

Why bother about Weyl geometry?
– a look at recent developments (not only)
in cosmology

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Workshop Cosmology Wuppertal, Feb 15, 2009

1. Intro
2. Structural observations
3. Weyl geometry in cosmology
4. Questions and conclusions

Weyl geometry ... I

... is a branch of differential geometry.

It generalizes (pseudo-)Riemannian geometry, with metric

$$\begin{aligned} ds^2 &= \sum g_{\mu\nu} dx_\mu dx_\nu \quad (\text{abbreviated}) = g_{\mu\nu} dx^\mu dx^\nu & (1) \\ g &= (g_{\mu\nu}) \end{aligned}$$

... by possibility to *freely chose units at every point* (“passive aspect”);
or by *rescaling the world content* (“active aspect”):

$$g \mapsto \tilde{g} = \Omega^2 g \quad \Omega \text{ real function, everywhere positive.} \quad (2)$$

Language: “(re-)scaling” will be used in the mathematical sense of equ. (2). Physical interpretation will be a context dependent task.

Important for Weyl: Comparison of metrical quantities immediately meaningful only “at one point” p (in $T_p M$).

Weyl geometry ... II

Comparison between 'infinitesimally close' points needs another symbolic gadget, a (real valued) differential 1-form φ , called **scale connection**:

$$\varphi = \sum \varphi_\mu dx_\mu = \varphi_\mu dx^\mu. \quad (3)$$

Rescaling $g \mapsto \tilde{g} = \Omega^2 g$ necessitates a **gauge transformation**
 $\varphi \mapsto \tilde{\varphi} = \varphi - d(\log \Omega) = \varphi - \frac{d\Omega}{\Omega}$.

Summing up: A **Weyl(ian) metric** on a differentiable manifold M is given by pairs $(g, \varphi) \sim (\tilde{g}, \tilde{\varphi})$ as above, more precisely by an equivalence class $[(g, \varphi)]$. Choice of (g, φ) means to **gauge** the metric.

A **Weylian manifold (mf)** is given by the (mathematical) data

$$(M, [(g, \varphi)])$$

Weyl geometry (WG) deals with Weylian manifolds,.

Important for Weyl geometry

- (1) Existence of unique compatible *affine connection* Γ , of its associated **covariant derivative** ∇ , and corresponding *curvature quantities/tensors* $Riem$, Ric , R .
- (2) Curvature of scale connection, $f := d\varphi$,
If $f = 0$ (φ closed) WG is locally reducible to Riemannian geom;
if $\varphi = d\alpha$ (φ exact) WG is globally reducible: **integrable WG (IWG)**.
Then exists gauge of form $(\tilde{g}, 0)$ – *Riemann gauge*.
- (3) **Scale covariant** quantities, i.e. scalars, vectors, tensors fields X , change like $\tilde{X} = \Omega^k X$; here $k = w(X)$ is *weight* of X .
They “need their own” **scale covariant derivative** D

$$DX = \nabla X + w(X)\varphi \otimes X,$$

because ∇X is in general *not* scale covariant, while DX is.

- (4) Natural to consider **geodesics γ of weight -1** . Then $g(\gamma', \gamma') = -1$ (for timelike geodesics), independent of scale. Geodesic equation written in terms of D ($w = -1$).

Historical reminder I

- (1) H. Weyl 1918: *Reine Infinitesimalgeometrie*, Math. Zs, GA II [30]
Gravitation und Elektrizität, Sitz.Ber. Berliner Ak.Wiss, GA II [31]
Identification of φ with e.m. potential ... *first gauge field theory*,
first geometrical UFT + dynamist matter theory.
- (2) Einstein: *rods and clocks* counterargument. Weyl *not* convinced.
But after Pauli's counterarguments Weyl **retreated** from UFT and
dynamist matter theory (1920/21).
- (3) After 1921 Weyl continued to insisted upon:
— **conceptual superiority** of WG over Riemannian geometry:
analysis of problem of space, 1920 – 1923 (no longer after 1926);
— **rejection of “rods and clocks”** as *fundamental* gadgets of GRT,
rather insistence on **adjustment** of mass m , Compton wave length,
spectral frequencies ... *to local field constellation* (until end of life).

Historical reminder II

- (4) Transfer of gauge theory of e.m. **from scale to phase** (1924 to 1929)
 - later standard paradigm for non-abelian gauge field theories (Yang-Mills).
- (5) Weyl's original scale gauge not dead: Ehlers/Pirani/Schild (**EPS**) (1972) emphasize **fundamental role of WG** in foundations of classical GRT, later Audretsch/Gähler/Straumann (1983) and others.
- (6) **Retake of Weyl geometry in 1970s:**
 - P.A.M. Dirac 1973: WG with scalar field, scale weight -1 , (\sim Weyl geometric version of Jordan-Brans-Dicke (**JBD**) theory) stick to *e.m. dogma* for φ ,
 - attempted applications by Canuto e.a. and Maeder e.a. .
 - continuation of **Dirac's theoretical approach** by *N. Rosen* and *M. Israelit* (until today), but not always convincing (*no systematic use of scale covariant Lagrangian and field equation, continuation of e.m. dogma, etc.*).

Weyl structures

Recent mathematical literature prefers to characterize WG by a **Weyl structure** on a differentiable manifold (mf), given by

- (i) a conformal (pseudo-)Riemannian structure $[g]$,
- (ii) a covariant derivative ∇ , or a linear (affine) connection Γ s.th.
 $\nabla = \nabla_\Gamma$,
- (iii) constrained by a compatibility condition (which we may skip here, but which ensures, among others, the affine nature of Γ)

Observation 1:

If you have a conformal structure and a uniquely determined (affine) connection respecting the constraint (iii) above,

you work in a Weyl structure – even if you don't know it.

Note: Data of Weylian mf and of Weyl structure are equivalent:

$$(M, [g, \varphi]) \sim (M, [g], \nabla)$$

Weyl structures, implicit in field theory

Jordan-Brans-Dicke (JBD) theory and conformal field theory (CFT)
– agree in coupling a scalar field to scalar curvature R in the Lagrangian of the action principle, and in transforming the metric conformally,
— differ in interpretation and technical handling of conformal rescaling.

JBD theory works in a conformal structure $[g]$ with specification of a (compatible) covariant derivative ∇ , once and for all. Not so in CFT, but for process of so-called “mass generation”.

Observation 2:

- ▶ JBD uses a Weyl structure from the outset (usually unknowingly).
- ▶ CFT enriches its framework from conformal geometry to Weyl structure (usually unknowingly), in a step often considered as a “scale symmetry break” (a misnomer, from the point of view of Weyl geometry) by a “Higgs field”.

WG in field theory I

Explicit use of Weyl geometry in field theory: Hung Cheng (1988), C. Castro e.a., here we follow **H. Tann, W. Drechsler** (1998, 1999).

Scalar field sector of CFT enriched to Weyl geometry.

Weyl geometric scale invariant Lagrangian:

$$\mathcal{L}_{WG}^{\phi, HE} = \left(\frac{1}{2} \xi |\phi|^2 R - \frac{1}{2} D^\mu \phi^* D_\mu \phi + V(\phi) + \dots \right) \sqrt{|\det g|} \quad (4)$$

R Weyl geometric scalar curvature, $D = (D^\mu)$ scale covariant derivative; dimensionless coupling constants; $\xi = \frac{1}{6}$ for $\dim M = 4$.

Scale co/invariant Einstein and scalar field equations:

$$\text{Ric} - \frac{R}{2} g = (\xi |\phi|^2)^{-1} T \quad (5)$$

$$D^\mu D_\mu \phi + \left(\xi R + \frac{2}{\phi} \frac{\partial V}{\partial \phi^*} \right) \phi = 0 \quad (6)$$

Scale covariant version of gravitational "constant" $(\xi |\phi|^2)^{-1} = 8\pi \tilde{G} [c^{-4}]$.

Energy momentum (E-M) tensor $T = T^{(\phi)} + T^m + \dots$, weight -2 .

WG in field theory II

E-M tensor of scalar field decomposes

$$(\xi|\phi|^2)^{-1} T^{(\phi)} = -\Lambda g_{\mu\nu} + (\xi|\phi|^2)^{-1} T^{(\phi, res)},$$

i.e., into a term which looks like “dark energy” and a residual term, where

$$\Lambda = (\xi|\phi|^2)^{-1} \left(\frac{1}{2} D^\lambda \phi^* D_\lambda \phi - \xi D^\lambda D_\lambda (\phi^* \phi) - V(\phi) \right) \quad (7)$$

$$T^{(\phi, res)} = D_{(\mu} \phi^* D_{\nu)} \phi - D_{(\mu} D_{\nu)} |\phi|^2 \quad (X \dots Y \text{ symmetrization}) \quad (8)$$

Observation 3:

- ▶ In the “dark energy” term, Λ is no constant.
- ▶ “Residual” contribution $T^{(\phi, res)}$ may acquire form of “dark matter”.
- ▶ Both **couple to curvature and thus to matter**.

Drechsler/Tann's "pseudo-Higgs" argument

Mass generation by "symmetry break" of scale ("Weyl") symmetry.
Assumption: Observable field values **adapt** to $|\phi|$ (like "rods and clocks"). Preferred scale gauge, in which $|\phi|$ becomes constant.

Drechsler (1999): **Extension to electroweak and fermionic sector.**

ϕ lifted to $\tilde{\phi}$ in trivial "scalar" \mathbb{C}^2 bundle (coordinate invariant, scaling), $(M, [(g, \varphi)])$ extended to principle bundle with ew group $SU_2 \times U(1)_Y$ plus fermionic bundle.

Fixing gauge in $SU_2 \times U(1)_Y$ bundle, $\tilde{\phi}_o = (0, \phi)$, similar to classical "Higgs mechanism", reduces symmetry to $U(1)_{em}$ (stabilizer of $\tilde{\phi}_o$) and **induces** terms in $U(1)_{em}$ extended covariant derivative.

That results in mass terms the EM tensor of ϕ due to ew field bosons.

Observation 4:

The arising mass terms agree formally with those of the classical "Higgs mechanism". But here they are induced by **coupling ew fields to gravity** via the scalar field.

Interesting: *No massive Higgs boson* expected. ϕ (and $\tilde{\phi}$) seem to be an effective field – *perhaps* with underlying quantum structure for curved φ close to Planck scale.

Basics of WG approach to cosmology I

For simple largest scale cosmological considerations, locally homogeneity and isotropy of spatial fibres is assumed, i.e. constant curvature “spaces”. Characterized by classical *Robertson Walker metric*:

$$\begin{aligned}\tilde{g} : \quad d\tilde{s}^2 &= -d\tau^2 + a(\tau)^2 d\sigma_\kappa^2 \\ d\sigma_\kappa^2 &= \frac{dr^2}{1 - \kappa r^2} + r^2(d\Theta^2 + \sin^2 \Theta d\phi^2)\end{aligned}$$

$a(\tau)$ warp (“expansion”) function.

Take it as Riemann gauge of an IWG with $(M, [(\tilde{g}, 0)])$.

Interesting, if there is a non-constant scalar field $\tilde{\phi}$ in this gauge (corresponds to “Jordan frame” in JBD theory).

Observer field U (timelike, unit) of weight -1 as $g(U, U) = -1$; in Riemann gauge $\tilde{U} = \frac{\partial}{\partial \tau}$.

This leads to **Robertson-Walker-Weyl (R-W-W) models** given by data:

$$(M, [(\tilde{g}, 0), \tilde{U}, \tilde{\phi}]).$$

Basics of WG approach to cosmology II

Cosmological redshift is encoded in the relation between null geodesics and observer field. Energy values e_o , e_1 of photon along null geodesic $\gamma(u)$ emitted at p_o observed at p_1 , with respect to observers U turn out to be **scale invariant** (geodesics of weight -1).

How possible, if warp function (“expansion”) can be flattened or steepened? — Reason: φ contributes to cosmological redshift.

Most markedly: Rescaling by $\Omega = \frac{1}{a}$ leads to gauge (g, φ) in which the Riemannian component g of the gauge looks static:

$$\begin{aligned} g : \quad ds^2 &= -dt^2 + d\sigma_\kappa^2 \quad \tau \text{ reparametrized to } t \\ \varphi &= d \log a \quad \text{with time component only } \neq 0. \end{aligned}$$

Here cosmological redshift is exclusively encoded in φ and given by

$$z + 1 = e^{\int_0^1 \varphi(\gamma'(u)) du} = \lambda(p_o, p_1) \quad \text{scale transfer function of WG}$$

Call this the **Hubble gauge**; its scale connection **Hubble connection**.

This indicates (formally only?) a kind of “*tiring light*” approach.

But note: WG framework leads to **different** properties from **classical** “*tired light*” hypothesis. In particular “time stretch” is inbuilt in Weylian scale transfer.

Energy momentum of scalar field in WG cosmology

E-M tensor of ϕ may shed new light on “dark energy” ...

Simple examples, **Weyl universes**, based upon following assumptions:

- ▶ scalar field gauge = Hubble gauge,
- ▶ linear warp function in Riemann gauge, $a(\tau) = H\tau$.

They look *time homogeneous* (“static”) in *scalar field gauge*:

$$g : \quad ds^2 = -dt^2 + d\sigma_\kappa^2, \quad \varphi = Hdt$$

Cosmological redshift $z + 1 = e^{H(t_1 - t_0)}$

$$Ric \doteq 2(\kappa + H^2)d\sigma_\kappa^2 \quad (\text{“}\doteq\text{” equality in special gauge})$$

$$R \doteq 6(\kappa + H^2), \quad \phi \doteq const.$$

With **classical**, although *scale invariantly formulated matter term*

$$\mathcal{L}^m = \mu[1 + \epsilon]\sqrt{|det g|} \quad \mu, \epsilon \text{ scalar function} \quad w(\mu) = -4, w(\epsilon) = 0,$$

constrained s.th. flow along U satisfies local energy conservation,
and $V(\phi) = \lambda_2|\phi|^2 + \lambda_4|\phi|^4$ ($w(\lambda_2) = -2, w(\lambda_4) = 0$), $\mu[1 + \epsilon] =: \rho$

$$\Lambda \doteq \rho - 6\lambda_2.$$

Weyl universes can be **balanced by scalar field** (Scholz 2009).

Supernovae data (SNIa) I

First empirical test of reliability of Weyl universes by magnitude (m) – redshift (z) characteristic of SNIa.

Here reduction of energy flux of cosmic light sources due to

- ▶ redshift (energy reduction of single photons)
- ▶ time stretch (reduction of photons per time unit)
- ▶ inverse area increase (later decrease !) of spheres of light cone
- ▶ extinction ϵ (?)

Resulting expression for relative magnitude m of sources with absolute magnitude M

$$m(z, \zeta, \epsilon, M) = \dots$$

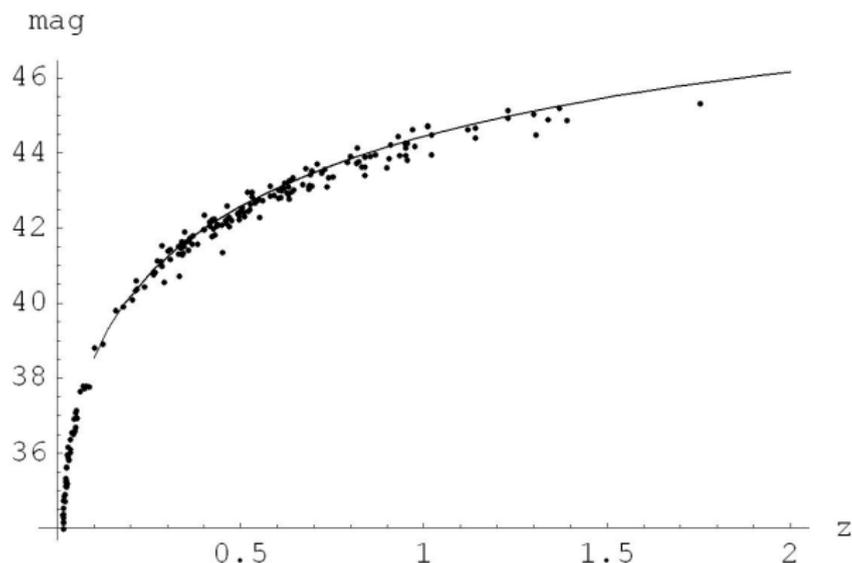
fitted to the data of 191 SNIa, $z \leq 1.755$ in (Riess e.a 2007)

leads to *positive* spatial curvature, i.e. **Einstein-Weyl universes** with

$$\text{roughly } 1 \leq \zeta \leq 3.6 \quad (\epsilon \approx 1).$$

Fit slightly better than in standard approach.

Supernovae data (SNIa) II



Magnitudes of 191 supernovae Ia (mag) $z \leq 1.755$ (Riess e.a. 2007), and “prediction” in Weyl geometric model $\zeta = 1$.

Empirical acceptability of Weyl universe ?

If Einstein-Weyl models were realistic, **matter density** had to be much **larger** than admitted at present

$$\zeta = 1, \quad \Omega_m = \frac{4}{3} \quad \Omega_\Lambda = \frac{2}{3}$$

Exclusion criterion for Weyl models ?

Be careful: Mass density estimations are *precarious data*.

E. Fischer estimates hot plasma density in static spherical geometry from diffuse X-ray background to be $\Omega_{m-Plasma} \approx \frac{2}{3}$.

R. Lieu argues that “dark matter” data for very large scales seems to be much larger than present ~ 0.3 , presumably even larger than 1.

Present consensus of “small” Ω_m may turn out to be strongly paradigm dependent.

Identical geometry in large cosmic means (“static geometry”), would be filled by **large scale matter cycles**:

... galaxy formation from plasma – galaxy aging – quasar formation – jets propagating high energy plasma in large scales – cooling of plasma ...

Questions

Weyl universes would indicate: **no big bang, no global evolution.**

To be discussed:

- ▶ CMB (cf. I.E. Segal's high entropy background state of quantized Maxwell field)
- ▶ Anisotropies, low multipoles: F. Steiner's method applicable to Weyl universes? (finite volume!)
- ▶ Anisotropies, higher momenta: Does lack of SZ effect indicate origin of CMB in low z -regions rather than at $z \sim 1500$? (T. Shanks)
- ▶ Metallicity studies of quasars (SDSS) and galaxies (Pagel) show **no correlation** of metallicity and redshift.
- ▶ Single extrem case: **X-ray quasar** (APM 08279+5255) with $z \approx 3.91$ observed by G. Hasinger and S. Komossa with **very high metallicity** ($Fe/O \sim 3$).
Breeding needs 3 GY, available time in standard model ≈ 1.7 GY.

Further questions:

- ▶ "Dark matter"?
- ▶ Stability of Weyl universes ? ...

Conclusions

Conclusions

- (I) Strong **arguments in favour of Weyl geometry** as a natural link between conformal field theory and gravitation:
- ▶ no CFT without conformal structure $[g]$, no gravity without affine connection ∇ ; both together (with compatibility) give Weyl structure,
 - ▶ in the “small” (high energies) Drechsler’s pseudo Higgs,
 - ▶ in the “large” Weyl geometric scalar field provides “dark energy” like term, coupled to matter.
- (II) Simple **examples** above (Einstein-Weyl models) **challenge** some of the deepest entrenched convictions of **present standard cosmology**.
No longer without alternative:
- ▶ *accelerated expansion* “proved” by SNIa data,
 - ▶ *redshift* as indicator of *real expansion*,
 - ▶ large scale *cosmic evolution*
 - ▶ CMB as view at *surface of last scattering*,
 - ▶ *big bang, inflation* and all that.
- (III) Maybe that is **too much** for the community ?
Surely it is not the “last word” ...
– But the standard model may be even less.

Appendix: A quote from Weyl 1949

“Classical physics derives the conservation of charge and mass from a tendency of perseverance, but permits bodies of arbitrary charge and mass to exist. This viewpoint is unsatisfactory as far as the fixed charges and masses of elementary particles are concerned. Their conservation must depend on *adjustment* rather than on *perseverance*. . . .

. . . The rigid rods and the clocks by which Einstein measures the fundamental quantity ds^2 of his metric theory of the gravitational field preserve their length and period on the last instance because charge e and mass m of the composing elementary particles are preserved. The systematic theory, however, proceeds in the opposite direction; it starts with the metrical ground form and thus introduces a primitive field quantity to which the Compton wave length m^{-1} of the particle adjusts itself in a definite proportion. . . . (Weyl: PMNS 1949, 288)

Appendix: Scale invariant “observables”

For any scale covariant quantity X (norm of vectors, component of E-M tensor ...) of weight $w(X) = k$ the quotient (proportion)

$$\hat{X} := \frac{X}{|\phi|^{-k}} = X |\phi|^{-k}$$

is **scale gauge independent**.

Formally this works like “adaptation” of atomic clocks etc. to the local field value of ϕ . Does it correspond to some physical “reality”?

In any case, \hat{X} will be considered as the **observable quantity of X** (measured in atomic clock units).

Obviously there is a gauge (g_*, φ_*) in which $|\phi|_*$ is constant,

$$g_* = |\phi|^2 g, \quad \varphi_* = \varphi - d \log |\phi|, \quad |\phi|_* = \frac{\phi}{|\phi|}.$$

Call it **scalar field gauge** (in JBD theory \sim *Einstein frame*).

It is “preferred” in the sense of

$$\hat{X} = X_* = \text{value of } X \text{ in scalar field gauge.}$$

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