Mach's Principle and a Relativistic Theory of Gravitation*

C. Brans† AND R. H. Dicke
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received June 23, 1961)

The role of Mach's principle in physics is discussed in relation to the equivalence principle. The difficulties encountered in attempting to incorporate Mach's principle into general relativity are discussed. A modified relativistic theory of gravitation, apparently compatible with Mach's principle, is developed.

INTRODUCTION

IT is interesting that only two ideas concerning the nature of space have dominated our thinking since the time of Descartes. According to one of these pictures, space is an absolute physical structure with properties of its own. This picture can be traced from Descartes vortices[1] through the absolute space of Newton,[2] to the ether theories of the 19th century. The contrary view that the geometrical and inertial properties of space are meaningless for an empty space, that the physical properties of space have their origin in the matter contained therein, and that the only meaningful motion of a particle is motion relative to other matter in the universe has never found its complete expression in a physical theory. This picture is also old and can be traced from the writings of Bishop Berkeley[3] to those of Ernst Mach.[4] These ideas have found a limited expression in general relativity, but it must be admitted that, although in general relativity spatial geometries are affected by mass distributions, the geometry is not uniquely specified by the distribution. It has not yet been possible to specify boundary conditions on the field equations of general relativity which would bring the theory into accord with Mach's principle. Such boundary conditions would, among other things, eliminate all solutions without mass present.

It is necessary to remark that, according to the ideas of Mach, the inertial forces observed locally in an accelerated laboratory may be interpreted as gravitational effects having their origin in distant matter accelerated relative to the laboratory. The imperfect expression of this idea in general relativity can be seen by considering the case of a space empty except for a lone experimenter in his laboratory. Using the traditional, asymptotically Minkowskian coordinate system fixed relative to the laboratory, and assuming a normal laboratory of

small mass, its effect on the metric is minor and can be considered in the weak-field approximation. The observer would, according to general relativity, observe normal behavior of his apparatus in accordance with the usual laws of physics. However, also according to general relativity, the experimenter could set his laboratory rotating by leaning out a window and firing his 22-caliber rifle tangentially. Thereafter the delicate gyroscope in the laboratory would continue to point in a direction nearly fixed relative to the direction of motion of the rapidly receding bullet. The gyroscope would rotate relative to the walls of the laboratory. Thus, from the point of view of Mach, the tiny, almost massless, very distant bullet seems to be more important that the massive, nearby walls of the laboratory in determining inertial coordinate frames and the orientation of the gyroscope.[5] It is clear that what is being described here is more nearly an absolute space in the sense of Newton rather than a physical space in the sense of Berkeley and Mach.

The above example poses a problem for us. Apparently, we may assume one of at least three things:

1. that physical space has intrinsic geometrical and inertial properties beyond those derived from the matter contained therein;

2. that the above example may be excluded as nonphysical by some presently unknown boundary condition on the equations of general relativity.

3. that the above physical situation is not correctly described by the equations of general relativity.

These various alternatives have been discussed previously. Objections to the first possibility are mainly philosophical and, as stated previously, go back to the time of Bishop Berkeley. A common inheritance of all present-day physicists from Einstein is an appreciation for the concept of relativity of motion.

As the universe is observed to be nonuniform, it would appear to be difficult to specify boundary conditions which would have the effect of prohibiting unsuitable mass distributions relative to the laboratory *arbitrarily placed*; for could not a laboratory be built near a massive star? Should not the presence of this massive star contribute to the inertial reaction?

The difficulty is brought into sharper focus by con-

[1] E. T. Whittaker, *History of the Theories of Aether and Electricity* (Thomas Nelson and Sons, New York, 1951).
[2] I. Newton, *Principia Mathematica Philosophiae Naturalis* (1686) (reprinted by University of California Press, Berkeley, California, 1934).
[3] G. Berkeley, *The Principles of Human Knowledge*, paragraphs 111–117, 1710-*De Motu* (1726).
[4] E. Mach, *Conservation of Energy*, note No. 1, 1872 (reprinted by Open Court Publishing Company, LaSalle, Illinois, 1911), and *The Science of Mechanics*, 1883 (reprinted by Open Court Publishing Company, LaSalle, Illinois, 1902), Chap. II, Sec. VI.

[5] Because of the Thirring-Lense effect, [H. Thirring and J. Lense, Phys. Zeits. **19**, 156 (1918)], the rotating laboratory would have a weak effect on the axis of the gyroscope.

sidering the laws of physics, including their quantitative aspects, inside a static massive spherical shell. It is well known that the interior Schwarzschild solution is flat and can be expressed in a coordinate system Minkowskian in the interior. Also, according to general relativity all Minkowskian coordinate systems are equivalent and the mass and radius of the spherical shell have no discernible effects upon the laws of physics as they are observed in the interior. Apparently the spherical shell does not contribute in any discernible way to inertial effects in the interior. What would happen if the mass of the shell were decreased, or its radius increased without limit? It might be remarked also that Komar[6] has attempted, without success, to find suitable boundary- and initial-value conditions for general relativity which would bring into evidence Mach's principle.

The third alternative is the subject of this paper. Actually the objectives of this paper are more limited than the formulation of a theory in complete accord with Mach's principle. Such a program would consist of two parts, the formulation of a suitable field theory and the formulation of suitable boundary- and initial-value conditions for the theory which would make the space geometry depend uniquely upon the matter distribution. This latter part of the problem is treated only partially.

At the end of the last section we shall briefly return again to the problem of the rotating laboratory.

A principle as sweeping as that of Mach, having its origins in matters of philosophy, can be described in the absence of a theory in a qualitative way only. A model of a theory incorporating elements of Mach's principle has been given by Sciama.[7] From simple dimensional arguments[8,9] as well as the discussion of Sciama, it has appeared that, with the assumption of validity of Mach's principle, the gravitational constant G is related to the mass distribution in a uniform expanding universe in the following way:

$$GM/Rc^2 \sim 1. \tag{1}$$

Here M stands for the finite mass of the visible (i.e., causally related) universe, and R stands for the radius of the boundary of the visible universe.

The physical ideas behind Eq. (1) have been given in references 7–9 and can be summarized easily. As stated before, according to Mach's principle the only meaningful motion is that relative to the rest of the matter in the universe, and the inertial reaction experienced in a laboratory accelerated relative to the distant matter of the universe may be interpreted equivalently as a gravitational force acting on a fixed laboratory

due to the presence of distant accelerated matter.[7] This interpretation of the inertial reaction carries with it an interesting implication. Consider a test body falling toward the sun. In a coordinate system so chosen that the object is not accelerating, the gravitational pull of the sun may be considered as balanced by another gravitational pull, the inertial reaction.[8] Note that the balance is not disturbed by a doubling of all gravitational forces. Thus the acceleration is determined by the mass distribution in the universe, but is independent of the strength of gravitational interactions. Designating the mass of the sun by m_s and its distance by r enables the acceleration to be expressed according to Newton as $a = Gm_s/r^2$ or, from dimensional arguments, in terms of the mass distribution as $a \sim mRc^2/Mr^2$. Combining the two expressions gives Eq. (1).

This relation has significance in a rough order-of-magnitude manner only, but it suggests that either the ratio of M to R should be fixed by the theory, or alternatively that the gravitational constant observed locally should be variable and determined by the mass distribution about the point in question. The first of these two alternatives is of course, in part, simply the limitation of mass distribution which it might be hoped would result from some boundary condition on the field equations of general relativity. The second alternative is not compatible with the "strong principle of equivalence"[10] and general relativity. The reasons for this will be discussed below.

If the inertial reaction may be interpreted as a gravitational force due to distant accelerated matter, it might be expected that the locally observed values of the inertial masses of particles would depend upon the distribution of matter about the point in question. It should be noted, however, that there is a fundamental ambiguity in a statement of this type, for there is no direct way in which the mass of a particle such as an electron can be compared with that of another at a different space-time point. Mass ratios can be compared at different points, but not masses. On the other hand, gravitation provides another characteristic mass

$$(\hbar c/G)^{\frac{1}{2}} = 2.16 \times 10^{-5} \text{ g}, \tag{2}$$

and the mass ratio, the dimensionless number

$$m(G/\hbar c)^{\frac{1}{2}} \cong 5 \times 10^{-23}, \tag{3}$$

provides an unambiguous measure of the mass of an electron which can be compared at different space-time points.

It should also be remarked that statements such as "$\hbar$ and c are the same at all space-time points" are in the same way meaningless within the same context until a method of measurement is prescribed. In fact, it should be noted that $\hbar$ and c may be defined to be constant. A set of physical "constants" may be defined as constant if they cannot be combined to form one or

[6] A. Komar, Ph.D. thesis, Princeton University, 1956 (unpublished).
[7] D. W. Sciama, Monthly Notices Roy. Astron. Soc. 113, 34 (1953); *The Unity of the Universe* (Doubleday & Company, Inc., New York, 1959), Chaps. 7–9.
[8] R. H. Dicke, Am. Scientist 47, 25 (1959).
[9] R. H. Dicke, Science 129, 621 (1959).
[10] R. H. Dicke, Am. J. Phys. 29, 344 (1960).

more dimensionless numbers. The necessity for this limitation is obvious, for a dimensionless number is invariant under a transformation of units and the question of the constancy of such dimensionless numbers is to be settled, not by definition, but by measurements. A set of such independent physical constants which are constant by definition is "complete" if it is impossible to include another without generating dimensionless numbers.

It should be noted that if the number, Eq. (3), should vary with position and h and c are defined as constant, then either m or G, or both, could vary with position. There is no fundamental difference between the alternatives of constant mass or constant G. However, one or the other may be more convenient, for the formal structure of the theory would, in a superficial way, be quite different for the two cases.

To return to Eq. (3), the odd size of this dimensionless number has often been noticed as well as its apparent relation to the large dimensionless numbers of astrophysics. The apparent relation of the square of the reciprocal of this number [Eq. (3)] to the age of the universe expressed as a dimensionless number in atomic time units and the square root of the mass of the visible portion of the universe expressed in proton mass units suggested to Dirac[11] a causal connection that would lead to the value of Eq. (3) changing with time. The significance of Dirac's hypothesis from the standpoint of Mach's principle has been discussed.[8]

Dirac postulated a detailed cosmological model based on these numerical coincidences. This has been criticized on the grounds that it goes well beyond the empirical data upon which it is based.[8] Also in another publication by one of us (R. H. D.), it will be shown that it gives results not in accord with astrophysical observations examined in the light of modern stellar evolutionary theory.

On the other hand, it should be noted that a large dimensionless physical constant such as the reciprocal of Eq. (3) must be regarded as either determined by nature in a completely capricious fashion or else as related to some other large number derived from nature. In any case, it seems unreasonable to attempt to derive a number like 10^{23} from theory as a purely mathematical number involving factors such as $4\pi/3$.

It is concluded therefore, that although the detailed structure of Dirac's cosmology cannot be justified by the weak empirical evidence on which it is based, the more general conclusion that the number [Eq. (3)] varies with time has a more solid basis.

If, in line with the interpretation of Mach's principle being developed, the dimensionless mass ratio given by Eq. (3) should depend upon the matter distribution in the universe, with h and c constant by definition, either the mass m or the gravitational constant, or both, must vary. Although these are alternative descriptions of the

same physical situation, the formal structure of the theory would be very different for the two cases. Thus, for example, it can be easily shown that uncharged spinless particles whose masses are position dependent no longer move on geodesics of the metric. (See Appendix I.) Thus, the definition of the metric tensor is different for the two cases. The two metric tensors are connected by a conformal transformation.

The arbitrariness in the metric tensor which results from the indefiniteness in the choice of units of measure raises questions about the physical significance of Riemannian geometry in relativity.[12] In particular the 14 invariants which characterize the space are generally not invariant under a conformal transformation interpreted as a redefinition of the metric tensor in the *same* space.[13] Matters are even worse, for a more general redefinition of the units of measure can be used to reduce all 14 invariants to zero. It should be said that these remarks should not be interpreted as casting doubt on the correctness or usefulness of Riemannian geometry in relativity, but rather that each such geometry is but a particular representation of the theory. It would be expected that the physical content of the theory should be contained in the invariants of the group of position-dependent transformations of units and coordinate transformations. The usual invariants of Riemannian geometry are not invariants under this wider group.

In general relativity the representation is one in which units are chosen so that atoms are described as having physical properties independent of location. It is assumed that this choice is possible!

In accordance with the above, a particular choice of units is made with the realization that the choice is arbitrary and without an invariant significance. The theoretical structure appears to be simpler if one defines the inertial masses of elementary particles to be constant and permits the gravitational constant to vary. It should be noted that this is possible only if the mass ratios of elementary particles are constant. There may be reasonable doubt about this.[9,10] On the other hand, it would be expected that such quantities as particle mass ratios or the fine-structure constant, if they depend upon mass distributions in the universe, would be much less sensitive in their dependence[9] rather than the number given by Eq. (3) and their variation could be neglected in a first crude theory. Also it should be remarked that the requirements of the approximate constancy of the ratio of inertial to passive gravitational mass,[14] and the extremely stringent requirement of spatial isotropy,[15] impose conditions so severe that it has been found to be difficult, if not impossible, to

[11] P. A. M. Dirac, Proc. Roy. Soc. (London) **A165**, 199 (1938).

[12] E. P. Wigner has questioned the physical significance of Riemannian geometry on other grounds [Relativity Seminar, Stevens Institute, May 9, 1961 (unpublished)].
[13] B. Hoffman, Phys. Rev. **89**, 49 (1953).
[14] R. Eötvös, Ann. Physik **68**, 11 (1922).
[15] V. W. Hughes, H. G. Robinson, and V. Beltran-Lopez, Phys. Rev. Letters **4**, 342 (1960).

construct a satisfactory theory with a variable fine-structure constant.

It should be emphasized that the above argument involving the large dimensionless numbers, Eq. (3), does not concern Mach's principle directly, but that Mach's principle and the assumption of a gravitational "constant" dependent upon mass distributions gives a reasonable explanation for varying "constants."

It would be expected that both nearby and distant matter should contribute to the inertial reaction experienced locally. If the theory were linear, which one does not expect, Eq. (1) would suggest that it is the reciprocal of the gravitational constant which is determined locally as a linear superposition of contributions from the matter in the universe which is causally connected to the point in question. This can be expressed in a somewhat symbolic equation:

$$G^{-1} \sim \sum_i (m_i/r_i c^2), \qquad (4)$$

where the sum is over all the matter which can contribute to the inertial reaction. This equation can be given an exact meaning only after a theory has been constructed. Equation (4) is also a relation from Sciama's theory.

It is necessary to say a few words about the equivalence principle as it is used in general relativity and as it relates to Mach's principle. As it enters general relativity, the equivalence principle is more than the assumption of the local equivalence of a gravitational force and an acceleration. Actually, in general relativity it is assumed that the laws of physics, including numerical content (i.e., dimensionless physical constants), as observed locally in a freely falling laboratory, are independent of the location in time or space of the laboratory. This is a statement of the "strong equivalence principle."[9,10] The interpretation of Mach's principle being developed here is obviously incompatible with strong equivalence. The local equality of all gravitational accelerations (to the accuracy of present experiments) is the "weak equivalence principle." It should be noted that it is the "weak equivalence principle" that receives strong experimental support from the Eötvös experiment.

Before attempting to formulate a theory of gravitation which is more satisfactory from the standpoint of Mach's principle than general relativity, the physical ideas outlined above, and the assumptions being made, will be summarized:

1. An approach to Mach's principle which attempts, with boundary conditions, to allow only those mass distributions which produce the "correct" inertial reaction seems foredoomed, for there do exist large localized masses in the universe (e.g., white dwarf stars) and a laboratory could, in principle, be constructed near such a mass. Also it appears to be possible to modify the mass distribution. For example, a massive

concrete spherical shell could be constructed with the laboratory in its interior.

2. The contrary view is that locally observed inertial reactions depend upon the mass distribution of the universe about the point of observation and consequently the quantitative aspects of locally observed physical laws (as expressed in the physical "constants") are position dependent.

3. It is possible to reduce the variation of physical "constants" required by this interpretation of Mach's principle to that of a single parameter, the gravitational "constant."

4. The separate but related problem posed by the existence of very large dimensionless numbers representing quantitative aspects of physical laws is clarified by noting that these large numbers involve G and that they are of the same order of magnitude as the large numbers characterizing the size and mass distribution of the universe.

5. The "strong principle of equivalence" upon which general relativity rests is incompatible with these ideas. However, it is only the "weak principle" which is directly supported by the very precise experiments of Eötvös.

A THEORY OF GRAVITATION BASED ON A SCALAR FIELD IN A RIEMANNIAN GEOMETRY

The theory to be developed represents a generalization of general relativity. It is not a completely geometrical theory of gravitation, as gravitational effects are described by a scalar field in a Riemannian manifold. Thus, the gravitational effects are in part geometrical and in part due to a scalar interaction. There is a formal connection between this theory and that of Jordan,[16] but there are differences and the physical interpretation is quite different. For example, the aspect of mass creation[17] in Jordan's theory is absent from this theory.

In developing this theory we start with the "weak principle of equivalence." The great accuracy of the Eötvös experiment suggests that the motion of uncharged test particles in this theory should be, as in general relativity, a geodesic in the four-dimensional manifold.

With the assumption that only the gravitational "constant" (or active gravitational masses) vary with position, the laws of physics (exclusive of gravitation) observed in a freely falling laboratory should be unaffected by the rest of the universe as long as self-gravitational fields are negligible. The theory should be constructed in such a way as to exhibit this effect.

If the gravitational "constant" is to vary, it should be

[16] P. Jordan, *Schwerkraft and Weltall* (Friedrich Vieweg and Sohn, Braunschweig, 1955); Z. Physik **157**, 112 (1959). In this second reference, Jordan has taken cognizance of the objections of Fierz (see reference 19) and has written his variational principle in a form which differs in only two respects from that expressed in Eq. (16). See also reference 20.

[17] For a discussion of this, see H. Bondi, *Cosmology*, 2nd edition, 1960.

a function of some scalar field variable. The contracted metric tensor is a constant and devoid of interest. The scalar curvature and the other scalars formed from the curvature tensor are also devoid of interest as they contain gradients of the metric tensor components, and fall off more rapidly than r^{-1} from a mass source. Thus such scalars are determined primarily by nearby mass distributions rather than by distant matter.

As the scalars of general relativity are not suitable, a new scalar field is introduced. The primary function of this field is the determination of the local value of the gravitational constant.

In order to generalize general relativity, we start with the usual variational principle of general relativity from which the equations of motion of matter and nongravitational fields are obtained as well as the Einstein field equation, namely,[18]

$$0=\delta \int [R+(16\pi G/c^4)L](-g)^{\frac{1}{2}}d^4x. \qquad (5)$$

Here, R is the scalar curvature and L is the Lagrangian density of matter including all nongravitational fields.

In order to generalize Eq. (5) it is first divided by G, and a Lagrangian density of a scalar field ϕ is added inside the bracket. G is assumed to be a function of ϕ. Remembering the discussion in connection with Eq. (4), it would be reasonable to assume that G^{-1} varies as ϕ, for then a simple wave equation for ϕ with a scalar matter density as source would give an equation roughly the same as (4).

The required generalization of Eq. (6) is clearly

$$0=\delta \int [\phi R+(16\pi/c^4)L-\omega(\phi,_i\phi^{,i}/\phi)](-g)^{\frac{1}{2}}d^4x. \quad (6)$$

Here ϕ plays a role analogous to G^{-1} and will have the dimensions $ML^{-3}T^2$. The third term is the usual Lagrangian density of a scalar field, and the scalar in the denominator has been introduced to permit the constant ω to be dimensionless. In any sensible theory ω must be of the general order of magnitude of unity.

It should be noted that the term involving the Lagrangian density of matter in Eq. (6) is identical with that in Eq. (5). Thus the equations of motion of matter in a given externally determined metric field are the same as in general relativity. The difference between the two theories lies in the gravitational field equations which determine g_{ij}, rather than in the equations of motion in a given metric field.

It is evident, therefore, that, as in general relativity, the energy-momentum tensor of matter must have a vanishing covariant divergence,

$$T^{ij}_{;j}=0, \qquad (7)$$

[18] L. Landau and E. Liftschitz, *Classical Theory of Fields* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1951).

where

$$T^{ij}=[2/(-g)^{\frac{1}{2}}](\partial/\partial g_{ij})[(-g)^{\frac{1}{2}}L]. \qquad (8)$$

It is assumed that L does not depend explicitly upon derivatives of g_{ij}.

Jordan's theory has been criticized by Fierz[19] on the grounds that the introduction of matter into the theory required further assumptions concerning the standards of length and time. Further, the mass creation aspects of this theory and the nonconservation of the energy-momentum tensor raise serious questions about the significance of the energy-momentum tensor. To make it clear that this objection cannot be raised against this version of the theory, we hasten to point out that L is assumed to be the normal Lagrangian density of matter, a function of matter variables and of g_{ij} only, *not* a function of ϕ. It is a well-known result that for *any* reasonable metric field distribution g_{ij} (a distribution which need not be a solution of the field equations of g_{ij}), the matter equations of motion, obtained by varying matter variables in Eq. (6), are such that Eq. (7) is satisfied with T^{ij} defined by Eq. (8). Thus Eq. (7) is satisfied and this theory does not contain a mass creation principle.

The wave equation for ϕ is obtained in the usual way by varying ϕ and $\phi,_i$ in Eq. (6). This gives

$$2\omega\phi^{-1}\Box\phi-(\omega/\phi^2)\phi^{,i}\phi,_i+R=0. \qquad (9)$$

Here the generally covariant d'Alembertian $\Box$ is defined to be the covariant divergence of $\phi^{,i}$:

$$\Box\phi=\phi^{,i}_{;i}=(-g)^{-\frac{1}{2}}[(-g)^{\frac{1}{2}}\phi^{,i}],_i. \qquad (10)$$

From the form of Eq. (9), it is evident that ϕR and the Lagrangian density of ϕ serves as the source term for the generation of ϕ waves. Remarkably enough, as will be shown below, this equation can be transformed so as to make the source term appear as the contracted energy-momentum tensor of matter alone. Thus, in accordance with the requirements of Mach's principle, ϕ has as its sources the matter distribution in space.

By varying the components of the metric tensor and their first derivatives in Eq. (6), the field equations for the metric field are obtained. This is the analog of the Einstein field equation and is

$$R_{ij}-\tfrac{1}{2}g_{ij}R=(8\pi\phi^{-1}/c^4)T_{ij}$$
$$+(\omega/\phi^2)(\phi,_i\phi,_j-\tfrac{1}{2}g_{ij}\phi,_k\phi^{,k})$$
$$+\phi^{-1}(\phi,_{i;j}-g_{ij}\Box\phi). \qquad (11)$$

The left side of Eq. (11) is completely familiar and needs no comment. Note that the first term on the right is the usual source term of general relativity, but with the variable gravitational coupling parameter ϕ^{-1}. Note also that the second term is the energy-momentum tensor of the scalar field, also coupled with the gravitational coupling ϕ^{-1}. The third term is foreign and results from the presence of second derivatives of the metric

[19] M. Fierz, Helv. Phys. Acta. **29**, 128 (1956).

tensor in R in Eq. (6). These second derivatives are eliminated by integration by parts to give a divergence and the extra terms. It should be noted that when the first term dominates the right side of Eq. (11), the equation differs from Einstein's field equation by the presence of a variable gravitational constant only.

While the "extra" terms in Eq. (12) may at first seem strange, their role is essential. They are needed if Eq. (7) is to be consistent with Eqs. (9) and (11). This can be seen by multiplying Eq. (11) by ϕ and then taking the covariant divergence of the resulting equation. The divergence of these two terms cancels the term $\phi_{,i}R_j{}^i = \phi^{,i}R_{ji}$. To show this, use is made of the well-known property of the full curvature tensor that it serves as a commutator for two successive gradient operations applied to an arbitrary vector.

If Eq. (11) is contracted there results

$$-R = (8\pi\phi^{-1}/c^4)T - (\omega/\phi^2)\phi_{,k}\phi^{,k} - 3\phi^{-1}\Box\phi. \quad (12)$$

Equation (12) can be combined with Eq. (9) to give a new wave equation for ϕ[20]:

$$\Box\phi = [8\pi/(3+2\omega)c^4]T. \quad (13)$$

With the sign convention

$$ds^2 = g_{ij}dx^i dx^j \quad \text{and} \quad g_{00} < 0,$$

for a fluid

$$T_{ij} = -(p+\epsilon)u_i u_j + p g_{ij}, \quad (14)$$

so that

$$T = -\epsilon + 3p, \quad (15)$$

where ϵ is the energy density of the matter in comoving coordinates and p is the pressure in the fluid. With this sign convention and ω positive, the contribution to ϕ from a local mass is positive. Note, however, that there is no direct electromagnetic contribution to T, as the contracted energy-momentum tensor of an electromagnetic field is identically zero. However, bound electromagnetic energy does contribute indirectly through the stress terms in other fields, the stresses being necessary to confine the electromagnetic field.[21] In conclusion, ω must be positive if the contribution to the inertial reaction from nearby matter is to be positive.

THE WEAK FIELD APPROXIMATION

An approximate solution to Eqs. (11) and (13) which is of first order in matter mass densities is now obtained. This weak-field solution plays the same important role that the corresponding solution fills in general relativity.

As in general relativity the metric tensor is written as

$$g_{ij} = \eta_{ij} + h_{ij}, \quad (16)$$

where η_{ij} is the Minkowskian metric tensor

$$\eta_{00} = -1, \quad \eta_{\alpha\alpha} = 1, \quad \alpha = 1, 2, 3. \quad (17)$$

h_{ij} is computed to the linear first approximation only. In similar fashion let $\phi = \phi_0 + \xi$, where ϕ_0 is a constant and is to be computed to first order in mass densities.

The weak-field solution to Eq. (13) is computed first. In this equation g_{ij} may be replaced by η_{ij}:

$$\Box\phi = \Box\xi = \frac{1}{(-g)^{\frac{1}{2}}}[(-g)^{\frac{1}{2}}g^{ij}\xi_{,i}]_{,j}$$

$$= \nabla^2\xi - \frac{\partial^2\xi}{\partial t^2} = \frac{8\pi T}{(3+2\omega)c^4}. \quad (18)$$

It is evident that a retarded-time solution to Eq. (18) can be written as

$$\xi = -[2/(3+2\omega)]\int T d^3x/rc^4, \quad (19)$$

where T is to be evaluated at the retarded time.

The weak-field solution to Eq. (11) is obtained in a manner similar to that of general relativity by introducing a coordinate condition that simplifies the equation. As a preliminary step let

$$\gamma_{ij} = h_{ij} - \tfrac{1}{2}\eta_{ij}h,$$
$$\sigma_i = \gamma_{ij,k}\eta^{jk}. \quad (20)$$

Equation (11) can be written to first order in h_{ij} and ξ as

$$-\tfrac{1}{2}\{\Box\gamma_{ij} - \sigma_{i,j} - \sigma_{j,i} + \eta_{ij}\sigma_{k,l}\eta^{kl}\}$$
$$= [\xi_{,i,j} - \eta_{ij}\Box\xi]\phi_0^{-1} + \frac{8\pi}{c^4}\phi_0^{-1}T_{ij}. \quad (21)$$

Equation (21) can now be simplified by introducing the four coordinate conditions

$$\sigma_i = \xi_{,i}\phi_0^{-1}, \quad (22)$$

and the notation

$$\alpha_{ij} = \gamma_{ij} - \eta_{ij}\xi\phi_0^{-1}. \quad (23)$$

Equation (21) then becomes

$$\Box\alpha_{ij} = -(16\pi/c^4)\phi_0^{-1}T_{ij}, \quad (24)$$

with the retarded-time solution

$$\alpha_{ij} = (4\phi_0^{-1}/c^4)\int (T_{ij}/r)d^3x. \quad (25)$$

From Eqs. (20) and (23),

$$h_{ij} = \alpha_{ij} - \tfrac{1}{2}\eta_{ij}\alpha - \eta_{ij}\xi\phi_0^{-1}. \quad (26)$$

Thus

$$h_{ij} = \frac{4\phi_0^{-1}}{c^4}\int \frac{T_{ij}}{r}d^3x - \frac{4\phi_0^{-1}}{c^4}\left(\frac{1+\omega}{3+2\omega}\right)\eta_{ij}\int \frac{T}{r}d^3x. \quad (27)$$

---

[20] There are but two formal differences between the field equations of this theory and those of the particular form of Jordan's theory given in Z. Physik **157**, 112 (1959). First, Jordan has defined his scalar field variable reciprocal to ϕ. Thus, the simple wave character of the scalar field equation [Eq. (13)] is not so clear and the physical arguments based on Mach's principle and leading to Eq. (4) have not been satisfied. Second, as a result of its outgrowth from his five-dimensional theory, Jordan has limited his matter variables to those of the electromagnetic field.

[21] C. Misner and P. Putnam, Phys. Rev. **116**, 1045 (1959).

For a stationary mass point of mass M these equations become

$$\phi = \phi_0 + \xi = \phi_0 + 2M/(3+2\omega)c^2 r, \tag{28}$$

$$g_{00} = \eta_{00} + h_{00} = -1 + (2M\phi_0^{-1}/rc^2)[1+1/(3+2\omega)],$$

$$g_{\alpha\alpha} = 1 + (2M\phi_0^{-1}/rc^2)[1-1/(3+2\omega)], \quad \alpha = 1, 2, 3, \tag{29}$$

$$g_{ij} = 0, \quad i \neq j.$$

The above weak-field solution is sufficiently accurate to discuss the gravitational red shift and the deflection of light. However, to discuss the rotation of the perihelion of Mercury's orbit requires a solution good to the second approximation for g_{00}.

The gravitational red shift is determined by g_{00} which also determines the gravitational weight of a body. Thus, there is no anomaly in the red shift. The strange factor $(4+2\omega)/(3+2\omega)$ in g_{00} is simply absorbed into the definition of the gravitational constant

$$G_0 = \phi_0^{-1}(4+2\omega)(3+2\omega)^{-1}. \tag{29a}$$

On the other hand, there is an anomaly in the deflection of light. This is determined, not by g_{00} alone, but by the ratio $g_{\alpha\alpha}/g_{00}$. It is easily shown that the light deflection computed from general relativity differs from the value in this theory by the above factor. Thus, the light deflection computed from this theory is

$$\delta\theta = (4G_0M/Rc^2)[(3+2\omega)/(4+2\omega)], \tag{30}$$

where R is the closest approach distance of the light ray to the sun of mass M. It differs from the general relativity value by the factor in brackets. The accuracy of the light deflection observations is too poor to set any useful limit to the size of ω.

On the contrary, there is fair accuracy in the observation of the perihelion rotation of the orbit of Mercury and this does serve to set a limit to the size of ω. In order to discuss the perihelion rotation, an exact solution for a static mass point will be written.

STATIC SPHERICALLY SYMMETRIC FIELD ABOUT A POINT MASS[22]

Expressing the line element in isotropic form gives

$$ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)], \tag{31}$$

where α and β are functions of r only. For $\omega > \frac{3}{2}$ the general vacuum solution can be written in the form

$$e^{2\alpha} = e^{2\alpha_0}[(1-B/r)/(1+B/r)]^{2/\lambda},$$

$$e^{2\beta} = e^{2\beta_0}(1+B/r)^4[(1-B/r)/(1+B/r)]^{2[(\lambda-C-1)/\lambda]}, \tag{32}$$

$$\phi = \phi_0[(1-B/r)/(1+B/r)]^{-C/\lambda},$$

where

$$\lambda = [(C+1)^2 - C(1-\tfrac{1}{2}\omega C)]^{\frac{1}{2}}, \tag{33}$$

and α_0, β_0, ϕ_0, B, and C are arbitrary constants. It may

[22] This form of solution was suggested to one of us (C. B.) by C. Misner.

be seen by substitution of Eqs. (31) and (32) into Eqs. (13) and (11) that this is the static solution for spherical symmetry when $T_{ij} = 0$.

To discuss the perihelion rotation of a planet about the sun requires a specification of the arbitrary constants in Eq. (32) in such a way that this solution agrees in the weak-field limit [first order in $M/(c^2 r\phi_0)$] with the previously obtained solution, Eqs. (28) and (29). It may be easily verified that the appropriate choice of constants is

ϕ_0 given by Eq. (29a);

$$\alpha_0 = \beta_0 = 0,$$

$$C \cong -1/(2+\omega),$$

$$B \cong (M/2c^2\phi_0)[(2\omega+4)/(2\omega+3)]^{\frac{1}{2}}, \tag{34}$$

with λ given by Eq. (33).

Remembering the previous discussion of Mach's principle, it is clear that the asymptotic Minkowskian character of this solution makes sense only if there is matter at great distance. Second, the matching of the solution to the weak-field solution is permissible only if the sun is a suitable mass distribution for the weak-field approximation. Namely, the field generated by the sun must be everywhere small, including the interior of the sun. With this assumption, the solution, Eqs. (31), (32), (33), and (34), is valid for the sun. It does not, however, justify its use for a point mass.

The question might be raised as to whether a matching of solutions, accurate to first order only in $M/(\phi_0 c^2 r)$, has a validity to the second order. It should be noted, however, that this matching condition is sufficient to assign sufficiently accurate values to all the adjustable parameters in Eqs. (32) except λB, and that we do not demand that λB be determined in terms of an integration over the matter distribution of the sun; it is determined from the observed periods of the planetary motion.

With the above solution, it is a simple matter to calculate the perihelion rotation. The labor is reduced if $e^{2\alpha}$ is carreid only to second order in $M/(c^2 r\phi_0)$, and $e^{2\beta}$ to first order. The result of this calculation is that the relativistic perihelion rotation rate of a planetary orbit is

$$[(4+3\omega)/(6+3\omega)] \times (\text{value of general relativity}). \tag{35}$$

This is a useful result as it sets a limit on permissible values of the constant ω. If it be assumed that the observed relativistic perihelion rotation agrees with an accuracy of 8% or less with the computed result of general relativity, it is necessary for ω in Eq. (35) to satisfy the inequality

$$\omega \geq 6. \tag{36}$$

The observed relativistic perihelion rotation of Mercury (after subtracting off planetary perturbations and other effects presumed known) is $42.6'' \pm 0.9''$/century.[23] For

[23] G. M. Clemence, Revs. Modern Phys. **19**, 361 (1947).

$\omega = 6$, the computed relativistic perihelion rotation rate is $39.4''$. The difference of $3.2''$ of arc per century is 3.3 times the formal probable error. It should also be remarked that Clemence[24] has shown that if some recent data on the general precession constant and the masses of Venus and the Earth-Moon system are adopted, the result is an increase in the discrepancy to $3.7''$ while decreasing the formal probable error by a factor of 2.

The formal probable error is thus substantially less than $3.2''$ arc, but it may be reasonable to allow this much to take account of systematic errors in observations and future modification of observations, adopted masses, and orbit parameters. Apparently there are many examples in celestial mechanics of quantities changing by substantially more than the formal probable errors. Thus, for example, the following is a list of values which have been assigned to the reciprocal of Saturn's mass (in units of the sun's reciprocal mass) by authors at various times:

$M^{-1} = 3501.6 \pm 0.8$, Bessel (1833) from the motion of Saturn's moon Titan;

$= 3494.8 \pm 0.3$, Jeffrey (1954) and G. Struve (1924–37) (Titan);

$= 3502.2 \pm 0.53$, Hill (1895) Saturn's perturbations of Jupiter;

$= 3497.64 \pm 0.27$, Hertz (1953) Saturn's perturbations of Jupiter;

$= 3499.7 \pm 0.4$, Clemence (1960) Saturn's perturbations of Jupiter.

While this example may be atypical, it does suggest that considerable caution be used in judging errors in celestial mechanics.

MACH'S PRINCIPLE

A complete analysis of Mach's principle in relation to the present scalar theory will not be attempted here. However, because of the motivation of this theory by Mach's principle, it is desirable to give a brief discussion. Having formulated the desired field equations, it remains to establish initial-value and boundary conditions to bring the theory in accord with Mach's principle. This will not be attempted in a general way, but in connection with special problems only.

The qualitative discussion in the Introduction suggested that for a static mass shell of radius R and mass M, the gravitational constant in its interior should satisfy the relation

$$GM/Rc^2 \sim 1. \tag{37}$$

Equivalently

$$\phi \sim M/Rc^2. \tag{38}$$

It may be noted that in a flat space, with the bound-

---

ary condition that $\phi = 0$ at infinity, Eq. (13) has as a solution for the interior $r < R$

$$\phi = 2M/(3+2\omega)Rc^2. \tag{39}$$

This is a hopeful sign and bodes well for Mach's principle within the framework of this theory. One should not be misled by this simple result, however. There are several factors which invalidate Eq. (39) as a quantitative result. First, space is not flat, but is warped by the presence of the mass shell. Second, the asymptotic zero boundary condition may be impossible for the exact static solution to the field equation. Third, it may be impossible to construct such a static massive shell in a universe empty except for the shell, without giving matter nonphysical properties. This third point is not meant to imply a practical limitation of real materials, but rather a fundamental limitation on the stress-energy tensor of matter. In this connection it should be noted that if Eq. (37) is to be satisfied, independent of the size and mass of the shell, the gravitationally induced stresses in the shell are enormous, of the order of magnitude of the energy density of the spherical shell. It is not possible to reduce the stress by decreasing M or increasing R, as the resulting change in the gravitational constant compensates for the change. We ignore here the above third point and assume for the moment that such a shell can be constructed in principle.

To turn now to the massive static shell, consider first the solution to the field equations in the exterior region, $r > R$. This solution is encompassed in the general solution Eqs. (32) and (33). Note that the boundary condition

$$\phi \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty \tag{40}$$

is not possible.

On the other hand, it is possible to change the sign in the brackets in Eq. (32) and absorb the complex factor into the constant before the bracket. These equations may then be assumed to hold for $r < B$ rather than for $r > B$ as in Eq. (32). The equations now have the form

$$e^{2\alpha} = e^{2\alpha_0}[(B/r-1)/(B/r+1)]^{2/\lambda},$$

$$e^{2\beta} = e^{2\beta_0}[1+B/r]^4[(B/r-1)/(B/r+1)]^{2[(\lambda-C-1)/\lambda]}, \tag{41}$$

$$\phi = \phi_0[(B/r-1)/(B/r+1)]^{C/\lambda}.$$

It may be noted that this solution, interesting for $r > R$ and $\lambda > 0$ only, results in space closure at the radius $r = B$ provided

$$(\lambda-C-1)/\lambda > 0. \tag{42}$$

In similar fashion at the closure radius, $\phi \rightarrow 0$, provided $C > 0$.

Equations (36) and (33) require that

$$C > 2/\omega. \tag{43}$$

That this boundary condition is appropriate to Mach's principle can be seen by an application of

Green's theorem. Introduce a Green's function η satisfying

$$\Box\eta = (-g)^{-\frac{1}{2}}[(-g)^{\frac{1}{2}}g^{ij}\eta_{,j}]_{,i} = (-g)^{-\frac{1}{2}}\delta^4(x-x_0), \quad (44)$$

also

$$\Box\phi = [8\pi/(3+2\omega)c^4]T. \quad (45)$$

Combining Eqs. (44) and (45) after the appropriate multiplications gives

$$[(-g)^{\frac{1}{2}}g^{ij}(\eta\phi_{,i}-\phi\eta_{,i})]_{,j}$$
$$= (-g)^{\frac{1}{2}}[8\pi/(3+2\omega)c^4]T\eta - \phi\delta^4(x-x_0). \quad (46)$$

It is assumed that η is an "advanced-wave" solution to Eq. (44), i.e., $\eta=0$ for all time future to t_0. The condition given by Eq. (42) implies a finite coordinate time for light to propagate from the radius B to R, the radius of the shell, hence to any interior point x_0.

Integrate Eq. (46) over the interior of the closed space $(r<B)$ between the time $t_2>t_0$ and the space like surface S_1 so chosen that the η wave starts out at the radius $r=B$ at times lying on this surface and that the normal to the surface at $r=B$ has no component in the r direction. The integral of the left side of Eq. (46), after conversion to a surface integral, vanishes, for η and η_0 both vanish on t_2, and both ϕ and $\phi_{,i}$ vanish on S_1 at $r=B$, with $i\neq 1$.

The integral over the right side of Eq. (46) yields

$$\phi(x_0) = [8\pi/(3+2\omega)c^4]\int \eta T(-g)^{\frac{1}{2}}d^4x, \quad (47)$$

or

$$\phi(x_0) \sim M/Rc^2. \quad (47a)$$

Note that this equation states that ϕ at the point x_0 is determined by an integral over the mass distribution, with each mass element contributing a wavelet which propagates to the point x_0. This is just the interpretation of Mach's principle desired.

COSMOLOGY

A physically more interesting problem to discuss from the standpoint of Mach's principle is the cosmological model derived from this theory. It will be recalled that the assumption of a uniform and isotropic space is supported to some extent by the observations of galaxy distribution. The kinematics of the comoving coordinate system is completely free of dynamical consideration. In spherical coordinates, a form of the line element is

$$ds^2 = -dt^2 + a^2(t)[dr^2/(1-\lambda r^2)+r^2(d\theta^2+\sin^2\theta d\phi^2)], \quad (48)$$

with $\lambda=+1$ for a closed space, $\lambda=-1$ for open, and $\lambda=0$ for a flat space, and where $r<1$ for the closed space. The Hubble age associated with the rate of expansion of the univese and the galactic red shift is $a/\dot{a}=a/(da/dt)$.

The substitution

$$r=\sin\chi \quad \text{(closed space, } \lambda=+1)$$

or

$$r=\sinh\chi \quad \text{(open space, } \lambda=-1) \quad (49)$$

simplifies the expression for line element somewhat:

$$ds^2 = -dt^2 + a^2[d\chi^2+\sin^2\chi(d\theta+\sin^2\theta d\phi^2)]$$
$$\text{(closed space).} \quad (50)$$

The most interesting case physically seems to be the closed universe.

Using Eq. (50) for interval and writing the (0,0) component of Eq. (11),

$$R_0^0 - \tfrac{1}{2}R = -(3/a^2)(\dot{a}^2\pm 1)$$

$$(+, \text{ space closed}; -, \text{ space open})$$

$$= \frac{8\pi\phi^{-1}}{c^4}T_0^0 - \frac{\omega}{2\phi^2}\dot{\phi}^2 + 3\frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi}. \quad (51)$$

Assuming negligible pressure in the universe we have $-T=-T_0^0=+\rho c^2$, where the mass density is ρ (observationally ρ seems to satisfy, $\rho>10^{-31}$ g/cm^3).

Again assuming negligible pressure, the energy density times a measure of the volume of the universe is constant. Hence

$$\rho a^3 = \rho_0 a_0^3 = \text{const.} \quad (52)$$

Substituting these results in Eq. (51) yields

$$\left(\frac{\dot{a}}{a}+\frac{1}{2}\frac{\dot{\phi}}{\phi}\right)^2 + \frac{\lambda}{a^2} = \tfrac{1}{4}(1+\tfrac{2}{3}\omega)\left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{8\pi}{3\phi}\rho_0\left(\frac{a}{a}\right)^3. \quad (53)$$

Here ρ_0 and a_0 refer to values at some arbitrary fixed time t_0. In similar fashion Eq. (13) becomes

$$(d/dt)(\dot{\phi}a^3) = [8\pi/(3+2\omega)]\rho_0 a_0^3. \quad (54)$$

After integration, Eq. (54) becomes

$$\dot{\phi}a^3 = [8\pi/(3+2\omega)]\rho_0 a_0^3(t-t_c). \quad (55)$$

The constant of integration, t_c in Eq. (55), can be evaluated by considerations of Mach's principle.

As before, we introduce Mach's principle into this problem by expressing $\phi(t)$ as an advanced-wave integral over all matter. Equations (46) and (47) require some assumption about the history of matter in the universe. We assume that the universe expands from a highly condensed state. It is possible that in the intense gravitational field of this condensed state, matter is created. For a closed universe, matter from a previous cycle may be regenerated in this high-temperature state. In view of our present state of ignorance, there seems to be little point in speculating about the processes involved. In any case the creation process lies outside the present theory.

We assume, therefore, an initial state at the beginning of the expansion ($t=0$) with $a\cong 0$ and matter already present. Although pressure would certainly be important in such a highly condensed state, with expansion the pressure would rapidly fall and no great harm is done to the model if pressure effects are neglected. In fact, an integration of the initial high-pressure phase for a
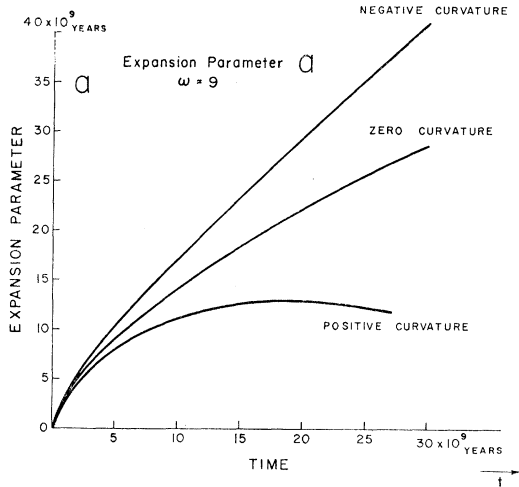
FIG. 1. The expansion parameter a as a function of t for the three cases, closed, open, and flat space with $\omega=9$.

particular cosmological model shows explicitly that it may be neglected to good approximation.

It is assumed that the inertial reaction, and hence ϕ, at time t_0 is determined uniquely by the matter distribution from $t=0$ to $t=t_0$. Hence, if Eq. (46) is integrated over all 3-space from $t=0$ to $t_1>t_0$, the surface integral obtained from the left-hand side should vanish. Initial conditions for Eqs. (53) and (55) in the form of values of a and ϕ at $t=0$ and a value of the constant t_c must be so chosen that the surface integral from Eq. (46) vanishes. In order for this surface integral to be meaningful at $t=0$, the a must be at least infinitesimally positive on the surface, otherwise the metric tensor is singular. If $t_c=0$ and $\phi=0$ on this surface, the surface integral vanishes. This follows because the vanishing factors ϕ and $a^3\phi_{,0}$ [see Eq. (55)] occur in the integral. It is concluded, therefore, that the appropriate initial conditions are $a=\phi=0$ with $t_c=0$. It should be noted
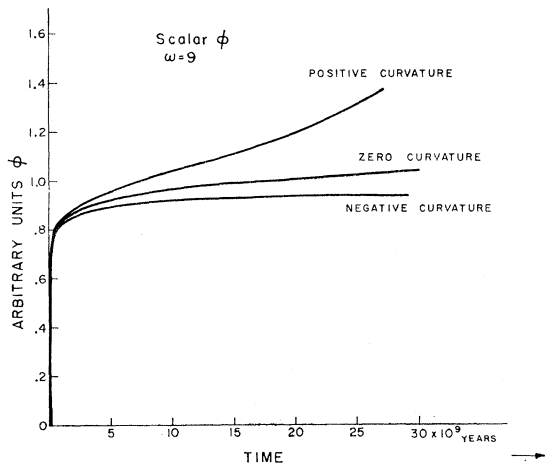

FIG. 2. The scalar ϕ, in arbitrary units, as a function of t for the three cases, closed, open, and flat space, with $\omega=9$.

that the other surface integral, over the surface $t=t_1$, vanishes since η and all its gradients are zero on this surface (advanced-wave solution).

Letting $t_c=0$ in Eq. (55) and combining with Eq. (53) gives

$$[(\dot{a}/a)+\tfrac{1}{2}(\dot{\phi}/\phi)]^2+\lambda a^{-2}$$
$$=\tfrac{1}{4}(1+\tfrac{2}{3}\omega)(\dot{\phi}/\phi)^2+(1+\tfrac{2}{3}\omega)(\dot{\phi}/\phi)(1/t), \quad (57)$$

$$\dot{\phi}a^3=[8\pi/(3+2\omega)]\rho_0 a_0{}^3 t. \quad (58)$$

It can be seen that for sufficiently small time the term $1/a^2$ in Eq. (57) is negligible and the solution differs only infinitesimally from the flat-space case. The resulting equations can be integrated exactly with the initial conditions

$$\phi=a=0; \quad t=0. \quad (59)$$

As both Eqs. (57) and (58) are now (in this approximation) homogeneous in (a,a_0), the solution is determined within a scale factor in a only.

This solution, good for the early expansion phases (i.e., $a\gg t$), is

$$\phi=\phi_0(t/t_0)^r,$$
$$a=a_0(t/t_0)^q, \quad (60)$$

with

$$r=2/(4+3\omega), \quad (61)$$

$$q=(2+2\omega)/(4+3\omega), \quad (62)$$

and

$$\phi_0=8\pi[(4+3\omega)/2(3+2\omega)]\rho_0 t_0{}^2. \quad (63)$$

For the flat-space case, the solution is exact for all $t>0$.

It should be noted that Eq. (63) is compatible with Eq. (1), for in Eq. (1) M is of the order of magnitude of $\rho_0 c^3 t_0{}^3$ and R is approximately ct_0. Thus, the initial conditions are compatible with Mach's principle as it has been formulated here.

For a nonflat space, the only feasible method of integrating Eqs. (57) and (58) beyond the range of validity of the above solution is numerical integration. An example of an integration is plotted in Figs. 1 and 2, where a and ϕ are plotted as a function of time for the three cases of positive, zero, and negative curvature with $\omega=9$.

It should be noted that for $\omega\geq6$, and the flat-space solution, the time dependence of a differs only slightly from the corresponding case in general relativity (Einstein-deSitter) where $a \sim t^{\frac{2}{3}}$. Consequently, it would be difficult to distinguish between the two theories on the basis of space geometry only. In similar fashion the mass density required for a particular Hubble age $a/\dot{a}$ (flat space) is the same as for general relativity if $\omega\gg1$. For $\omega=6$ there is only a 2% difference between the two theories.

On the other hand, stellar evolutionary rates are a sensitive function of the gravitational constant, and this makes an observational test of the theory possible.

This matter is discussed in a companion article by one of us (R. H. D.).[25]

At the beginning of this article a problem was posed, to understand within the framework of Mach's principle the laws of physics seen within a laboratory set rotating within a universe otherwise almost empty. We are now in a position to begin to understand this problem. Consider a laboratory, idealized to a spherical mass shell with a mass m and radius r, and stationary in the comoving coordinate system given by Eqs. (50) with Eqs. (60), (61), (62), and (63) satisfied. Imagine now that the laboratory is set rotating about an axis with an angular velocity α_0. This rotation affects the metric tensor inside the spherical shell in such a way as to cause the gyroscope to precess with an angular velocity[5] [also see Eq. (27)]

$$\alpha = (8m/3rc^2\phi_0)\alpha_0, \qquad (64)$$

where ϕ_0 is given by Eq. (63). Equation (64) is valid in the weak-field approximation only for which $m/(rc^2\phi_0) \ll 1$. Substituting Eq. (63) in Eq. (64) gives

$$\alpha = [2(3+2\omega)/3\pi(4+3\omega)] \cdot (m/rc^2\rho_0 t_0^2)\alpha_0. \qquad (65)$$

It may be noted that if the matter density ρ_0 of the universe is decreased, with t_0 const, α increases. Thus, as the universe is emptied, the Thirring-Lense precession of the gyroscope approaches more closely the rotation velocity α_0 of the laboratory. Unfortunately, the weak-field approximation does not permit a study of the limiting process $\rho_0 \to 0$.

In another publication by one of us (C. B.) other aspects of the theory, including conservation laws, will be discussed.

ACKNOWLEDGMENTS

[25] R. H. Dicke, Revs. Modern Phys. (to be published).

APPENDIX

In general relativity the equation of motion of a point particle, without spin, moving in a gravitational field only, may be obtained from the variational principle

$$0 = \delta \int m(g_{ij}u^i u^j)^{\frac{1}{2}} ds, \qquad (66)$$

or

$$(d/ds)(mu_i) - \tfrac{1}{2}mg_{jk,i}u^j u^k = 0. \qquad (67)$$

If the mass in Eq. (66) is assumed to be a function of position,

$$m = m_0 f(x), \qquad (68)$$

an added force term appears and

$$(d/ds)(mu_i) - \tfrac{1}{2}mg_{jk,i}u^j u^k - m_{,i} = 0. \qquad (69)$$

Both equations are consistent with the constraint condition $u^i u_i = 1$. It should be noted that because of the added force term in Eq. (69), the particle does not move on a geodesic of the geometry.

If now the geometry is redefined in such a way that the new metric tensor is (conformal transformation)

$$\bar{g}_{ij} = f^2 g_{ij}, \qquad (70)$$

and

$$d\bar{s}^2 = f^2 ds^2, \quad \bar{u}^i = f^{-1}u^i.$$

Equation (69) may be written as

$$(d/d\bar{s})(m_0\bar{u}_i) - \tfrac{1}{2}m_0\bar{g}_{jk,i}\bar{u}^i\bar{u}^k = 0. \qquad (71)$$

The particle moves on a geodesic of the new geometry. With the new units of length, time, and mass appropriate for this new geometry, the mass of the particle is m_0, a constant.