From Schwarzschild de-Sitter to Mannheim-Kazanas

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Any Weyl-invariant relativistic theory that admits a Schwarzschild de-Sitter metric solution equally well admits the Mannheim-Kazanas metric. This statement is shown explicitly via a combined coordinate and Weyl transformation.

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I. GENERATING THE MANNHEIM-KAZANAS FROM THE SCHWARZSCHILD DE-SITTER METRIC

We start off with the standard Schwarzschild de-Sitter metric. This metric is a well-known solution of both GR and fourth-order Weyl gravity. In the latter case it is a special case ($\gamma = 0$) of the Mannheim-Kazanas metric, which by itself is a solution of fourth-order Weyl gravity. Extending the symmetry of GR to allow for Weyl invariance we obtain a Weyl-invariant scalar-tensor theory described by an action $\mathcal{I} = \int(\frac{1}{6}R\phi^2 + \phi_{,\mu}\phi^{,\mu} + \mathcal{L}_m)\sqrt{-g}d^4x$.

In the r' frame the infinitesimal interval that describes the Schwarzschild de-Sitter metric reads

$$ds'^2 = -(1 - \frac{2\beta'}{r'} + \Lambda' r'^2)d\eta^2 + \frac{dr'^2}{1 - \frac{2\beta'}{r'} + \Lambda' r'^2} + r'^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

where β' & Λ have their usual meaning. Again, this metric is a solution of both fourth-order Weyl gravity and the scalar-tensor extension of GR (in case that ϕ is a constant the theory reduces exactly to GR for which the Schwarzschild de-Sitter metric is a solution). Since both theories are invariant under both general coordinate transformations, as well as under Weyl transformations, we can generate new solutions by applying a combined coordinate and weyl transformation. Specifically, we are seeking a spherically static solution of the form

$$ds^2 = -A(r)d\eta^2 + \frac{dr^2}{A(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{2}$$

in the, new, r-system. Rather than going through a tedious solution of the field equations we employ the symmetry of the theory to explicitly derive $A(r)$ from Eq. (1) and show that it is exactly the corresponding quantity found by Mannheim & Kazanas (1989) from solving the fourth-order Bach equations within the Weyl gravity framework or by solving the coupled scalar-tensor field equations of $\mathcal{I} = \int(\frac{1}{6}R\phi^2 + \phi_{,\mu}\phi^{,\mu} + \mathcal{L}_m)\sqrt{-g}d^4x$. Eqs. (1) & (2) are related via a conformal transformation $ds'^2 = \Omega^2 ds^2$. From the the angular, radial and time components of the metric we obtain, respectively

$$\Omega r' = r \tag{3}$$

$$\frac{\Omega^2 dr'^2}{1 - \frac{2\beta'}{r'} + \Lambda' r'^2} = \frac{dr^2}{A} \tag{4}$$

$$\Omega^2\left(1 - \frac{2\beta'}{r'} + \Lambda' r'^2\right) = A. \tag{5}$$

Using Eq. (3) to eliminate Ω from Eqs. (4) & (5) and comparing between them we obtain that $dr'/r'^2 = dr/r^2$ which integrates to

$$\frac{1}{r'} = \frac{1}{r} + C \tag{6}$$

where C is an integration constant. Plugging this back in Eq. (5) we obtain

$$A(r) = 1 - 6C\beta' - \frac{2\beta'}{r} + (2C - 6\beta' C^2)r + (C^2 - 2\beta' C^3 + \Lambda')r^2. \tag{7}$$

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Comparison to the Mannheim-Kasanas solution [1989]

$$A(r) = 1 - 3\beta\gamma - \frac{\beta(2 - 3\beta\gamma)}{r} + \gamma r - kr^2 \tag{8}$$

where β, γ & k are integration constants immediately shows that setting $-k = \Lambda' + \frac{\gamma^2(1-\beta\gamma)}{4(1-\frac{3\beta\gamma}{2})^2}$, $\beta' = \beta(1 - \frac{3\beta\gamma}{2})$ & $C = \frac{\gamma}{2(1-\frac{3\beta\gamma}{2})}$ renders our metric solution, Eq. (7), equivalent to the Mannheim-Kazanas metric. Since both theories are Weyl-invariant all the fields, not only the metric, should be appropriately locally-rescaled, e.g. $\phi \to \Omega^{-1}\phi$, $\psi \to \Omega^{-3/2}\psi$, $A_\mu \to A_\mu$, $A^\mu \to \Omega^{-2}A^\mu$, etc.

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