A REMARKABLE PROPERTY OF THE RIEMANN-CHRISTOFFEL TENSOR IN FOUR DIMENSIONS

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Introduction. If the geometry of nature is Riemannian and the field equations of this geometry are controlled by a scale-invariant action principle, there are four a priori possible and algebraically independent invariants which may enter in the integrand of the action principle. This abundance of invariants hampers the mathematical development and the logical appeal of the theory. The present paper shows that two of these invariants are inactive in the formation of field equations and thus may be omitted. Only the two invariants

$$I_1 = R_{\alpha\beta}R^{\alpha\beta} \quad \text{and} \quad I_2 = R^2$$

remain effective which are formed with the help of the contracted curvature tensor R_{ik} alone, while the complete curvature tensor $R_{\alpha\beta\mu\nu}$ drops out from the action principle.

1. In the attempt to find the fundamental geometrical laws of nature, the general possibilities can be greatly reduced by a small number of reasonable a priori assumptions:

a) The geometry of nature shall be of the *Riemannian type*.

b) This geometry shall be characterized by *field equations* for the metrical tensor.

c) In view of the universal appearance of the principle of least action in all branches of physics: the field equations shall be deducible from an *action principle*.

d) The existence of an action principle reduces the problem of geometry to the determination of *one fundamental invariant I*, viz. the integrand of the action principle:

$$\delta \int I d\tau = 0, \tag{1.1}$$

($d\tau$ = four-dimensional volume element) that controls the field equations of geometry. This fundamental invariant has to be composed out of the fundamental quantities of Riemannian geometry and these are, in addition to the g_{ik}, the components of the curvature tensor $R_{\alpha\beta\mu\nu}$ called the Riemann-Christoffel tensor. It is feasible to assume that the fundamental invariant I shall be merely an *algebraic* function of the components of the curvature tensor, not involving any differentiation. As far as the *form* of this function is concerned, the principle of "scale-independence" may be assumed: the value of the action-integral that has to be minimized should not depend on the arbitrary units employed in measuring the lengths of the space-time manifold. Owing to this principle, the invariant I has to be a *quadratic* function of the curvature components $R_{\alpha\beta\mu\nu}$.[1]

2. The conditions a), b), c), and d) permit one to restrict to a large degree, in a purely logical fashion, the laws of geometry presumably realized in the physical universe. In a number of papers[2] the author investigated the general nature of field equations which are derived from an action principle with a quadratic invariant. He has also applied the Hamiltonian method of doubling the number of independent variables of the given action principle, treating the g_{ik} and the R_{ik} as two independent sets of variables and considering the field equations, in generalization of Einstein's gravitational equations, as a mutual interaction of matter tensor and metrical tensor.[3] There are indications in this theory for the appearance of the vector potential and thus for the possibility of explaining the electromagnetic phenomena. But the mathematical difficulty of obtaining "regular solutions" of the field equations, to be correlated to the physical behavior of single particles, is still a serious objection against the plausibility of these considerations.[4]

There is, however, an even more serious obstacle that has handicapped the

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[1] As it is well known, H. Weyl has developed an ingenious theory (see Ann d. Physik *59*, 101, (1919); W. Pauli: *Relativitäts-theorie* (Teubner, 1921), p. 759), which generalizes the Riemannian basis of geometry and builds up geometry in a "purely infinitesimal" fashion. In this geometry the ordinary principle of invariance is completed by the principle of "gauge-invariance," which requires invariance with respect to the substitution:

$$\bar{g}_{ik} = \lambda g_{ik} \tag{1.2}$$

where λ is an arbitrary function of the coördinates. If we do not forsake the basis of Riemannian geometry, which postulates the existence of an infinitesimal *and transportable* calibrated yard-stick, the principle of gauge-invariance has still its logical significance in a *limited* sense, viz. for *constant* λ. The substitution (1.2) with constant λ has the significance of *changing the scale of calibration* of the infinitesimal yard-stick. Since this scale may be chosen arbitrarily, the factor λ is undetermined. In general, an arbitrary invariant of Riemannian geometry will be a "dimensioned" quantity, i.e. it will depend on the value of the scale-constant λ. It is feasible to assume that the fundamental integral, which has to be minimized according to the action principle, shall have the "dimension zero," i.e. it shall not depend on the arbitrary scale-factor we employ in measuring lengths of the space-time manifold. Otherwise any arbitrary small or large value can be assigned to the fundamental integral by merely changing the unit of the length—except if that value happens to be zero.

[2] Phys. Rev. *39*, 716 (1932); Zeits. f. Physik *73*, 147 (1931) and *75*, 63 (1932).

[3] Zeits. f. Physik. *96*, 76 (1935).

[4] Some recent advances in the field of approximation methods led to a new approach of the problem and seem to indicate that the required "proper solutions" are actually present. The author hopes to report about these results in the near future.

progress along these lines considerably. The investigations of the author are based on a linear combination of the two invariants

$$I_1 = R_{\alpha\beta}R^{\alpha\beta}, \tag{2.1}$$

and

$$I_2 = R^2. \tag{2.2}$$

These are, however, not the *only* invariants satisfying the condition of gauge-invariance. Also the following invariant satisfies all conditions:

$$I_3 = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}. \tag{2.3}$$

This invariant involves the complete curvature tensor of 4[th] order and seems mathematically of a much more cumbersome nature than the invariants (2.1) and (2.2). This, however, is no inherent reason for excluding it from the participation in the action integral.[5]

Even more discouraging is the fact that still more invariants of quadratic nature can be constructed if we make use of the completely anti-symmetric "determinant tensor"

$$\delta^{\alpha\beta\mu\nu} = \frac{1}{\sqrt{g}}\,\epsilon^{\alpha\beta\mu\nu}, \tag{2.4}$$

where the symbol $\epsilon^{\alpha\beta\mu\nu}$ has the value $+1$ for any even permutation of the four indices α, β, μ, ν and the value -1 for any odd permutation, provided that no two indices are equal. In the latter case $\epsilon^{\alpha\beta\mu\nu}$ vanishes.

With the help of the tensor (2.4) we can form the "simply dual" and the "doubly dual" curvature tensors:

$$\overset{*}{R}_{ik}{}^{mn} = R_{ik\alpha\beta}\,\delta^{\alpha\beta mn}, \tag{2.5}$$

and

$$\overset{**}{R}{}^{ikmn} = R_{\alpha\beta\mu\nu}\,\delta^{\alpha\beta ik}\,\delta^{\mu\nu mn}, \tag{2.6}$$

which give rise to two more invariants:

$$K_1 = R_{\alpha\beta\mu\nu}\overset{*}{R}{}^{\alpha\beta\mu\nu}, \tag{2.7}$$

and

$$K_2 = R_{\alpha\beta\mu\nu}\overset{**}{R}{}^{\alpha\beta\mu\nu}. \tag{2.8}$$

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[5] The alternative combination $R_{\alpha\beta\mu\nu}R^{\alpha\mu\beta\nu}$, mentioned by Weyl (see reference 1, p. 133), is reducible to (2.3) owing to the algebraic identity:

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0 \tag{2.9}$$

One can prove that similarly also in case of the two invariants (2.7) and (2.8), in consequence of the same identity, the *transposition of indices does not produce new invariants.*

Hence there are altogether 5 apparently independent invariants which satisfy all our conditions and we may choose an arbitrary linear combination of them, with some numerical factors, as our action integrand I. This abundance of equivalent elements, among which no a priori choice seems to be possible, impairs greatly the logical persuasiveness of the theory.

The present investigation shows that this surplus of invariants is only apparent. We shall see that *the invariants K_1 and K_2 do not contribute any terms to the field equations* since their variation reduces to a mere boundary term. Moreover, we shall prove *that the invariant I_3, as far as field equations are concerned, is reducible to the two invariants I_1 and I_2*. These invariants were chosen previously owing to their property to depend only on the contracted curvature tensor R_{ik} of Einstein and not on the general Riemannian tensor $R_{\alpha\beta\mu\nu}$ of 4^{th} order. The present study yields the deeper evidence for the fundamental nature of Einstein's curvature tensor R_{ik}, proving that those invariants which involve the complete curvature tensor of the 4^{th} order are reducible, as far as the forming of field equations is concerned, to the two simpler invariants I_1 and I_2 which are constructed with the help of the contracted tensor R_{ik} alone.

3. The following formulae belong to the common arsenal of tensor-calculus and may be found more or less completely in almost any of the common textbooks on tensor-calculus or relativity. They are listed here for later references. The notations are self-explanatory and agree with the standard methods of writing tensor-equations. In lack of a universally accepted practical symbol for the process of "covariant differentiation" the ordinary symbol of partial differentiation: $\partial/\partial x_k$ shall be adopted for this purpose, in view of the fact that the ordinary differentiation will not enter in our present considerations. In accordance with the "sum-convention" of Einstein, equal indices shall automatically mean summation indices.

a) The Gaussian integral transformation:

$$\int \frac{\partial V^\alpha}{\partial x_\alpha}\, dt = \text{surface integral.} \tag{3.1}$$

(3.2) b)
$$\int \frac{\partial A_{\alpha\beta\cdots\mu}}{\partial x_\nu} B^{\alpha\beta\cdots\mu\nu}\, d\tau = \text{surface integral} - \int A_{\alpha\beta\cdots\mu} \frac{\partial B^{\alpha\beta\cdots\mu\nu}}{\partial x_\nu}\, d\tau.$$

c) Bianchi's identity for the Riemann-Christoffel tensor:

$$\frac{\partial R_{\alpha\beta\mu\nu}}{\partial x_\rho} + \frac{\partial R_{\alpha\beta\nu\rho}}{\partial x_\mu} + \frac{\partial R_{\alpha\beta\rho\mu}}{\partial x_\nu} = 0. \tag{3.3}$$

d) As it is well known, the quantities Γ^α_{ik} do not form the components of a tensor of third order. But the *variation* of the Γ^α_{ik}, produced by an infinitesimal variation of the g_{ik}, is actually a genuine tensor of third rank, covariant and symmetric in i, k, contravariant in α:

$$\delta\Gamma^\alpha_{ik} = \gamma^\alpha_{ik}. \tag{3.4}$$

e) The variation of the curvature tensor $R^{\alpha}_{\beta\mu\nu}$ can be expressed by means of the tensor γ^{α}_{ik} in the following form:

$$\delta R^{\alpha}_{\beta\mu\nu} = \frac{\partial \gamma^{\alpha}_{\beta\mu}}{\partial x_\nu} - \frac{\partial \gamma^{\alpha}_{\beta\nu}}{\partial x_\mu}. \tag{3.5}$$

f) The "dual transposition" of an anti-symmetric pair of indices may be obtained from the following table:

$$\begin{matrix} 12 & 34 \\ 13 & 42 \\ 14 & 23. \end{matrix} \tag{3.6}$$

The table reads in both directions and contains thus all possible combinations of two different indices. Exchange of the sequence of an index-pair means the corresponding exchange in the correlated pair.

The process of "dual transposition" of an anti-symmetric pair of indices shall be denoted by *underlining* these indices. Thus e.g. $R_{\alpha\beta\underline{12}}$ shall mean $R_{\alpha\beta34}$, etc. With this notation we may write:

$$\overset{*}{R}_{\alpha\beta}{}^{\mu\nu} = \frac{1}{\sqrt{g}} R_{\alpha\beta\underline{\mu\nu}} \tag{3.7}$$

$$\overset{**}{R}{}^{\alpha\beta\mu\nu} = \frac{1}{g} R_{\underline{\alpha\beta}\,\underline{\mu\nu}} \tag{3.8}$$

4. We deal at first with the two invariants K_1 and K_2 and show that the terms, obtained from them by variation of the g_{ik}, all vanish identically. This is in the case of K_1 due to Bianchi's differential identity, while in the case of K_2 we have to add the algebraic symmetry-properties of the Riemann-Christoffel tensor, together with the fact that the number of variables is 4.

The Bianchi-identity (3.3) may be written for the special case of 4 dimensions in the following form:

$$\frac{\partial R_{ik\alpha\beta}}{\partial x_\gamma} \delta^{\alpha\beta\gamma m} = 0. \tag{4.1}$$

Due to this relation we get for the two dual tensors (2.5) and (2.6):

$$\frac{\partial \overset{*}{R}_{ikm}{}^{\alpha}}{\partial x_\alpha} = 0, \tag{4.2}$$

$$\frac{\partial \overset{**}{R}_{ikm}{}^{\alpha}}{\partial x_\alpha} = 0. \tag{4.3}$$

We now consider the action integral

$$A = \int R_{\alpha\beta\mu\nu} \overset{*}{R}{}^{\alpha\beta\mu\nu}\, d\tau. \tag{4.4}$$

Since the volume-element $d\tau$ has the significance

$$d\tau = \sqrt{g}\, dx_1 \cdots dx_4, \tag{4.5}$$

we may write (4.4), making use of (3.7), in the following form:

$$A = \int R^{\alpha}{}_{\beta\mu\nu} R_{\alpha}{}^{\beta}{}_{\underline{\mu}\underline{\nu}}\, dx_1 \cdots dx_4. \tag{4.6}$$

Owing to the complete mutual symmetry of the two factors the variation of the second factor reproduces the result of the variation of the first factor. Thus it suffices to vary the first factor alone and multiply the result by 2. Applying the formula (3.5) for the variation of the curvature tensor and making use of the integral-transformation (3.2) we obtain:

$$\begin{aligned} \delta A &= 2 \int \left(\frac{\partial \gamma^{\alpha}_{\beta\mu}}{\partial x_\nu} - \frac{\partial \gamma^{\alpha}_{\beta\nu}}{\partial x_\mu} \right) R_{\alpha}{}^{\beta}{}_{\underline{\mu}\underline{\nu}}\, dx_1 \cdots dx_4 \\ &= 4 \int \frac{\partial \gamma^{\alpha}_{\beta\mu}}{\partial x_\nu} R_{\alpha}{}^{\beta}{}_{\underline{\mu}\underline{\nu}}\, dx_1 \cdots dx_4 \\ &= \text{boundary term} - 4 \int \gamma^{\alpha}_{\beta\mu} \frac{\partial \overset{*}{R}_{\alpha}{}^{\beta\mu\nu}}{\partial x_\nu}\, d\tau. \end{aligned} \tag{4.7}$$

The second term vanishes owing to (4.2). Thus δA reduces to a boundary term and does not yield any terms toward the field equations.

We now turn to the second invariant K_2 and handle the variation problem in a similar manner. The action integral

$$A = \int R_{\alpha\beta\mu\nu} \overset{**}{R}{}^{\alpha\beta\mu\nu}\, d\tau \tag{4.8}$$

may now be written as follows:

$$A = \int R_{\alpha\beta\mu\nu} R_{\underline{\alpha}\underline{\beta}\underline{\mu}\underline{\nu}}\, dx_1 \cdots dx_4 = R^{\rho}{}_{\beta\mu\nu} R^{\sigma}{}_{\underline{\beta}\underline{\mu}\underline{\nu}} \frac{g_{\rho\alpha} g_{\sigma\underline{\alpha}}}{\sqrt{g}}\, dx_1 \cdots dx_4. \tag{4.9}$$

Again the variation of the first and the second factor yields the same, due to the symmetry of their mutual relation. The process (4, 7) again yields a pure boundary term in view of the fact that not only $\overset{*}{R}_{\alpha\beta\mu\nu}$ but also $\overset{**}{R}_{\alpha\beta\mu\nu}$ satisfies the divergence equation (4.3). Hence only the variation of the *third* factor will contribute something to the field equations. Here no derivatives of the g_{ik} enter any more, and thus the field equations deduced from the invariant K_2 have the remarkable property of being not higher than *second* order in the g_{ik}. We will prove, however, that *these terms all vanish identically*, owing to the algebraic symmetry-properties of the tensor $R_{\alpha\beta\mu\nu}$.

Performing the variation of the g_{ik} in the third factor, we obtain the following symmetric tensor of second order:

$$S_{ik} = R_{i\alpha\beta\gamma} \overset{**}{R}_{k}{}^{\alpha\beta\gamma} + R_{k\alpha\beta\gamma} \overset{**}{R}_{i}{}^{\alpha\beta\gamma} - \tfrac{1}{2} K_2 g_{ik}. \tag{4.10}$$

Now a tensor-equation may be proved in *any* reference-system. Thus we may introduce a local reference-system in which the g_{ik} have the normal values δ_{ik}. Moreover, by a suitable rotation of our reference-system, the symmetric tensor S_{ik} may be transformed into a *purely diagonal* form. Hence it suffices to show that all the *diagonal* terms of the expression (4.10) vanish. This guarantees the vanishing of the *complete* tensor.

Now in our local reference system co-variant and contra-variant components are equal and the diagonal terms of S_{ik} may be written as follows:

$$\tag{4.11} \tfrac{1}{2}S_{ii} = R_{i\alpha\beta\gamma}\overset{**}{R}_{i\alpha\beta\gamma} - \tfrac{1}{4}K_2 = R_{i\alpha\beta\gamma}R_{\underline{i}\underline{\alpha}\underline{\beta}\underline{\gamma}} - \tfrac{1}{4}K_2$$

The index i is here exceptionally *no* sum-index, but a fixed index which assumes successively the values 1, 2, 3, 4, while α, β, γ are, as usual, sum-indices.

We examine particularly the expression

$$\rho_i = R_{i\alpha\beta\gamma}R_{\underline{i}\underline{\alpha}\underline{\beta}\underline{\gamma}} ,$$

the second term of (4.11) being the same for all the four S_{ii}. Let us evaluate the contributions of the different curvature-components of ρ_i *separately*. We may have an index-combination of the type R_{1212} with two equal pairs of indices. This component combines with R_{3434} so that the effect of these two components has to be considered simultaneously. Now in the first row ($i = 1$) the product $(12, 12)\cdot(34, 34)$ enters[6], omitting for a moment the letter R and denoting the various components of R merely by the corresponding index-combinations. In the second row ($i = 2$) the same product enters, due to the combination $(21, 21)\cdot(43, 43)$. The same happens in the third and fourth rows, with changed sequence of the factors. Thus all the four ρ_i are equal.

We prove the same for a combination of the type (12, 13); (only *two* equal indices). Here the dual partner is (34, 42). In the first row we get two terms since 1 may combine with either 2 or 3:

$$(12, 13)\cdot(34, 42) + (13, 12)\cdot(42, 34) = 2\cdot(12, 13)\cdot(34, 42).$$

In the second row 2 may combine with either 1 or 4 and we obtain:

$$(21, 13)\cdot(43, 42) + (24, 34)\cdot(12, 31) = 2\cdot(12, 13)\cdot(34, 42).$$

The result for the third and fourth row is similar. Again all the four ρ_i are the same.

Finally we consider a combination of the type (12, 34), where all the four indices are different. Here the square of (12, 34) enters into the first row, but the same happens in all the other rows. The contributions to all four rows are the same.

Hence we have proved for any possible index-combinations that all the four ρ_i

[6] Actually also $(12, 21)\cdot(34, 43)$ should be considered which only duplicates the result. We consistently omit here and in the next similar instances the interchange of the second pair of indices.

come out as equal and then the same holds, due to (4.11), for the four S_{ii}. We may put

$$S_{ii} = C. \tag{4.13}$$

But the expression (4.10) shows directly that the scalar

$$S^{\alpha}_{\alpha} = 0. \tag{4.14}$$

This means for our reference system:

$$4C = 0. \tag{4.15}$$

We thus arrive at the conclusion that all the diagonal components of the tensor S_{ik} vanish which leads to the complete vanishing of the tensor S_{ik}.

The result of this paragraph is that *neither the invariant K_1 nor the invariant K_2 is able to contribute any terms to the field-equations.*

5. The tensor $\overset{**}{R}_{\alpha\beta\mu\nu}$ is not independent of the original tensor $R_{\alpha\beta\mu\nu}$ but can be reduced to it without referring to the anti-symmetric tensor $\delta^{\alpha\beta\mu\nu}$. Let us form the difference

$$R_{\alpha\beta\mu\nu} - \overset{**}{R}_{\alpha\beta\mu\nu}. \tag{5.1}$$

This tensor has only 9 independent components and can be produced explicitly with the help of the contracted tensor R_{ik}. We can prove by direct inspection in a local reference-system $g_{ik} = \delta_{ik}$ the validity of the following tensor relation:

$$R_{\alpha\beta\mu\nu} - \overset{**}{R}_{\alpha\beta\mu\nu} = B_{\alpha\mu}g_{\beta\nu} - B_{\alpha\nu}g_{\beta\mu} + B_{\beta\nu}g_{\alpha\mu} - B_{\beta\mu}g_{\alpha\nu}, \tag{5.2}$$

where we have put

$$B_{ik} = R_{ik} - \tfrac{1}{4}Rg_{ik}. \tag{5.3}$$

Hence

$$\overset{**}{R}_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - B_{\alpha\mu}g_{\beta\mu} + B_{\alpha\nu}g_{\beta\mu} - B_{\beta\nu}g_{\alpha\mu} + B_{\beta\mu}g_{\alpha\nu}. \tag{5.4}$$

Multiplying by $R^{\alpha\beta\mu\nu}$ we get the following linear relation between the invariants I_1, I_2, I_3 and K_2:

$$K_2 = I_3 - 4R^{\alpha\mu}B_{\alpha\mu} = I_3 - 4I_1 + I_2. \tag{5.5}$$

The fact that the field equations deduced from K_2 vanish identically may now be expressed in the following manner: *The field equations deduced from the action integral*

$$\int I_3 d\tau \tag{5.6}$$

are identical with the field-equations deduced from the action-integral

$$\int (4I_1 - I_2)d\tau. \tag{5.7}$$

6. Conclusion. If we want to investigate the complete group of Riemannian geometries deducible from an action principle with an integrand which is quadratic in the curvature components and thus scale-invariant, we have to start out with a general expression of the following form:

$$\delta \int (I_1 + \alpha I_2 + \beta I_3 + \gamma K_1 + \epsilon K_2) d\tau = 0 \tag{6.1}$$

with the five invariants I_1, I_2, I_3, K_1, K_2 and four numerical constants α, β, γ, ϵ. Due to a linear relation between these five invariants I_3 may be eliminated and thus the action integral reduced to but four independent invariants. It has been shown that for any variations between definite limits we get identically:

$$\delta \int K_1 d\tau = 0, \tag{6.2}$$

$$\delta \int K_2 d\tau = 0, \tag{6.3}$$

so that the action principle (6.1) reduces in fact to the simple form

$$\delta \int (I_1 + \alpha I_2) d\tau = 0. \tag{6.4}$$

The two remaining invariants

$$I_1 = R_{\alpha\beta} R^{\alpha\beta}, \qquad I_2 = R^2, \tag{6.5}$$

are characterized by the property that they are formed exclusively by means of the contracted curvature tensor R_{ik} while the general Riemann-Christoffel tensor $R_{\alpha\beta\mu\nu}$ does not enter any more into the action-principle. Thus the difficulty of an abundance of invariants, coupled with each other by arbitrary numerical constants, is removed and the action integral reduced to a form in which but *one* unknown numerical constant remains.

This result confirms in a very satisfactory manner the fundamental significance of the contracted curvature tensor of Einstein for the geometry of nature. It also confirms the logical reasonableness of the attempt to deduce the field equations of nature from a scale-invariant action principle.

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