + (\sigma_1 x_2 - \sigma_2 x_1)^2 = r^2,$$

which gives

$$\{(j+1)x_3 - x_3 j\}\{x_3(j+1) - j x_3\} = r^2.$$

If we equate the (j, j) matrix elements of each side of this equation, we get on the left-hand side the sum of three terms, namely, the $(j, -j)$ matrix element of the first $\{\}$ bracket times the $(-j, j)$ element of the second, the $(j, j+1)$ element of the first times the $(j+1, j)$ element of the second, and the $(j, j-1)$ element of the first times the $(j-1, j)$ element of the second. The second of these three terms vanishes, leaving

$$(2j+1)^2|x_3(j, -j)|^2 + 4|x_3(j, j-1)|^2 = r^2.$$

Hence

$$|x_3(j, j-1)|^2 = \tfrac{1}{4}r^2\left\{1 - \frac{u^2}{(j-\frac{1}{2})^2}\right\} = \tfrac{1}{4}r^2\,\frac{(j+u-\frac{1}{2})(j-u-\frac{1}{2})}{(j-\frac{1}{2})^2}. \tag{13}$$

Writing $-j$ for j, we get

$$|x_3(-j, -j-1)|^2 = \tfrac{1}{4}r^2\,\frac{(j+u+\frac{1}{2})(j-u+\frac{1}{2})}{(j+\frac{1}{2})^2}. \tag{14}$$

The three matrix elements of x_3 given in (12), (13) and (14) are associated with the three components of the multiplet formed by the combination of two doublets. The ratios of these matrix elements will, to a first approximation, remain unchanged when one makes a transformation from the matrix scheme in which r, j, u, ρ_3 are diagonal to a scheme in which the Hamiltonian is diagonal, and will therefore give the relative intensities of the Zeeman components $\Delta u = 0$ of the lines in a combination doublet. These ratios are in agreement with those of previous theories based on the spinning electron model.

§ 4. *The Zeeman Effect.*

If there is a uniform magnetic field of intensity H in the direction of the x_3 axis, we can take the magnetic potentials to be

$$A_1 = -\tfrac{1}{2}Hx_2, \qquad A_2 = \tfrac{1}{2}Hx_1 \qquad A_3 = 0.$$

The additional terms appearing in the Hamiltonian F will now be

$$\Delta F = \rho_1 e' (\boldsymbol{\sigma}, \mathbf{A}) = -\tfrac{1}{2}He'\rho_1 (\sigma_1 x_2 - \sigma_2 x_1).$$

From (10) it follows that $\rho_3 (\sigma_1 x_2 - \sigma_2 x_1)$ or $(\sigma_1 x_2 - \sigma_2 x_1)$, like x_3, contains only matrix elements of the type $(j, -j)$ or $(j, j \pm 1)$. Now ρ_1 anticommutes with j, and therefore contains only matrix elements of the type $(j, -j)$. Hence ΔF contains only matrix elements of the type (j, j) or $(j, -j \pm 1)$.

In *loc. cit.*, § 6, it was found [see equation (24)] that the Hamiltonian could be expressed as

$$F \equiv p_0 + V + \varepsilon p_r + i\varepsilon\rho_3 jh/r + \rho_3 mc. \tag{15}$$

It follows from (10) that $(\sigma_1 x_2 - \sigma_2 x_1)$ anticommutes with $(\boldsymbol{\sigma}, \mathbf{x})$, and therefore also with ε. Thus if we put

$$\Delta F = ih\varepsilon\rho_3\eta r,$$

so that

$$\eta = \tfrac{1}{2}He/ch \,.\, \varepsilon\rho_2 (\sigma_1 x_2 - \sigma_2 x_1)/r,$$

η commutes with ε. Further, η commutes with ρ_3, r and p_r, so that it commutes with all the variables occurring in (15) except j. If we now express η as a matrix in j, we shall have obtained an expression for ΔF in terms of the variables occurring in (15). We have from (10) and (13)

$$|\rho_3 (\sigma_1 x_2 - \sigma_2 x_1)(j, j-1)|^2 = |x_3 (j, j-1)|^2 = \tfrac{1}{4}r^2 \frac{(j+u-\frac{1}{2})(j-u-\frac{1}{2})}{(j-\frac{1}{2})^2},$$

and similarly

$$\rho_3 (\sigma_1 x_2 - \sigma_2 x_1)(j, j+1)|^2 = |x_3 (j, j+1)|^2 = \tfrac{1}{4}r^2 \frac{(j+u+\frac{1}{2})(j-u+\frac{1}{2})}{(j+\frac{1}{2})^2}.$$

We have seen that the matrix elements of $\varepsilon\rho_1$, all of which are of the type $(j, -j)$, must be of modulus unity. Hence

$$\left.\begin{aligned} |\eta (j, -j-1)|^2 &= \left(\frac{He}{2chr}\right)^2 |i\varepsilon\rho_1 (j, -j)|^2 |\rho_3 (\sigma_1 x_2 - \sigma_2 x_1)(-j, -j-1)|^2 \\ &= \left(\frac{He}{4ch}\right)^2 \frac{(j+u+\frac{1}{2})(j-u+\frac{1}{2})}{(j+\frac{1}{2})^2} \\ \text{and similarly} \\ |\eta (j, -j+1)|^2 &= \left(\frac{He}{4ch}\right)^2 \frac{(j+u-\frac{1}{2})(j-u-\frac{1}{2})}{(j-\frac{1}{2})^2} \end{aligned}\right\}. \tag{16}$$

Again, from (10) and (11)

$$\rho_3 (\sigma_1 x_2 - \sigma_2 x_1)(-j, j) = -2ij \,.\, x_3 (-j, j) = -\frac{u}{(j^2 - \frac{1}{4})} irj \,.\, (\varepsilon\rho_1)(-j, j),$$

so that

$$\eta (j, j) = \frac{He}{2ch} \frac{uj}{j^2 - \frac{1}{4}}. \tag{17}$$

If we now write down in full, as in *loc. cit.*, the wave equation corresponding to (15), and include the extra term ΔF, we shall have

$$[(F + \Delta F)\,\psi]_\alpha = (p_0 + V)\,\psi_\alpha - h\frac{\partial}{\partial r}\psi_\beta - \left(\frac{j}{r} + \eta r\right) h\psi_\beta + mc\psi_\alpha = 0,$$

$$[(F + \Delta F)\,\psi]_\beta = (p_0 + V)\,\psi_\beta + h\frac{\partial}{\partial r}\psi_\alpha - \left(\frac{j}{r} + \eta r\right) h\psi_\alpha - mc\psi_\beta = 0,$$

where η is now an operator, operating on ψ_α and ψ_β, that commutes with everything except j. On eliminating ψ_α this gives, corresponding to (25) of *loc. cit.*,

$$\frac{\partial^2}{\partial r^2}\psi_\beta + \left[\frac{(p_0 + V)^2 - m^2c^2}{h^2} - \frac{j(j+1)}{r^2} + \eta - \eta j - j\eta - \eta^2 r^2\right]\psi_\beta - \frac{1}{p_0 + V + mc}\frac{\partial V}{\partial r}\left[\frac{\partial}{\partial r} + \frac{j}{r} + \eta r\right]\psi_\beta = 0.$$

We can neglect the $\eta^2 r^2$ term, which is proportional to the square of the field strength, and also the ηr term in the last bracket, which is of the order of magnitude of field strength times spin correction. The only first order effect of the field is the insertion of the terms $\eta - \eta j - j\eta$ in the first bracket. This bracket may now be written as

$$\left[\frac{2mE}{h^2} + \frac{E^2}{c^2h^2} + \frac{2\cdot(E + mc^2)}{ch^2}V + \frac{V^2}{h^2} - \frac{j(j+1)}{r^2} + \eta - \eta j - j\eta\right], \quad (18)$$

where E is the energy level, equal to $p_0c - mc^2$.

If the field is weak compared with the doublet separation, we can obtain a first approximation to the change in the energy levels by neglecting the non-diagonal matrix elements of ΔF or of η. The extra terms $\eta - \eta j - j\eta$ in (18) are now a constant instead of an operator, namely, the constant

$$-(2j-1)\,\eta\,(j,j) = -\frac{He}{ch}\frac{uj}{j + \frac{1}{2}}$$

from (17). The energy levels will be reduced by $h^2/2m$ times this constant, if we neglect the fact that the characteristic E occurs in (18) in other places besides the term $2mE/h^2$, which means neglecting the interaction of the magnetic field with the relativity variation of mass with velocity. The increase in the energy levels caused by the magnetic field is thus

$$\frac{He}{2mc}\frac{j}{j + \frac{1}{2}}uh = \omega guh$$

where ω is the Larmor frequency $He/2mc$, and g, the Landê splitting factor, has the value

$$g = j/(j+\tfrac{1}{2}).$$

For the succession of j-values, -1, 1; -2, 2, -3 ... g has the values, 2, $\frac{2}{3}$, $\frac{4}{5}$, $\frac{4}{5}$, $\frac{6}{5}$ ..., in agreement with Landê's formula for alkali spectra.

We now take the case of a magnetic field that is strong compared with the doublet separation, but weak compared with the separations of terms of different series. This requires that the matrix elements of η of the type $\eta(j, -j-1)$ with $j > 0$ shall be taken into account, although those of the type $\eta(j, -j+1)$ can still be neglected. The reduction in the energy levels will now be approximately $h^2/2m$ times one or other of the characteristic values of the extra terms $\eta - \eta j - j\eta$ in (18). These characteristic values are the roots ξ of the equation

$$\begin{vmatrix} (\eta - \eta j - j\eta)(j, j) - \xi & (\eta - \eta j - j\eta)(j, -j-1) \\ (\eta - \eta j - j\eta)(-j-1, j) & (\eta - \eta j - j\eta)(-j-1, -j-1) - \xi \end{vmatrix} = 0.$$

or

$$\begin{vmatrix} -(2j-1) \cdot \eta(j, j) - \xi & 2\eta(j, -j-1) \\ 2\eta(-j-1, j) & (2j+3) \cdot \eta(-j-1, -j-1) - \xi \end{vmatrix} = 0.$$

This gives, with the help of (16) and (17)

$$\xi^2 + \frac{He}{ch}\left[\frac{uj}{j+\frac{1}{2}} + \frac{u(j+1)}{j+\frac{1}{2}}\right]\xi + \left(\frac{He}{ch}\right)^2\left[\frac{u^2 j(j+1)}{(j+\frac{1}{2})^2} - \frac{(j+\frac{1}{2})^2 - u^2}{4(j+\frac{1}{2})^2}\right] = 0,$$

which reduces to

$$\xi^2 + \frac{He}{ch} 2u\xi + \left(\frac{He}{ch}\right)^2 (u^2 - \tfrac{1}{4}) = 0.$$

Hence

$$\xi = -\frac{He}{ch}(u \pm \tfrac{1}{2}).$$

The increase in the energy levels due to the magnetic field is therefore

$$-\frac{h^2}{2m}\xi = \frac{h^2}{2m}\frac{He}{ch}(u \pm \tfrac{1}{2}) = \omega(u \pm \tfrac{1}{2})\, h,$$

in agreement with the previous spinning electron theory of the Paschen-Back effect.

One might expect that with still stronger magnetic fields the matrix elements $(j, -j+1)$ of η would come into play, and would cause interference between the Zeeman patterns of terms whose quantum numbers k in the usual notation differ by 2. The matrix elements $(j, -j+1)$ of $\eta - \eta j - j\eta$, however, vanish for arbitrary η, so that no effect of this nature occurs.