

Bell's Inequality and the Collapse of Objective Reality

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I first read about Bell's inequality and the "demise of locality" in a *Scientific American* article in the late 1970s, and I remember being very much impressed with its oddball conclusions regarding the fate of objective reality (what E. Squires has more recently referred to as "the ultimate silliness of the quantum world"). Nowadays I am much less impressed, but I don't know why; maybe it's only because I am now more accustomed to quantum mechanics than I was then. Nevertheless, the quantum mechanical violation of Bell's inequality, which was proven experimentally in the 1980s, is yet another example of how God has cooked up a universe that is stranger than anything we could have imagined.

If you have experienced problems deriving Bell's inequality and/or its quantum variant, the following discussion should be of help to you. The only prerequisite is elementary quantum mechanics and an open mind. And although I will share some of my personal thoughts about what I think it all means, you'll have to decide for yourself if there's anything really profound involved.

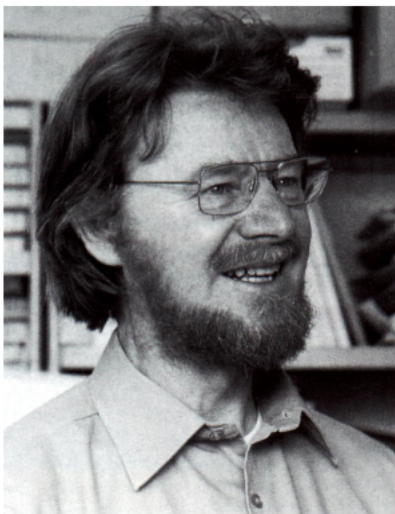
Bohr v. Einstein

You may have read about the friendly but earnest "war" that Niels Bohr and Albert Einstein waged from the 1920s on regarding whether nature was deterministic or probabilistic. Briefly stated, Einstein never made the "quantum leap," believing instead that God didn't play dice with the universe in the sense that God didn't leave important stuff to probability (like an atom radiating purely by chance). In contrast, his good friend Bohr believed that nature was probabilistic at its core because quantum mechanics said it was. The two great physicists bickered good-naturedly between themselves for years; Einstein would propose some hypothetical thought experiment designed to trip up quantum theory, but the oft-worried though ultimately assured Bohr would always find a way to evade Einstein's trap and secure another win for quantum mechanics. Eventually, Einstein had to concede that quantum mechanics was a pretty good description of nature, but he still felt that it was fundamentally incomplete. In 1933 Einstein came up with an argument that he believed would finally win him a victory against Bohr, and in 1935 Einstein and his colleagues Boris Podolsky and Nathan Rosen published a paper describing an effect that has since become famously known as the Einstein-Podolsky-Rosen (or EPR) paradox.

To rephrase Einstein's original thought experiment, the EPR effect involves an unstable, spin-zero particle which spontaneously decays into two spin-1/2 particles that then fly away from each other. Quantum physics says that initially each daughter particle is in a superposition of spin eigenstates, existing as a kind of strange hybrid creature with both spin-up and spin-down characteristics (like Schrödinger's half-living, half-dead cat). Let each particle be accompanied by an observer until a great distance has accumulated between the particles. Now if one of the observers measures the spin of the particle she has been following, she will get one of the two possible results, $\pm 1/2 \hbar$. Einstein argued that since the total spin must be conserved, the wave function of the other particle (whether it is measured or not) must collapse instantaneously to give a spin of $\mp 1/2 \hbar$, regardless of the fact that it might be light years away from its partner. Bohr did not (and could not) disagree with this reasoning because it is allowed by quantum mechanics. However, Einstein pointed out that this would require that the first particle send out a faster-than-light signal to its partner so that its spin would agree with conservation of spin, thus violating a major tenet of the principle of special relativity (in modern language, we would say that the principle of *locality* had been violated). Einstein claimed that quantum theory must therefore be wrong (or at least incomplete) lest the cherished notion of locality be overturned. He consequently refused to believe that nature would allow what he called this kind of "spooky action at a distance" (yes, he actually used that term).

After having survived many such battles with Einstein, Bohr had enough confidence in quantum mechanics to remain calm and collected. However, his "explanation" upset a lot of people at the time (and maybe even himself). Bohr stated that, *until an observation is made*, no one is allowed to make any definitive statements about a physical system, regardless of its provenance. In essence, Bohr postulated that the two particles are initially "entangled" quantum mechanically, and that this quantum connectedness persists until a measurement is made. He believed that when the first particle's spin is measured, it in fact represents a measurement on the *entire* system,

regardless of how far apart the particles had moved from one another. In this sense, Bohr refused to ascribe to either particle an identity separate from its partner. Furthermore, Bohr believed that special relativity would not be technically violated provided that no physical signal was transferred from the first particle to the second when the measurement is performed. However, Bohr realized that the principle of locality and quantum mechanics were indeed fundamentally incompatible, and as one of the founders of quantum physics he believed that quantum theory would ultimately be proved correct. In order to fully appreciate the EPR paradox, it is absolutely essential that you understand the superposition principle; otherwise, you will be tempted to say that the first particle's spin had been determined at the instant it was created, and that no therefore paradox actually exists. It's the same argument that says that if I have a closed box with a black marble and a red marble in it and I take out a red marble, the remaining marble must be black. But particle spin is not as simple as this because it can be measured in many different orientations, up, down, sidewise, diagonally, and so on, while colored marbles can only be "measured" one way.



J.S. Bell, 1928-1990

At the time Einstein and Bohr were haggling with each other there was no way to test whether the EPR effect was a deterministic or probabilistic phenomenon (or indeed, whether there even was a phenomenon). Then in 1964 the Irish physicist John Stewart Bell established a clever mathematical criterion that could be used to determine which explanation was correct (interestingly, Bell originally sided with Einstein on the matter). However it was not until the 1970s that the criterion could be tested experimentally. These early results were rather inconclusive, but in 1982 the French physicist Alain Aspect and his colleagues devised a vastly more accurate experiment that finally provided the unequivocal answer. More experiments followed in the 1990s which removed all doubt.

Bell's Inequality

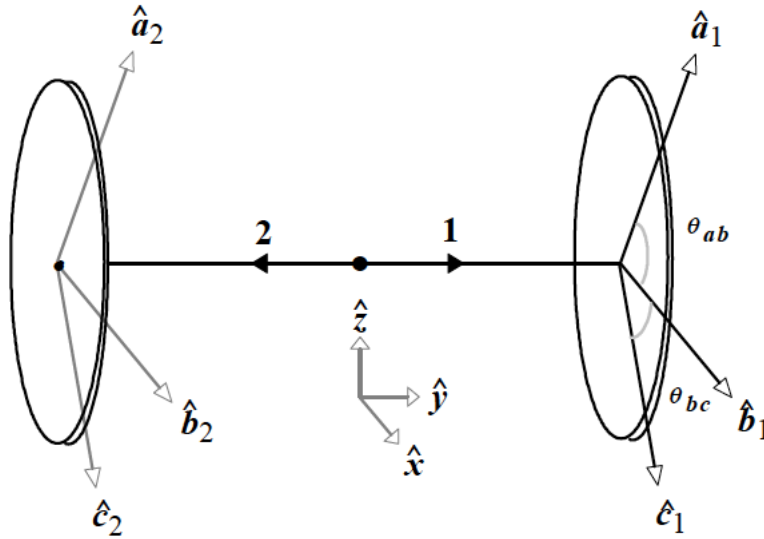
To begin, let us take an unstable spin-zero particle and let it decay into two spin-1/2 fermions (we can use an η meson, which occasionally decays into a muon pair μ^+ and μ^- , but we can assume a purely hypothetical decay if we want). The major requirement is that the two daughter particles must be in what is known as a *spin-singlet state*, which means that the total spin of the system (like the parent particle) is always zero. The spin-zero requirement is needed to enforce conservation of angular momentum, which you should already be familiar with. Anyway, after the particles are created let them separate and move in opposite directions at high velocity, while we follow each one with a spin measuring device. When the particles have gotten to a nice distance (say, one light year), we measure the particles' spins, first one and then the other.

Immediately prior to any measurement, quantum theory states that each particle is in a superposition of possible spin states; for example, Particle 1's spin state in some arbitrary direction can be expressed as

$$|1\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

so it has an equal chance of being caught in the spin-up state (+) or the spin down state (-). Particle 2 finds itself in a similar superposition of states. However, the instant that the spin of Particle 1 is determined, Particle 2's spin state must collapse to give an opposing spin. How does the first particle instantaneously "inform" the second so that the total spin is zero, despite the fact that they are a light year apart? Einstein believed that the particles must carry some additional information, or hidden variables (which determine the spin of each at the point of their creation) that quantum mechanics has hitherto concealed from us. But Bohr stuck to his guns, saying that the two-particle system could be viewed as a single quantum system and that a measurement on one particle was the same as a measurement on the entire system, regardless of how far apart the particles are. So who's right?

As you are already aware, electron spin ordinarily comes in three varieties, S_x , S_y and S_z . But this does not mean that spin measurements have to be made along Cartesian axes. Instead, we can set up each spin-measuring device in a planar system defined by the directions a , b and c , all of which are separated by some predetermined angular specification for both particles. (Why not just use a Cartesian system to start with? Because Bell's inequality will not be violated by quantum mechanics, and I wouldn't be writing this.) So let's set it up this way: after the particles are created, they fly off to the left and right, where we have our spin measuring devices waiting for them. Let left and right be aligned along the y -axis, and let the z - x plane represent the planar system in which the measurements will be made:



If we make a measurement for Particle 1 along the a_1 direction and find that it is spin down (a_1^-), then we can know with absolute certainty that the spin of Particle 2 taken in the same direction will be a_2^+ , while the spin along direction b_2 or c_2 remains to be seen. So let's do a bunch of experiments using random combinations of $a_1 b_1 c_1$ and $a_2 b_2 c_2$. We can summarize the data we get using the following simple correlation table (the subscripts have been dropped for brevity):

Particle 1	Particle 2	Number of Observations
$a^+ b^+ c^+$	$a^- b^- c^-$	N_1
$a^+ b^+ c^-$	$a^- b^- c^+$	N_2
$a^+ b^- c^+$	$a^- b^+ c^-$	N_3
$a^+ b^- c^-$	$a^- b^+ c^+$	N_4
$a^- b^+ c^+$	$a^+ b^- c^-$	N_5
$a^- b^+ c^-$	$a^+ b^- c^+$	N_6
$a^- b^- c^+$	$a^+ b^+ c^-$	N_7
$a^- b^- c^-$	$a^+ b^+ c^+$	N_8

The information in the table accounts for every possible measurement outcome. I absolutely guarantee that the result of any spin measurement performed on any connected two-particle system is included in the above table. In fact, if the table had consisted of only measurements for Particle 1, the corresponding results for Particle 2 could be written down without actually bothering to make any measurements.

If we perform a sufficient number of these experiments, then we can do some simple statistical analyses. Let

us do a total of N measurements covering every spin direction. The results are shown in the third column, which records the number N_k corresponding to the associated combinations of spins for each particle. If we divide each N_k by N , we'll have (to a high degree of accuracy) the probability that each of the eight combinations will occur.

In an actual experiment, we can measure only one spin direction at a time for each particle. In view of this, we need to consider the probabilities for seeing different combinations of paired spin measurements, such as $a_1^+ c_2^+$ (from now on, I'll drop the subscripts altogether, so this will now be written as $a^+ c^+$, etc.). This notation means that we found Particle 1 to have spin up in the a direction while Particle 2 was spin-up in the c direction. This approach allows us to decide ahead of time which direction we will measure for each particle. Let us call the probability of this combination $P(a^+ c^+)$, and so on for all other combinations. The information in the above table is still valid, and using it we can confidently write expressions like

$$\begin{aligned} P(a^+ c^+) &= (N_2 + N_4)/N \\ &= n_2 + n_4 \end{aligned}$$

where n_k is the frequency of the particular combination. Similarly, we can write expressions for $P(a^+ b^+)$ and $P(b^+ c^+)$:

$$\begin{aligned} P(a^+ b^+) &= n_3 + n_4 \\ P(b^+ c^+) &= n_2 + n_6 \end{aligned}$$

We now add the last two expressions to get

$$\begin{aligned} P(a^+ b^+) + P(b^+ c^+) &= n_2 + n_4 + n_3 + n_6 \\ &= P(a^+ c^+) + n_3 + n_6 \end{aligned}$$

Since all the n_k are positive (or zero at the least), we can unambiguously write

$$P(a^+ c^+) \leq P(a^+ b^+) + P(b^+ c^+) \tag{1}$$

This is one version of Bell's inequality. You can write down other versions corresponding to other possible combinations of two-particle measurements if you like, but we'll stick with this one as it is the only one we'll need.

Carefully review the logic I have used in developing (1). When I first encountered Bell's inequality, I could see nothing wrong with the logic, nor was I able to discern any fatal assumption that might disprove it. If you're like most people, you won't see anything amiss, either. But I will tell you right now that although we've made no mathematical mistakes or errors in judgment, we did interject one extremely subtle and seemingly innocent assumption into the overall logic: *we presumed to believe that the particles exist as separate entities, each with its own physical reality*. Because of this, we have assumed that we can know the spin of Particle 2 without actually measuring it. I will return to this apparently irrelevant issue after we have reproduced Bell's inequality using quantum mechanics.

Bell's Inequality in Quantum Mechanics

Now let us see what quantum mechanics has to say about all this. Consider the probability $P(a^+ b^+)$. We will assume that the two particles start out in the singlet spin eigenstate $|\psi_i\rangle$ defined by

$$|\psi_i\rangle = \frac{1}{\sqrt{2}} [|a^+ a^-\rangle - |a^- a^+\rangle] \tag{2}$$

where $|a^+ a^-\rangle = |a^+\rangle|a^-\rangle$, or $|a^+\rangle \otimes |a^-\rangle$ if you're a purist (if you're unsure of where (2) came from, look it up in any elementary quantum mechanics text. It just ensures that a two-particle system state will have zero total spin). When the measurements have been made, and we've ascertained that the combination $a^+ b^+$ has been obtained, then we know that the system's final state $|\psi_f\rangle$ is just

$$|\psi_f\rangle = |a^+ b^+\rangle$$

Therefore, the probability amplitude \mathcal{M} for the two-particle system to go from the initial state to the final state can be written as

$$\begin{aligned}\mathcal{M}(a^+b^+) &= \langle \psi_f | \psi_i \rangle \\ &= \frac{1}{\sqrt{2}} \langle a^+b^+ | a^+a^- \rangle - \frac{1}{\sqrt{2}} \langle a^+b^+ | a^-a^+ \rangle\end{aligned}$$

Now look at the first bra and recall that it really stands for $\langle a^+ | \langle b^+ |$, while the adjacent ket stands for $|a^+ \rangle |a^- \rangle$. Thus, Particle 1's $\langle a^+ |$ bra reaches through Particle 2's $\langle b^+ |$ bra and meets up with Particle 1's $|a^+ \rangle$ ket (a better way to see this is to note that the order of the bras is unimportant; we could have written the term as $\langle b^+ | \langle a^+ |$ instead, but then you would have to keep track of which bra belongs to which particle). Anyway, using the fact that $\langle a^+ | a^+ \rangle = 1$, the first term becomes $\frac{1}{\sqrt{2}} \langle b^+ | a^- \rangle$, where the b^+ and a^- now both belong to Particle 2. Using this same approach, we see that the second term includes the vanishing quantity $\langle a^+ | a^- \rangle$, leaving us with

$$\mathcal{M}(a^+b^+) = \frac{1}{\sqrt{2}} \langle b^+ | a^- \rangle$$

We now need to simplify this to get a numerical result. As the primary difference between b^+ and a^- is the direction, we can use the unitary rotation operator to write $|a^- \rangle$ as the rotated form of $|b^- \rangle$ (we have to use $|b^- \rangle$ to begin with, because rotation cannot change spin-up into spin-down). You may recall that the rotation operator \hat{D} in quantum mechanics is

$$\hat{D}(\hat{n}\theta) = \exp\left[-\frac{i}{\hbar} \hat{S} \cdot \hat{n}\theta\right]$$

where \hat{S} represents the 2×2 spin operators and $\hat{n}\theta$ is the angular argument. Since we're working in the z - x plane, all rotations are performed around the y -axis, so

$$\begin{aligned}\hat{S} &= \hat{S}_y = \frac{1}{2} \hbar \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ &= \frac{1}{2} \hbar \sigma_y\end{aligned}$$

where σ_y is the y -form Pauli spin matrix. Using the familiar identity

$$\begin{aligned}\exp\left[-\frac{i}{2} \sigma_y \theta\right] &= \cos \frac{1}{2} \theta - i \sigma_y \sin \frac{1}{2} \theta \\ &= \begin{bmatrix} \cos \frac{1}{2} \theta & -\sin \frac{1}{2} \theta \\ \sin \frac{1}{2} \theta & \cos \frac{1}{2} \theta \end{bmatrix}\end{aligned}$$

we can write

$$\begin{aligned}|a^- \rangle &= \exp\left[-\frac{i}{2} \sigma_y \theta\right] |b^- \rangle \\ &= \begin{bmatrix} \cos \frac{1}{2} \theta & -\sin \frac{1}{2} \theta \\ -\sin \frac{1}{2} \theta & \cos \frac{1}{2} \theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\end{aligned}$$

where I have used a spin-down column matrix for the ket $|b^- \rangle$. Multiplying this out, we have

$$\begin{aligned}|a^- \rangle &= \begin{bmatrix} -\sin \frac{1}{2} \theta \\ \cos \frac{1}{2} \theta \end{bmatrix} \\ &= -\sin \frac{1}{2} \theta |b^+ \rangle + \cos \frac{1}{2} \theta |b^- \rangle\end{aligned}$$

We then finally have

$$\begin{aligned}\mathcal{M}(a^+b^+) &= \frac{1}{\sqrt{2}} \langle b^+ | a^- \rangle \\ &= -\frac{1}{\sqrt{2}} \sin \frac{1}{2} \theta\end{aligned}$$

The square of this quantity is the probability, so

$$P(a^+b^+) = \frac{1}{2} \sin^2 \frac{1}{2} \theta_{ab}$$

where I have now explicitly indicated that the angle we're dealing with is the one between a and b . This result makes sense, for if $\theta_{ab} = 0$ then we find that the probability of both particles having identical spins in the same direction is identically zero. Similarly, if $\theta_{ab} = \pi$ then the probability is exactly 50% because the chances of Particle 1's spin being a^+ is also 50%. So far so good.

The exact same calculations can be done for all the other probabilities, so Bell's inequality according to quantum mechanics can be expressed as

$$\sin^2 \frac{1}{2} \theta_{ac} \leq \sin^2 \frac{1}{2} \theta_{ab} + \sin^2 \frac{1}{2} \theta_{bc} \quad (3)$$

(Note that if we had used orthogonal Cartesian coordinates to start with, all the angles used in this argument would be $\pi/2$; the inequality would always hold, and all of this would be very boring.) Now let's assume some plausible values for the angles. Let $\theta_{ac} = \pi/2$ (just like the real Cartesian z - x plane), and let $\theta_{ab} = \theta_{bc} = \theta_{ac}/2 = \pi/4$. The b direction thus bisects the right angle, and (3) then gives us

$$0.5 \stackrel{?}{\leq} 0.293$$

Thus, Bell's inequality is violated in quantum mechanics.

Experimental Verification

We have seen that quantum mechanics disagrees with the approach used in the beginning of this discussion. But which view is correct?

In the early 1980s the French physicist Alain Aspect and his colleagues devised several clever experiments to decide the issue once and for all. Their laboratory investigations conclusively demonstrated that Bell's inequality was violated (in one experimental run it was violated by nine standard deviations), so quantum mechanics was proved to be correct. Subsequent investigations were conducted in which the spin direction was not decided upon in advance, but chosen while the particles (actually, polarized photons) were still in flight. These "delayed decision" experiments again confirmed that Bell's inequality was violated.

What Does It All Mean?

Locality is a term physicists use to describe allowed causal effects of one object or condition on another. Basically, it states that no physical effect can be transferred from one place to another unless there is a clear, unambiguous physical connection involved that does not rely on superluminal velocities. Einstein argued that the act of observing the spin of a particle quantum mechanically "entangled" with another distant particle could not cause the collapse of the other's spin state because it violated locality. In establishing his argument, Einstein attributed a separate physical reality to each particle, at least partly because they could be physically separated by an arbitrarily large distance. This point of view is known as "local realism."

The violation of Bell's inequality shows that this kind of thinking is wrong. Regardless of the extent of their physical separation, two entangled particles in fact belong to the same system and a measurement performed on one constitutes a measurement performed on the entire system. The mechanism that lies behind their connectedness is an enigma, although it is almost certainly quantum mechanical in nature. Whatever it is and however it works is a mystery. Locality is certainly dead, but this realization now brings into question the role of the observer in what is being observed. Einstein fervently believed that physical reality is objective in the sense that things can exist whether they are observed or not. This point of view also appears to be incorrect.

What really happens when an observation is made? For a sentient observer, information that was not hitherto available is registered somewhere in the brain. Somehow this causes wave functions outside of the brain to collapse. Does this mean that the brains of conscious observers are connected in some way to what they're looking at? What happens when the observer is not sentient? Can a cat collapse a wave function? If so, can a mouse or

a microorganism? We can pose similar questions concerning the nature of reality itself. Does the universe exist simply because there are observers to look at it? Can we be sure that a physical object exists if it is never actually observed? The old yarn about a tree falling silently in a forest with no one around may be more profound than we think.

Knowledge and intelligence seem to play very important roles in our understanding of objective reality. Let's say that a friend has a hydrogen atom that resides in a super position of energy states. In your absence, she carries out an experiment and finds that the atom has collapsed into the $n = 3$ energy eigenstate. She tells you this, and you conduct your own experiment – sure enough, it's in the $n = 3$ state. But what if someone had done an experiment on the atom just before your friend came into possession of it and didn't tell her – was it always $n = 3$?

Here's an interesting thought – When you perform experiments and make observations, wave functions collapse accordingly, and they reflect truthfully what actually occurred. If you choose to lie about your findings, it doesn't change anything in itself, but the transference of false information to others can result in unpredictable consequences. Suppose your friend is a pathological liar and tells you that her atom is “fresh” – if you then observe its energy state, what will you see? Can the lie force the atom's eigenstate back into a superposition of states? If so, then just what is the reality behind the first measurement? On the other hand, suppose your friend doesn't lie, but goes crazy or gets Alzheimer's disease or dies, and you inherit the atom – is it still $n = 3$, or does the atom regress into a superposition again? Does God (or Satan or whoever) collapse wave functions, and if so, what are the consequences for us? Maybe these questions are all a waste of time, and maybe not. I certainly don't know.

It gets weirder. The noted American physicist John Archibald Wheeler has mused over the very real possibility that the universe came into being simply because it needed to have observers around to look at it and give it reality. To me, the philosophical and religious aspects of this kind of thinking are mind boggling. My mind immediately turns to the concept of free will and how God, by endowing us with this gift, knowingly set into motion a physical reality that had to be consistent with our ability to willingly choose between good and evil, truth and lies. Did Jesus Christ have to die for us? Because of free will, it was probably inevitable, and God knew it from the beginning. Or maybe not. I certainly don't know that, either.

My own personal view is that, whatever reality is, it is non-local and involves the acquisition and/or exchange of information. This is hardly a unique idea, and I haven't really given it much thought, but the concept of sentient awareness seems to be a good candidate as the basis of things. I encourage you to read up on the ideas of David Bohm (hidden variables and the quantum potential force), Hugh Everett (the many worlds theory), and Stephen Hawking (the information paradox), all of which have been formulated to help us better understand the world we live in. Maybe you can make more sense out of all this than I've been able to.

References

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5. E. Squires, *The Mystery of the Quantum World*. 2nd Ed. IOP Publishing, London (1994).
6. E. H. Walker, *The Physics of Consciousness*. Perseus Publishing, Cambridge, Massachusetts (2000). I was enlightened and extremely moved by this book, which weaves a Christian physicist's views on quantum physics, consciousness and the nature of reality with truly touching reminiscences of the author's long-lost teenage love, Marilyn, who died tragically at an early age. Unlike the pretentious *The Tao of Physics*, it is one of those uncommon books that speaks to both scientists and non-scientists alike (and the author is not afraid to use equations). Walker's 368-page book includes detailed discussions of Bell's inequality, Schrödinger's cat, the concept of free will, the structure of matter, and many other related topics.