A Conformally Invariant Lagrangian for Schrödinger Geometry

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October 8, 2016

Abstract

In 1950 the Austrian physicist Erwin Schrödinger suggested a non-Riemannian geometry that he considered to be the most general among all possible non-symmetrical metric affinities. In doing so, he also came upon a class of purely *symmetric* connections that necessarily violated the notion of *metricity*; that is, the non-vanishing of the covariant derivative of the metric tensor. While Schrödinger did not pursue this geometry to any significant extent, we show in the following that it has a number of theoretical advantages over a similar geometry put forward years earlier by the German mathematical physicist Hermann Weyl in his effort to develop a purely geometrical basis for electromagnetism from an extension of general relativity. In particular, Schrödinger's geometry provides a means of avoiding certain objections Einstein had voiced against Weyl's theory (objections that effectively killed the theory) and, unlike Weyl's effort, Schrödinger's geometry leads to a unique, conformally invariant action Lagrangian that may have application to the problems of dark matter and dark energy.

1. Introduction

Currently there are no generally accepted theories of gravity that are scale (or conformally) invariant. All attempts to date have been dependent on the Ricci scalar R in linear combination with various *ad hoc* scalar, tensor and spinor quantities designed to avoid action Lagrangians that are higher than second order in the metric tensor and its derivatives. Today the only purely scale-invariant (and Riemannian) Lagrangian remains that based on the *conformal tensor*, a fourth-order quantity first derived by the German mathematical physicist Hermann Weyl in the early 1920s.

Several years earlier (in 1918), Weyl proposed a theory of gravitation based on the fourth-order quantity R^2, which he concurrently used in an attempt to develop a unified theory of electromagnetism and Einstein's theory of general relativity. Weyl's proposal was based on the assumed invariance of physics with respect to a conformal (or scale) transformation of the metric tensor $g_{\mu\nu} \rightarrow e^{\pi(x,t)} g_{\mu\nu}$, where $\pi(x,t)$ is an arbitrary scalar function of space and time. A decade later Weyl's ideas were recast as *gauge invariance*, a symmetry that subsequently became a cornerstone of modern quantum theory. More recently, the notion of conformal symmetry has been explored in numerous cosmological models, and there is increasing speculation that a conformally invariant theory may indeed underlie all of physics.

Weyl's theory, which introduced a non-Riemannian geometry in an effort to embed electromagnetism into general relativity as a purely geometrical construct, necessarily relied upon a Lagrangian that was invariant with respect to the local rescaling of the metric tensor $g_{\mu\nu}$. Weyl believed that the scalar parameter π might be related to the gauge transformation property of electromagnetism $(A_\mu \rightarrow A_\mu + \partial_\mu \pi)$, and thus provide an opportunity for deriving Maxwell's equations from geometry. The theory failed, but it has since spurred a considerable amount of interest in gravitational theories based on conformal invariance. That interest has continued to this day, with many researchers contributing to the topic, now properly called *Weyl conformal gravity*.

Meanwhile, a similar non-Riemannian geometry was almost accidentally suggested by Erwin Schrödinger in the late 1940s that in some ways paralleled Weyl's effort. While Schrödinger did not pursue this geometry to the extent that Weyl did, his approach nevertheless represents an arguably superior formalism, as it appeared to surmount a key objection Einstein had expressed regarding Weyl's earlier work. Although Schrödinger spent much of his time investigating geometries that were non-symmetric in the metric tensor and the connection term, his geometry is simpler and arguably more elegant than Weyl's.

In his 1950 book *Space-Time-Structure*, Schrödinger outlined an effort to develop a metric affinity (or connection) $\Gamma^\lambda_{\mu\nu}$ that was as general as possible. While the form he arrived at was non-symmetric, he was ultimately led to a purely symmetric connection that incorporated a non-Riemannian generality that was similar to but notably distinct from the one Weyl had proposed over thirty years earlier. While Schrödinger did not examine this new connection in any detail, we shall refer to his work here as *Schrödinger geometry* and present some of its consequences, particulary its application to conformal gravity. Unlike Weyl's theory, however, we will assume no connection between Schrödinger's work and electromagnetism, although — as in Weyl's geometry — there are some notable similarities.

2. Motivation for a Non-Riemannian Geometry

The predictions of Einstein's theory of general relativity, which is rooted in Riemannian geometry, have never failed. So why seek an alternative geometry?

The quest for a grand unification of all of Nature's laws has been an inspiration for all physicists since James Clerk Maxwell first showed that the forces associated with electricity and magnetism, once considered separate phenomena, were in fact the same thing. When Einstein announced his gravity theory in 1915, many physicists (Einstein included) naturally believed that the theory could be expanded to include Maxwell's laws, the two forces — gravity and electromagnetism — being the only forces of Nature known at the time. This belief was also motivated by the fact that the classical inverse-square expressions for gravity and electromagnetism were essentially identical, leading many to believe that some tweaking or approximation of Einstein's theory would bring Maxwell's equations to the fore. In addition, it was widely suspected at the time that gravity propagated at the speed of light, like electromagnetism. Furthermore, the sheer mathematical beauty of Einstein's theory appeared to provide sufficient motivation to seek a unification of the two forces.

It was quickly realized, however, that Einstein's theory was not wide enough to include electromagnetism as a geometrical phenomenon, and so physicists began efforts to generalize the theory in the hope that electromagnetism would spring from the theory. A number of these efforts — particularly those of Hermann Weyl, A. S. Eddington, Theodor Kaluza, Oskar Klein and Einstein himself — were ingenious, bolstering hopes that unification was close at hand. However, these efforts all failed, and when physicists subsequently realized that gravity and electromagnetism were not the only forces in Nature's arsenal, attention turned away from geometry to the then-emerging quantum theory.

Today, it is still not known what role, if any, spacetime geometry plays in electromagnetism. But surely Riemannian geometry is not the final word on our classical description of the world, and it may still be possible that past efforts to generalize that geometry have overlooked some opportunities that will lead to future enlightenment.

However, it is not the purpose of the following discussion to attempt any unification of gravity with electromagnetism, but to point out two aspects of Riemannian geometry that continue to call out for attention: the apparently unnecessary invariance of vector length or magnitude under parallel transport, and the need to develop a working theory of gravity that is conformally invariant. In a very real sense, the two problems result from the same limitation of Riemannian geometry, that of *metricity*.

3. Mathematical Preliminaries

We will be using old-style tensor notation throughout. It is assumed the reader is familiar with the del (δ) variational notation, which will be briefly reviewed in what follows.

3.1. Notation

Following Adler et al., ordinary partial and covariant differentiation of scalars, vectors and tensors will be denoted with a single subscripted bar and double subscripted bar, respectively, as in

$$A^{\alpha}_{\ \mu\nu||\lambda} = A^{\alpha}_{\ \mu\nu|\lambda} + A^{\beta}_{\ \mu\nu}\Gamma^{\alpha}_{\beta\lambda} - A^{\alpha}_{\ \beta\nu}\Gamma^{\beta}_{\lambda\mu} - A^{\alpha}_{\ \mu\beta}\Gamma^{\beta}_{\lambda\nu}$$

with the signs of the various terms following basic convention; unless denoted otherwise, the *connections* $\Gamma^{\alpha}_{\mu\nu}$ are symmetric with respect to their lower indices.

3.2. Gauge Variations

Gauge theory in physics is based on some defined change in the mathematical description of a system that doesn't result in any observable overall description of the system. For example, in quantum mechanics the wave function of some system can undergo a change of phase $\Psi \to e^{i\pi}\Psi$ (where the phase parameter π is a local function of space and time) without changing any of the system's underlying physics. For whatever reason, *gauge invariance* in physics seems to be a favorite characteristic of Nature, and today it underlies all of modern physics.

The demand of gauge invariance is just one step in the development of an *action Lagrangian* that can be used to derive equations of motion. In that sense, gauge invariance, like all the other traditional notions of continuous and discrete physical invariances (momentum, angular momentum, charge, etc.), is simply a condition that needs to be imposed beforehand. In the end, one takes the variation of a suitable gauge-invariant Lagrangian L with respect to the metric tensor, or

$$\frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{-g} L \, d^4x = 0,$$

which presumably provides the equations of motion.

The traditional free-space Einstein-Hilbert action,

$$\int \sqrt{-g} R \, d^4x$$

has been amazingly successful in predicting all manner of gravitational phenomena. However, it is not conformally invariant, and it can be shown mathematically that any such Lagrangian that is of only second order in the metric tensor and its derivatives (like the Ricci scalar R) cannot be made conformally invariant.

The gauge variations we will be dealing with in the following all involve an infinitesimal (real) change in the metric tensor $g_{\mu\nu}$ which determines, among other things, the local or global length or magnitude of a vector. Such a change is commonly referred to as a *conformal variation*, since the angle between two rescaled vectors does not change. We will also encounter other quantities that may undergo gauge variations, such as the well-known gauge transformation of the electromagnetic field.

We then have

$$\begin{aligned} g'_{\mu\nu} &= e^{\epsilon\pi(x)} g_{\mu\nu} \\ &\approx (1 + \epsilon\pi) g_{\mu\nu} \end{aligned}$$

or

$$\begin{aligned} \delta g_{\mu\nu} &= g'_{\mu\nu} - g_{\mu\nu}, \\ \delta g_{\mu\nu} &= \epsilon\pi g_{\mu\nu} \end{aligned}$$

Similarly, we have

$$\delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$$

The del operator δ will be used extensively in determining how the Riemann-Christoffel curvature tensor $R^{\lambda}_{\mu\nu\alpha}$ changes under a conformal transformation. In most cases we will use *Palatini's identity*, which specifies how the curvature tensor varies:

$$\delta R^{\lambda}_{\mu\nu\alpha} = \left(\delta\Gamma^{\lambda}_{\mu\nu}\right)_{||\alpha} - \left(\delta\Gamma^{\lambda}_{\mu\alpha}\right)_{||\nu},$$

which incidentally holds for *any* variation. We then also have

$$\delta R_{\mu\nu} = \left(\delta\Gamma^{\lambda}_{\mu\lambda}\right)_{||\nu} - \left(\delta\Gamma^{\lambda}_{\mu\nu}\right)_{||\lambda}$$

and

$$\begin{aligned} \delta R &= \delta\left(g^{\mu\nu} R_{\mu\nu}\right) \\ &= -\epsilon\pi R + g^{\mu\nu}\delta R_{\mu\nu} \end{aligned}$$

Another quantity whose variation will be required is the metric determinant $\sqrt{-g}$. It can be shown that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}$$

or, or four dimensions,

$$\delta\sqrt{-g} = 2\epsilon\pi\sqrt{-g}$$

3.3. Integration By Parts

Physical theories are derived by taking variational derivatives of integral scalar densities. This formalism invariably involves integration by parts under the integral, a basic mathematical technique used extensively in quantum mechanics and general relativity. To see how this is applied to gauge variations, a single example should suffice. Consider the integral

$$\int \sqrt{-g}R^2\, d^4x,$$

which incidentally appears in Weyl's gravity theory. Conformal variation then gives, using the above definitions,

$$\delta\int \sqrt{-g}R^2\, d^4x = \int \left(R^2\delta\sqrt{-g} + 2\sqrt{-g}\,R\,\delta R\right) d^4x$$

This collapses to

$$\begin{aligned} 2\int \sqrt{-g}\,g^{\mu\nu}R\,\delta R_{\mu\nu}\,d^4x &= 2\int \sqrt{-g}\,g^{\mu\nu}R\left[\left(\delta\Gamma^{\lambda}_{\mu\lambda}\right)_{||\nu} - \left(\delta\Gamma^{\lambda}_{\mu\nu}\right)_{||\lambda}\right] d^4x \\ &= -2\int \left(\sqrt{-g}\,g^{\mu\nu}R\right)_{||\nu}\delta\Gamma^{\lambda}_{\mu\lambda}\,d^4x + 2\int \left(\sqrt{-g}g^{\mu\nu}R\right)_{||\alpha}\,\delta\Gamma^{\lambda}_{\mu\nu}\,d^4x \end{aligned}$$

Integration by parts using the covariant derivative is allowed, since for any vector quantity A we have $\left(\sqrt{-g}A^{\lambda}\right)_{||\lambda} = \left(\sqrt{-g}A^{\lambda}\right)_{|\lambda}$. In Weyl's theory the connection terms are already conformally invariant, so the above expression is identically zero. But in general the connections are not invariant, so to proceed we will need to know exactly what $\delta\Gamma^{\lambda}_{\mu\nu}$ is.

4. Metricity

The formalism of Riemannian geometry is succinctly characterized by the vanishing of the covariant derivative of the metric tensor $g_{\mu\nu||\beta}$, or

$$g_{\mu\nu||\beta} = g_{\mu\nu|\beta} - g_{\mu\lambda}\Gamma^{\lambda}_{\nu\beta} - g_{\lambda\nu}\Gamma^{\lambda}_{\mu\beta} = 0 \tag{4.1}$$

This condition is known as *metricity*. It is easy to see that this condition necessarily forces the connection terms to be the usual Levi-Civita (or Christoffel) terms defined by

$$\Gamma^{\lambda}_{\mu\nu} \to \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} = \frac{1}{2}g^{\lambda\beta}\left(g_{\mu\beta|\nu} + g_{\beta\nu|\mu} - g_{\mu\nu|\beta}\right) \tag{4.2}$$

This identification also results in the invariance of vector magnitude under parallel transport, so that vectors can change direction or orientation but not length.

Both the Weyl and Schrödinger theories require that the so-called *non-metricity* tensor $g_{\mu\nu||\beta}$ be non-zero. This requirement not only allows vector magnitude to vary under parallel transport, but it surprisingly affects the conventional symmetry properties of the Riemann-Christoffel curvature tensor $R^{\ \lambda}_{\mu\nu\beta}$ as well. Given the arbitrary vector one-form ξ_μ, the curvature tensor is typically introduced by taking the difference of the double covariant derivatives of the vector, or

$$\xi_{\mu||\alpha||\beta} - \xi_{\mu||\beta||\alpha} = -\xi_\lambda R^{\ \lambda}_{\mu\alpha\beta}$$

The same formalism can be used on any arbitrary tensor. Of particular interest to us here is the identity

$$\begin{aligned} g_{\mu\nu||\alpha\beta} - g_{\mu\nu||\beta\alpha} &= -g_{\mu\lambda} R^{\lambda}{}_{\nu\alpha\beta} - g_{\lambda\nu} R^{\lambda}{}_{\mu\alpha\beta} \\ &= -\left(R_{\mu\nu\alpha\beta} + R_{\nu\mu\alpha\beta}\right) \end{aligned} \tag{4.3}$$

This neatly explains the antisymmetry of the first two indices of $R_{\mu\nu\alpha\beta}$ when space is Riemannian. Another useful identity can be derived when considering the pair of expressions

$$\begin{aligned} R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} &= 0 \\ R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} &= 0 \end{aligned}$$

It is simple to show that these can be combined to give

$$\begin{aligned} -\left(R_{\alpha\mu\beta\nu} + R_{\mu\alpha\beta\nu}\right) &= \left(R_{\mu\nu\alpha\beta} - R_{\alpha\beta\mu\nu}\right) + \left(R_{\alpha\nu\mu\beta} - R_{\mu\beta\alpha\nu}\right) \\ &= g_{\mu\alpha||\beta||\nu} - g_{\mu\alpha||\nu||\beta} \end{aligned} \tag{4.4}$$

which similarly explains the "pair exchange" symmetry $R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$ of Riemannian geometry. When non-metricity holds, these traditional Riemannian identities fail, while other identities — notably the Bianchi identities — fail to hold as well. This alone serves to motivate an examination of non-Riemannian geometry, at least with respect to non-metricity, and its impact on the gravitational field equations. Indeed, non-metricity is arguably the most natural approach to generalizing Einstein's theory.

5. Overview of Weyl's 1918 Theory

We begin with Weyl's theory of 1918, not only because it was the first comprehensive effort to generalize Einstein's 1915 theory but because it contains a key element that will be of use when we turn to Schrödinger's ideas. The literature on Weyl's 1918 theory, now nearly one hundred years old, is so extensive that we will provide only the basics of his approach, starting with a cursory review of the notion of *parallel transport*.

5.1. Parallel Transport, the Connection, and Vector Magnitude

One of Weyl's major contributions to early differential geometry was his development of the formalism of *parallel transport*, which allows two vectors at neighboring points to be compared in a covariant manner. This formalism introduced the notion of the connection $\Gamma^{\alpha}_{\mu\nu}$, which is of critical importance in differential geometry. If a vector ξ^{α} is parallel-transported from the point x to the point $x + dx$, it can be linked covariantly with its value $\xi^{\alpha}(x + dx) = \xi^{\alpha}(x) + \xi^{\alpha}_{|\mu}$ at the new point with the quantity $\mathscr{D}\xi^{\alpha}$, where

$$\mathscr{D}\xi^{\alpha} = -\Gamma^{\alpha}_{\mu\nu}\xi^{\mu} dx^{\nu} \tag{5.1.1}$$

The differential operator $\mathscr{D}$ links the vector $\xi^{\alpha}(x + dx)$ with the parallel "twin" of $\xi^{\alpha}(x)$ at that point, a concept that is more fully explained in any elementary text on differential geometry (it should be distinguished from the total differential operator d, although many authors use the latter notation for both operators). The connection term $\Gamma^{\alpha}_{\mu\nu}$ is not a tensor since it transforms in a non-covariant manner under a change of coordinates, but it is otherwise completely arbitrary.

The length or magnitude L^2 of the vector ξ^{μ} is given by $L^2 = g_{\mu\nu}\xi^{\mu}\xi^{\nu}$, and in Riemannian geometry it is assumed to be invariant with respect to parallel transport. Using (5.1.1), a straightforward calculation shows that this is equivalent to

$$\mathscr{D}L^2 = 2L\mathscr{D}L = g_{\mu\nu||\alpha} A^{\mu} A^{\nu} dx^{\alpha} \tag{5.1.2}$$

where $g_{\mu\nu||\alpha}$ is the covariant derivative of the metric tensor. Also known as the *non-metricity tensor*, it necessarily vanishes in a Riemannian space. By considering a cyclic combination of the indices in (4.1), it is easy to show that in a Riemannian space

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\}$$

Thus, this connection, like the metric tensor from which it is constructed, is symmetric in its lower indices. In a general non-Riemannian space the tensor $g_{\mu\nu||\alpha}$ does not vanish, nor is the connection (or the metric tensor) necessarily required to be symmetric. The consequences of non-symmetric connections (particularly the concept known as *torsion*) have been explored by many researchers (including Einstein, Eddington and Schrödinger) and continue to this day. Whether or not Nature employs torsion in physics (particularly with respect to particle spin) remains a topic of speculation.

One confusing aspect of parallel transport concerns whether geometric quantities other than vectors exhibit changes similar to that given by (5.1.1). While a vector may change in both direction and magnitude under parallel transport, it is difficut to imagine what it even means for an arbitrary tensor (like $g_{\mu\nu}$) to undergo a change in "direction" or "length." For this reason we will assume that only vector quantities behave according to (5.1.2) under parallel transport, while scalar fields and tensors of rank higher than one change only according to $\mathscr{D}\phi \rightarrow \phi_\alpha dx^\alpha$ and $\mathscr{D}g_{\mu\nu} \rightarrow g_{\mu\nu|\alpha}dx^\alpha$, etc.

5.2. Weyl's Theory

For a flat space in Riemannian geometry the metric tensor can be expressed as a constant, the connection vanishes and a vector parallel-transports to the new location $x + dx$ unchanged. When the space is not flat, the vector ξ^α can only change in direction, while the square of its magnitude $L^2 = g_{\mu\nu}\xi^\mu\xi^\nu$ remains fixed. Weyl's idea was to remove this constraint by allowing vector magnitude to change under transport as well. While perhaps counterintuitive, this is arguably the simplest path to a classical non-Riemannian geometry.

In order to proceed, Weyl had to assume a form for the change in vector magnitude under parallel transport, which he believed should be structurally similar to that of the vector itself. So he simply wrote

$$\mathscr{D}L = \phi_\alpha L dx^\alpha \tag{5.2.1}$$

where the as-yet undefined vector field ϕ_α acts somewhat like a connection term. At the same time Weyl noted that the change in vector magnitude L^2 could be determined by direct calculation which, for the arbitrary vector ξ^μ is given by

$$2L\mathscr{D}L = g_{\mu\nu||\alpha}\xi^\mu\xi^\nu dx^\alpha \tag{5.2.2}$$

Comparing these two expressions, we see that Weyl's definition for the change in vector magnitude is then equivalent to

$$g_{\mu\nu||\alpha} = 2g_{\mu\nu}\phi_\alpha \tag{5.2.3}$$

As a result, in Weyl's geometry any and all vectors are necessarily obliged to undergo a change in magnitude whenever the Weyl vector field ϕ_μ does not vanish. This places a rather harsh condition on vectors, as it is known that the magnitude of some vectors, like the unit vector dx^μ/ds, must remain constant.

5.3. Weyl's Connection

By expansion of $g_{\mu\nu||\alpha}$ and using cyclic permutations of the above expression it is a simple matter to show that the connection term in Weyl's geometry is

$$\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} - \delta^\alpha_\mu\phi_\nu - \delta^\alpha_\nu\phi_\mu + g_{\mu\nu}g^{\alpha\beta}\phi_\beta \tag{5.3.1}$$

At this point Weyl made an interesting observation. Let the metric tensor $g_{\mu\nu}$, which determines vector magnitude, undergo an infinitesimal rescaling (or *regauging*) given by $g_{\mu\nu} \rightarrow e^{\epsilon\pi(x)}g_{\mu\nu}$, or $\delta g_{\mu\nu} = \epsilon\pi g_{\mu\nu}$ (similarly, $\delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$). Weyl noticed that if the Weyl vector ϕ_μ also undergoes the transformation $\delta\phi_\mu = \frac{1}{2}\epsilon\pi_{|\mu}$, then the connection $\Gamma^\alpha_{\mu\nu}$ remains *unchanged*. Although Weyl believed that Nature should be invariant with regard to a rescaling of the metric (also known as *conformal invariance*), more importantly he recognized that the transformation of ϕ_μ was the same as that of the electromagnetic four-potential of electrodynamics. For this reason, Weyl believed he had discovered a way to unify the forces of gravitation and electromagnetism using a purely geometrical approach.

There is perhaps a much more convincing reason why Weyl might have believed his theory involved electromagnetism. In that theory, the Ricci tensor $R_{\mu\nu}$ is not symmetric; we have instead, by direct calculation using (5.3.1),

$$\begin{aligned} R_{\mu\nu} - R_{\nu\mu} &= \Gamma^{\lambda}_{\mu\lambda|\nu} - \Gamma^{\lambda}_{\mu\nu|\lambda} \\ &= -4(\phi_{\mu|\nu} - \phi_{\nu|\mu}) \\ &= -4F_{\mu\nu} \end{aligned}$$

where $F_{\mu\nu}$ is the antisymmetric electromagnetic tensor. In addition to the traditional list of identities exhibited by the curvature tensor $R^{\lambda}_{\ \mu\nu\alpha}$, there is a rather obscure identity attributed to Veblen which appeared in 1922 (but surely was known earlier to Weyl). It is

$$R^{\lambda}_{\ \mu\beta\nu||\alpha} + R^{\lambda}_{\ \beta\alpha\mu||\nu} + R^{\lambda}_{\ \alpha\nu\beta||\mu} + R^{\lambda}_{\ \nu\mu\alpha||\beta} = 0$$

(We omit the proof, as it is trivial.) Setting $\lambda = \beta$ and using the Bianchi identity, it is straightforward to show that the reduced Veblen identity becomes

$$(R_{\mu\nu} - R_{\nu\mu})_{||\alpha} + (R_{\alpha\mu} - R_{\mu\alpha})_{||\nu} + (R_{\nu\alpha} - R_{\alpha\nu})_{||\mu} = 0$$

Because of antisymmetry, the covariant derivatives are replaced by ordinary partial derivatives, and we have

$$(R_{\mu\nu} - R_{\nu\mu})_{|\alpha} + (R_{\alpha\mu} - R_{\mu\alpha})_{|\nu} + (R_{\nu\alpha} - R_{\alpha\nu})_{|\mu} = 0 \tag{5.3.2}$$

In Weyl's theory, this expression represents the set of homogeneous Maxwell's equations

$$F_{\mu\nu|\alpha} + F_{\alpha\mu|\nu} + F_{\nu\alpha|\mu} = 0$$

For example, setting $\mu = 0$, $\nu = i$ we have the familiar expression

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

It seems remarkable that Weyl's theory does indeed appear to link electromagnetism to geometry in this way, although any geometry in which $R_{\mu\nu}$ is non-symmetric might produce a similar result.

5.4. The Metric Determinant in Weyl's Theory

In addition to the metric tensor, rescaling of the metric determinant quantity defined as $\sqrt{-|g_{\mu\nu}|} = \sqrt{-g}$ deserves mention at this point, since it defines the notion of a *scalar density*. It is a simple matter to show that for any variation, this quantity changes in accordance with

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}$$

so that, with the conformal variation $\delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$, we have

$$\delta\sqrt{-g} = \frac{1}{2}\epsilon n\pi\sqrt{-g} \tag{5.4.1}$$

where n is the dimension of space. In four dimensions this is simply $\delta\sqrt{-g} = 2\epsilon\sqrt{-g}\,\pi$. It is highly significant that the Lagrangian $\sqrt{-g}\, F_{\mu\nu}F^{\mu\nu}$ of classical electromagnetism exhibits conformal invariance only in a four-dimensional space. This property certainly bolstered Weyl's belief that his theory could successfully link electromagnetism with geometry.

Since the Weyl connection is fully conformally invariant, it is easy to see from (5.3.1) that the righthand side of

$$\Gamma^{\alpha}_{\mu\alpha} = \ln(\sqrt{-g})_{|\mu} - 4\phi_{\mu}$$

is also invariant provided that $\delta\phi_\mu = \frac{1}{2}\epsilon\pi_{|\mu}$, which is indeed the case.

Thus, in Weyl's geometry the connection exhibits scale or conformal invariance, a type of symmetry that Weyl believed should be a basic principle not only in general relativity but in all of Nature. Although the Einstein-Hilbert Lagrangian $\sqrt{-g}\,R$ (from which one derives the traditional free-space Einstein gravitational field equations) is not conformally invariant, Weyl noted that the quantity $\sqrt{-g}\,R^2$ is fully invariant, and he subsequently used this Lagrangian to derive a set of equations that appeared to provide not only an alternative version of Einstein's gravitational field equations but the entirety of Maxwell's equations as well.

5.6. The Einstein and Weyl Equations of Motion

Equations of motion are derived by taking the variation of a suitable action Lagrangian with respect to the metric tensor $g^{\mu\nu}$. For comparison purposes, we will derive these equations for both the Einstein and Weyl gravity theories.

Let us first consider Einstein's equation with a cosmological constant term Λ, which is derived from the action

$$I_E = \int \sqrt{-g}\,(R - 2\Lambda)\, d^4x$$

Using the variational identities outlined earlier, the variation is straightforward, and we get

$$\delta I_E = \int \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}\right)\delta g^{\mu\nu}\, d^4x$$

Einstein's free-space equations are then

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 \tag{5.6.1}$$

Neglecting the cosmological term, the spherically symmetric solution gives the standard Schwarzschild metric, the derivation of which can be found in any text on general relativity. We thus have

$$g_{00} = 1 - \frac{2GM}{c^2 r}, \quad g_{11} = -1/g_{00}$$

However, let us retain the cosmological term and multiply (5.6.1) by $g^{\mu\nu}$, which then shows that $R = 4\Lambda$. Equation (5.6.1) can then be written as the *traceless* expression

$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = 0, \tag{5.6.2}$$

(which is known as the *Schouten tensor*). Again, the solution for this expression can be found in any text, and is

$$g_{00} = 1 - \frac{2GM}{c^2 r} + \frac{1}{3}\Lambda r^2, \quad g_{11} = -\frac{1}{g_{00}}$$

The r^2 term represents an anti-gravitational *acceleration* effect, which explains why the cosmological constant Λ is often used to explain the phenomenon known as *dark energy*.

Let us now consider Weyl's theory without the cosmological constant; for simplicity we will also ignore the Weyl vector ϕ_μ. We thus have the purely Riemannian action

$$I = \int \sqrt{-g}\,R^2\, d^4x$$

Variation of this quantity with respect to $g^{\mu\nu}$ is straightforward, giving

$$R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) + R_{|\mu||\nu} - g_{\mu\nu}g^{\alpha\beta}R_{|\alpha||\beta} = 0$$

But contraction with $g^{\mu\nu}$ shows that we have the side condition $g^{\mu\nu}R_{|\mu||\nu} = 0$, so that the (5.6.2) is just

$$R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) + R_{|\mu||\nu} = 0$$

One solution is obviously $R = 0$, but the more general solution is the non-zero constant $R = 4\lambda$, so that

$$g_{00} = 1 - \frac{2GM}{c^2 r} + \frac{1}{3}\lambda r^2, \quad g_{11} = -\frac{1}{g_{00}}$$

which is identical to the previous Einstein case with a cosmological constant. Weyl's theory thus appears to include the cosmological term automatically as a constant of integration.

What is missing in all this is the fact that (5.6.1) and (5.6.2) are incompatible with the Bianchi identities, which require the upper-index form of the Einstein field equations $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ to be divergenceless:

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{||\nu} = 0$$

Although the mathematical problems associated with strict conservation of energy-momentum in general relativity have never been resolved, the Bianchi identities are nevertheless valid, making the above arguments questionable.

5.7. Summary of Weyl's Geometry

To summarize, Weyl's geometry is characterized by the following:

1. There is a non-zero non-metricity tensor that is proportional to a field quantity that Weyl associated with the electromagnetic four-potential, or $g_{\mu\nu||\alpha} = 2g_{\mu\nu}\phi_\alpha$. From this definition, the covariant derivative of the metric determinant is also non-zero, and can be expressed as $(\ln\sqrt{-g})_{||\mu} = 4\sqrt{-g}\,\phi_\mu$.
2. The geometry utilizes a symmetric, non-Riemannian connection consisting of the Christoffel term and the Weyl vector field, or $\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} - \delta^\alpha_\mu\phi_\nu - \delta^\alpha_\nu\phi_\mu + g_{\mu\nu}g^{\alpha\beta}\phi_\beta$
3. Under an infinitesimal conformal variation of the metric tensor $g_{\mu\nu} \to (1+\epsilon\pi)g_{\mu\nu}$ (or $\delta g_{\mu\nu} = \epsilon\pi g_{\mu\nu}, \delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$), the Weyl connection remains unchanged provided the Weyl vector field varies according to $\delta\phi_\alpha = \frac{1}{2}\epsilon\pi_{|\alpha}$
4. The magnitude of *any* vector changes under parallel transport according to $\mathscr{D} = \phi_\alpha L dx^\alpha$. In Weyl's geometry vector length is *obliged* to change under parallel transport; there are no truly fixed-length vectors
5. The Weyl action $\int \sqrt{-g}\,R^2 d^4x$ is conformally invariant. While of fourth order, this action leads to equations of motion that make the same predictions as those of the Einstein free-space field equations
6. In a Weyl space the quantities $\sqrt{-g}\,R_{\mu\nu}R^{\mu\nu}$ and $\sqrt{-g}\,R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ are also conformally invariant. Thus, the Weyl action is not unique, since any linear combination of these quantities with $\sqrt{-g}R^2$ can be employed.

6. Einstein's Objection

Although Weyl's theory was able to reproduce the classical predictions of Einstein's 1915 theory (perihelion advance of the planet Mercury, gravitational redshift, deflection of starlight, etc.), Einstein — an initial admirer of Weyl's work — objected to the theory on more fundamental grounds. Under a rescaling of the metric tensor, the invariant line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ also undergoes rescaling via $ds \to \exp\frac{1}{2}\pi ds$. Einstein argued that since ds can be associated with the ticking of a clock or the spacings of atomic spectral lines, then many basic physical quantities (Compton wavelength, electron mass, etc.) would vary arbitrarily with time and location if ds were not conformally invariant. Weyl tried valiantly to refute Einstein's argument, and even undertook numerous efforts to make ds a true invariant, but to no avail. Within a few years after its proposal, Weyl's theory was considered a dead end.

However, let us consider the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ again, which we now write as

$$1 = g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{6.1}$$

which merely expresses the fact that the magnitude or length of the unit vector dx^μ/ds is unity. Since its magnitude is a pure number, it cannot change under parallel transport. But as noted previously, in Weyl's theory the magnitude of *any* vector *must* change according to

$$\mathscr{D}L = \phi_\alpha L dx^\alpha$$

Indeed, in Weyl's geometry there can be are no truly "constant" vectors provided that $\phi_\mu \neq 0$, in which case we merely revert back to the Riemannian case. However, we will see that there is a way to preserve the notion of constant vectors while allowing other vector quantities to change under physical transplantation.

For the unit vector dx^μ (or any vector proportional to it) we can demand that $\mathscr{D}L = 0$ if

$$g_{\mu\nu||\alpha}dx^\mu dx^\nu dx^\alpha = 0$$

This presents two possibilities: either the non-metricity tensor $g_{\mu\nu||\alpha}$ vanishes identically (and we again have Riemannian geometry), or it satisfies the cyclic property

$$g_{\mu\nu||\alpha} + g_{\alpha\mu||\nu} + g_{\nu\alpha||\mu} = 0 \tag{6.2}$$

We note at this point that Weyl's definition of the non-metricity tensor in (5.2.3) cannot satisfy this condition, so if we are to maintain the notion of a non-zero $g_{\mu\nu||\alpha}$ then we must revise the geometry. In particular, it remains to be seen what the non-metricity tensor really represents, what forms it can take, and whether — as in Weyl's theory — it involves some vector quantity that we might still attribute to more familiar physics. As we shall see, the geometry that Schrödinger happened upon allows for vectors that can be constant as well as variable under parallel transport. That geometry is completely dependent upon the property given by (6.2).

7. Schrödinger's Geometry

We now turn to Schrödinger's work of 1944-1950, a late period in the great Austrian physicist's life when he too considered non-Riemannian connections as a route to unification. He seems to have been especially interested in purely *affine* theories (connections without metrics), and his investigations included ideas that would seem odd to a classical relativist today. For example, he explored replacing the metric determinant $\sqrt{-g}$ with its Ricci equivalent $\sqrt{|R_{\mu\nu}|}$, although similar ideas had already been proposed by Einstein and Eddington. Schrödinger also considered metrics and connections that were both symmetric and non-symmetric, again mirroring ideas that had been considered years earlier by others.

Much of this work was summarized in a series of papers Schrödinger wrote in 1947-48, but in 1950 he published his famous book *Space-Time Structure*, which included more conventional approaches to generalizations of Einstein's foundational 1915 theory. In that book he happened upon what he considered at the time to be the most general possible symmetric connection (although he was initially motivated by the non-symmetric case). What he found was a rank-three tensor $T_{\mu\nu\alpha}$ that satisfies the symmetry properties of both $g_{\mu\nu}$ and $g_{\mu\nu||\alpha}$ as given in (6.2). Although Schrödinger did not explicitly identify his T-tensor with the non-metricity tensor at the time, that identification is unavoidable, as we now show.

Schrödinger started with a symmetric metric tensor and a non-symmetric connection, but with a vanishing non-metricity tensor $g_{\mu\nu||\alpha}$. First, Schrödinger wrote the three permutations

$$\begin{aligned} g_{\mu\nu||\alpha} &= g_{\mu\nu|\alpha} - g_{\mu\lambda}\Gamma^\lambda_{\alpha\nu} - g_{\lambda\nu}\Gamma^\lambda_{\alpha\mu} \\ g_{\alpha\mu||\nu} &= g_{\alpha\mu|\nu} - g_{\alpha\lambda}\Gamma^\lambda_{\nu\mu} - g_{\lambda\mu}\Gamma^\lambda_{\nu\alpha} \\ g_{\nu\alpha||\mu} &= g_{\nu\alpha|\mu} - g_{\nu\lambda}\Gamma^\lambda_{\mu\alpha} - g_{\lambda\alpha}\Gamma^\lambda_{\mu\nu} \end{aligned}$$

Subtracting the first expression from the sum of the other two, and setting all the permuted $g_{\mu\nu||\alpha}$ terms to zero, he arrived at

$$\frac{1}{2}\left(\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\nu\mu}\right) = \left\{ \begin{matrix} \alpha \\ \mu\,\nu \end{matrix} \right\} + \frac{1}{2} g^{\alpha\beta} g_{\mu\lambda} \left(\Gamma^\lambda_{\nu\alpha} - \Gamma^\lambda_{\alpha\nu}\right) + \frac{1}{2} g^{\alpha\beta} g_{\nu\lambda} \left(\Gamma^\lambda_{\mu\alpha} - \Gamma^\lambda_{\alpha\mu}\right)$$

Using the notation

$$\Gamma^\lambda_{(\mu\nu)} = \frac{1}{2}\left(\Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\mu}\right), \quad \Gamma^\lambda_{[\mu\nu]} = \frac{1}{2}\left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right)$$

Schrödinger was able to write the above expression more succintly as

$$\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\,\nu \end{matrix} \right\} + g^{\alpha\beta} g_{\mu\lambda} \Gamma^\lambda_{[\alpha\nu]} + g^{\alpha\beta} g_{\nu\lambda} \Gamma^\lambda_{[\beta\mu]} + \Gamma^\alpha_{[\mu\nu]} \tag{7.1}$$

Recognizing that the antisymmetric (or skew) aspect of the last term causes it to vanish when the geodesic equations are considered, Schrödinger simply scrapped it, and wrote

$$\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\,\nu \end{matrix} \right\} + g^{\alpha\beta} \left(g_{\mu\lambda} \Gamma^\lambda_{[\alpha\nu]} + g_{\nu\lambda} \Gamma^\lambda_{[\beta\mu]} \right) \tag{7.2}$$

Note that this expression now represents a family of purely *symmetric* connections in the indices μ and ν, which he now wrote as

$$\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\,\nu \end{matrix} \right\} + g^{\alpha\beta} T_{\mu\nu\beta} \tag{7.3}$$

where the quantity $T_{\mu\nu\beta} = T_{\nu\mu\beta}$ consists of a combination of two antisymmetric connections whose sum is nevertheless a legitimate tensor quantity. This connection was, in Schrödinger's opinion, the most general symmetric connection possible. As we will show, it is distinctly different from Weyl's' connection, and thus represents a new geometry in its own right.

Parallel transport of an arbitrary vector ξ^α in Schrödinger's geometry is now expressible using the new differential quantity

$$\mathscr{D}\xi^\alpha = -\Gamma^\alpha_{\mu\nu} \xi^\mu dx^\nu = -\left(\left\{ \begin{matrix} \alpha \\ \mu\,\nu \end{matrix} \right\} + g^{\alpha\beta} T_{\mu\nu\beta} \right) \xi^\mu dx^\nu \tag{7.4}$$

We can safely presume that Schrödinger also recognized that the unit vector dx^μ/ds must not change under parallel transport. Repeating the calculations above with the revised differential operator in (7.4) we have, after some simple algebra,

$$\mathscr{D}L^2 = 2L\mathscr{D}L = -2T_{\mu\nu\alpha} dx^\mu dx^\nu dx^\alpha = 0$$

from which Schrödinger surmised the condition

$$T_{\mu\nu\alpha} + T_{\alpha\mu\nu} + T_{\nu\alpha\mu} = 0 \tag{7.5}$$

which strongly resembles the cyclic property exhibited in (6.2). However, the connection in (7.3) presents a problem, for if we continue to assume that $g_{\mu\nu||\alpha} = 0$ for this connection, then $T_{\mu\nu\alpha}$ itself must vanish identically, as a quick calculation shows. To remedy this, let us expand and cyclically permute the identity

$$g_{\mu\nu||\alpha} = g_{\mu\nu|\alpha} - g_{\mu\lambda} \Gamma^\lambda_{\nu\alpha} - g_{\lambda\nu} \Gamma^\lambda_{\mu\alpha}$$

using (7.3) where the connection, as Schrödinger assumed, is fully symmetric. The Christoffel terms drop out, and we are left with $g_{\mu\nu||\alpha} = T_{\mu\nu\alpha}$. The conclusion is inescapable: *Schrödinger's T-tensor is the non-metricity tensor*. We can now write Schrödinger's connection formally as

$$\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\,\nu \end{matrix} \right\} + g^{\alpha\beta} g_{\mu\nu||\beta} \tag{7.6}$$

8. Explicit Identification of the Non-Metricity Tensor

In his his 1918 theory, Weyl was motivated to introduce a vector field ϕ_α that he subsequently identified as the electromagnetic four-potential. In his view, the presence of an electromagnetic field causes the ordinary Christoffel connection of Riemannian geometry to acquire terms proportional to ϕ_α, resulting in a non-zero non-metricity tensor. No such field is apparent in Schrödinger's connection, but it contains a link to Weyl's theory that allows a plausible definition of the connection in terms of such a field. To see this, let us contract (6.2) with $g^{\mu\nu}$:

$$g^{\mu\nu} g_{\mu\nu||\alpha} + 2 g^{\mu\nu} g_{\mu\alpha||\nu} = 0$$

or

$$\frac{1}{2} g^{\mu\nu} g_{\mu\nu||\alpha} = -g^{\mu\nu} g_{\mu\alpha||\nu}$$

Both sides of this expression are obviously vector one-forms, and for definiteness we make the definition

$$\varphi_\alpha = \frac{1}{2} g^{\mu\nu} g_{\mu\nu||\alpha} \tag{8.1}$$

(Note that we would have arrived at essentially the same identification by contracting Weyl's non-metricity tensor.) Expanding, we have

$$\varphi_\alpha = \frac{1}{2} g^{\mu\nu} \left(g_{\mu\nu|\alpha} - g_{\mu\lambda} \Gamma^\lambda_{\nu\alpha} - g_{\lambda\nu} \Gamma^\lambda_{\mu\alpha} \right)$$

which reduces to

$$\Gamma^\mu_{\alpha\mu} = (\ln \sqrt{-g})_{|\alpha} - \varphi_\alpha \tag{8.2}$$

We thus have a definition for the contracted variant of Schrödinger's connection in terms of the covariant vector φ_α, one that is very similar to Weyl's. But since this vector was defined in terms of the connection itself, we would seem to have a circular definition yielding an empty formalism. To avoid this we assume, contrary to Weyl's approach, that the corresponding vector in Schrödinger's geometry is merely a *shorthand notation* for the cyclic property given by (6.2). Thus, Schrödinger's geometry stands fully distinct from Riemannian geometry and that of Weyl while avoiding any immediate association with electromagnetism.

We now seek a definition of the non-metricity tensor in terms of the metric tensor and this vector field. Following Weyl, the most plausible approach is to assume an expression of the form

$$g_{\mu\nu||\alpha} = A g_{\mu\nu} \varphi_\alpha + B g_{\alpha\mu} \varphi_\nu + C g_{\nu\alpha} \varphi_\mu$$

where A, B, C are constants. From the symmetry of $g_{\mu\nu}$ we must have $B = C$, while from (6.2) we see that $A = -2B$. By considering our definition of φ_α in (8.1) we have $A = 2/3$, so we can finally write

$$g_{\mu\nu||\alpha} = \frac{2}{3} g_{\mu\nu} \varphi_\alpha - \frac{1}{3} g_{\alpha\mu} \varphi_\nu - \frac{1}{3} g_{\nu\alpha} \varphi_\mu \tag{8.3}$$

as the most general definition of the non-metricity tensor in Schrödinger's geometry. From (7.6), we can now write Schrödinger's connection as

$$\Gamma^\alpha_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\} - \frac{1}{3} \delta^\alpha_\mu \varphi_\nu - \frac{1}{3} \delta^\alpha_\nu \varphi_\mu + \frac{2}{3} g_{\mu\nu} g^{\alpha\beta} \varphi_\beta \tag{8.4}$$

which is already very similar to Weyl's connection. Again, it is emphasized that the vector φ_μ is to be regarded merely as a convenient shorthand for the cyclic property of the non-metricity tensor, and no association with electromagnetism is implied.

9. Conformal Invariance in Riemannian, Weyl and Schrödinger Geometries

Although Weyl's 1918 theory failed, his basic ideas were vindicated in 1929 when the notion of gauge invariance was applied to quantum physics. Weyl himself recognized that it was not the rescaling of the metric tensor that mattered in physics, but a *regauging* of the wave function with a *complex scalar phase factor* that was physically meaningful. As Weyl showed, the invariance of quantum physics with respect to such a phase factor did indeed result in a new conservation theorem (per Noether), that of the conservation of electric charge. Gauge or phase invariance today represents a profoundly fundamental and important symmetry in quantum mechanics, while the importance of conformal invariance in general relativity is still largely unconfirmed. Consequently, the conformal properties of classical general relativity, which rely primarily on the conformal aspects of the connection itself, remain to be more fully explored. Here we will consider how things change when the metric tensor undergoes the infinitesimal local change of scale defined by $\delta g_{\mu\nu} = \epsilon\pi g_{\mu\nu}, \delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$). First we consider conformal invariance in Riemannian geometry.

9.1. Conformal Invariance in Riemannian Geometry: Weyl's Conformal Tensor

In 1921 Weyl derived a unique action Lagrangian that is conformally invariant in ordinary Riemannian geometry. He found that the quantity

$$C^{\lambda}{}_{\nu\alpha\beta} = R^{\lambda}{}_{\nu\alpha\beta} + \frac{1}{2}\left(\delta^{\lambda}_{\beta} R_{\nu\alpha} - \delta^{\lambda}_{\alpha} R_{\nu\beta} + g_{\nu\alpha} R^{\lambda}_{\beta} - g_{\nu\beta} R^{\lambda}_{\alpha}\right) + \frac{1}{6}\left(\delta^{\lambda}_{\alpha}\, g_{\beta\nu} - \delta^{\lambda}_{\beta}\, g_{\alpha\nu}\right) R \qquad (9.1.1)$$

is fully invariant in four dimensions, and that the corresponding Lagrangian density

$$I = \sqrt{-g}\, C_{\mu\nu\alpha\beta}\, C^{\mu\nu\alpha\beta}, \qquad (9.1.2)$$

is likewise invariant (this quantity is often independently attributed to the German mathematician Rudolf Bach). The quantity $C_{\mu\nu\alpha\beta}$ is known as the *Weyl conformal tensor*; it represents a unique opportunity to study scale-invariant gravity, even though it is of fourth order in the metric tensor and its derivatives. However, the presence of the curvature term in (9.1.1) greatly complicates any attempt to derive equations of motion from the associated Lagrangian (9.1.2). Fortunately, there is an ingenious simplification due to Lanczos, who in 1938 found a way to eliminate the term entirely. Because we will use a similar simplification when we look at the conformal tensor in Schrödinger's geometry, we'll spend some time here on a variation of Lanczos's argument.

9.2. Simplification of the Conformal Lagrangian

A rather tedious expansion of the terms in (9.1.2) shows that it reduces to

$$\int \sqrt{-g}\, C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} = \int \sqrt{-g}\left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2\right) d^4x \qquad (9.2.1)$$

By taking a gauge variation of each of these terms, and integratiing by parts once when needed, it is easy to show that

$$\begin{aligned}
\delta\left(\sqrt{-g} R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}\right) &\to -4\epsilon\sqrt{-g}\, R^{\mu\nu}{}_{||\nu} \pi_{|\mu}, \\
\delta\left(\sqrt{-g} R_{\mu\nu} R^{\mu\nu}\right) &\to -2\epsilon\sqrt{-g}\, R^{\mu\nu}{}_{||\nu} \pi_{|\mu} - \epsilon\sqrt{-g}\, g^{\mu\nu} R_{|\nu} \pi_{|\mu}, \\
\delta\left(\sqrt{-g} R^2\right) &\to -6\epsilon\sqrt{-g}\, g^{\mu\nu} R_{|\nu} \pi_{|\mu},
\end{aligned}$$

where the arrow represents the associated resulting quantity in the integrand. Using these results in the variation of

$$\int \sqrt{-g}\left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + A R_{\mu\nu} R^{\mu\nu} + B R^2\right) d^4x \qquad (9.2.2)$$

(where A, B are constants), dropping $\epsilon\sqrt{-g}\pi_{|\mu}$ and setting the total integrand to zero, we have

$$2(A+2) R^{\mu\nu}{}_{||\nu} + (A + 6B)\, g^{\mu\nu} R_{|\nu} = 0$$

One solution is obviously $A = -2, B = 1/3$, but this only reproduces (9.2.1). But if we set $A \neq -2$ and divide out $A + 2$, this can be written as

$$R^{\mu\nu}{}_{||\nu} + \frac{A + 6B}{2(A+2)} g^{\mu\nu} R_{|\nu} = 0$$

If we now set $B = -(A+1)/3$, this becomes

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{||\nu} = 0 \tag{9.2.3}$$

which represents the Bianchi identities. Thus, by a suitable choice of the constants A, B the conformal Lagrangian in Riemannian space can be written either of two ways. The difference between (9.2.1) and (9.2.3) provides yet another Lagrangian, which is simply

$$\sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)$$

We have thus gotten rid of the curvature term. This is the simplified form that Lanczos discovered (it is often asserted that the action in (9.2.1) hides a pure divergence under the integral that can be ignored, but in reality the action hides the Bianchi identities, which automatically vanish). The action

$$\int \sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) d^4x \tag{9.1.4}$$

is today considered the standard action for conformal gravity. It was used by Mannheim and Kazanas in 1989 to derive a Schwarzschild-like solution to the associated free-space field equations, where several of the terms may be related to dark matter and/or dark energy.

Lastly, we note that the conformal tensor is of little use in Weyl's 1918 theory, since all of the terms in that tensor are already conformally invariant and no special combination is required. However, we might consider the possibility of using a similar approach to that above in developing a scale-invariant action Lagrangian involving Schrödinger'a geometry, which would then yield equations of motion specific to Schrödinger's formalism. This is really no more difficult than the Riemannian case, and we will outline the steps in detail after a brief sojourn into the Bianchi identities associated with Weyl's geometry.

9.3. The Bianchi Identities in Weyl's Geometry

As noted earlier, Weyl's connection is automatically conformally invariant, so the Riemann-Christoffel curvature tensor and the Ricci tensor are also automatically invariant. But because the scalar R involves $g^{\mu\nu}$, Weyl was motivated to consider the quadratic quantity $\sqrt{-g}\,R^2$ as a suitable Lagrangian for his gravity theory. Weyl's approach was able to replicate the successes of Einstein's 1915 theory (at least in source-free space), but his real goal was to embed electromagnetism into the theory as a natural consequence of the geometry. This aspect of Weyl's theory failed, not only because of Einstein's objection, but because the theory could not make any testable predictions involving electromagnetism.

It is, however, interesting to consider the Bianchi identities in Weyl geometry, which are easily derived. Using (4.3), we have

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} + 2g_{\mu\nu}F_{\alpha\beta},$$

where $F_{\alpha\beta} = \phi_{\alpha||\beta} - \phi_{\beta||\alpha}$. A straightforward reduction of the contracted Bianchi identities

$$R^{\lambda}_{\ \mu\nu\alpha||\lambda} + R_{\mu\nu||\alpha} - R_{\mu\alpha||\nu} = 0$$

then gives

$$\begin{aligned}\left[\sqrt{-g}(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\right]_{||\nu} &= \left(\sqrt{-g}F^{\mu\nu}\right)_{||\nu} \\ &= \sqrt{-g}s^{\mu}\end{aligned}$$

where s^{μ} is the electromagnetic source vector! In Weyl's theory, the Bianchi identities provide yet another connection between geometry and electromagnetism in Weyl's theory, although it is doubtful if this argument is physically meaningful. It is also interesting that this form of Einstein's equations do not involve an energy-momentum term on the righthand side, but an electromagnetic source term.

9.4. Conformal Invariance in Schrödinger's Geometry

While Schrödinger's geometry might appear to be a more eloquent approach compared to that of Weyl, it suffers from the fact that several traditional properties of the curvature tensor are no longer valid, preventing even a straightforward derivation of Bianchi's identity. While this problem also exists in Weyl's formalism, we've seen that he did not need to consider these properties because all the scalar densities in his theory are already gauge invariant. Consequently, Weyl didn't need to worry about taking tedious variations of terms like $\sqrt{-g}\, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ because they are automatically gauge invariant.

We will see that in Schrödinger's geometry this is not so (and it has to do with the sequence of indices in $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$). In addition, while Weyl was able to choose the variation of his vector one-form ϕ_α to be such that the connection $\Gamma^\alpha_{\mu\nu}$ was itself invariant to a conformal variation, a quick glance at the Schrödinger's connection shows this to be impossible. This is because the variation of the Christoffel term, which gives

$$\delta \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} = \frac{1}{2}\epsilon\delta^\alpha_\mu \pi_{|\nu} + \frac{1}{2}\epsilon\delta^\alpha_\nu \pi_{|\mu} - \frac{1}{2}\epsilon g_{\mu\nu} g^{\alpha\beta}\pi_{|\beta},$$

is nullified in Weyl's connection only if we choose $\delta\phi_\mu = \frac{1}{2}\epsilon\pi$, whereas Schrödinger's connection is incompatible with such a choice. We can, however, still straightforwardly determine how the connection changes under a change of scale. To see this, let us conduct a conformal variation of the Schrödinger connection as given in (7.6):

$$\delta\Gamma^\lambda_{\mu\nu} = \delta \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} + g_{\mu\nu||\alpha}\delta g^{\alpha\lambda} + g^{\alpha\lambda}\delta g_{\mu\nu||\alpha}$$

Using $\delta g^{\alpha\lambda} = -\epsilon\pi g^{\alpha\lambda}$, expanding $g_{\mu\nu||\alpha}$ and collecting terms, we can write this as

$$\delta\Gamma^\lambda_{\mu\nu} + g_{\mu\alpha}g^{\lambda\beta}\delta\Gamma^\alpha_{\beta\nu} + g_{\nu\alpha}g^{\lambda\beta}\delta\Gamma^\alpha_{\beta\mu} = \frac{1}{2}\epsilon g_{\mu\nu}g^{\lambda\beta}\pi_{|\beta} + \frac{1}{2}\epsilon\delta^\lambda_\mu\pi_{|\nu} + \frac{1}{2}\epsilon\delta^\lambda_\nu\pi_\mu$$

Term-by-term comparison shows that

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\epsilon g_{\mu\nu}g^{\lambda\beta}\pi_{|\beta} \tag{9.4.1}$$

with the variation of the contracted form being

$$\delta\Gamma^\lambda_{\mu\lambda} = \frac{1}{2}\epsilon\pi_\mu \tag{9.4.2}$$

Since, from (8.2),

$$\Gamma^\alpha_{\mu\alpha} = \left(\ln\sqrt{-g}\right)_{|\mu} - \varphi_\mu,$$

it is an easy matter to show that

$$\delta\varphi_\mu = \frac{3}{2}\epsilon\pi_{|\mu}$$

This demonstrates that a scale variation of Schrödingr's φ_μ, like Weyl's vector, is a pure gradient, and any suspected relationship with the electromagnetic four-potential would seem to be preserved in the Schrödinger case. However, unlike Weyl's connection, the quantity in (8.4) is not invariant with respect to rescaling. Schrödinger's formalism can nevertheless be carried over as a candidate for a modern conformal gravity theory using a different approach, which we explore in the next section.

10. The Schrödinger Lagrangian

As was mentioned earlier, Weyl's conformally-invariant action Lagrangian is the simple quantity $\sqrt{-g}\, R^2$, which results from the fact that the Weyl connection is itself conformally invariant (as noted, he could have also used $\sqrt{-g}\, R_{\mu\nu}R^{\mu\nu}$ and $\sqrt{-g}\, R_{\mu\nu\alpha\beta}\, R^{\mu\nu\alpha\beta}$ or any combination of these three scalar densities).This allowed Weyl to straightforwardly derive equations of motion by considering arbitrary independent variations of the metric tensor $\delta g^{\mu\nu}$ and the Weyl vector $\delta\phi_\mu$, which led him to expressions similar to Einstein's gravitational field equations and those of Maxwell's electrodynamics. But none of these densities are conformally invariant in Schrödinger's geometry, if for no other reason than the fact that his connection $\Gamma^\lambda_{\mu\nu}$ is not scale invariant. However, by considerating a linear combination of the three densities we can find a Lagrangian that is not only suitable but unique.

We remind ourselves that, in view of (4.3) and (4.4), the symmetry properties of the curvature tensor in Schrödinger geometry diverge from their Riemannian counterparts. As a result, it should be apparent that the scalar quantities $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ and $R_{\mu\nu\alpha\beta}R^{\nu\mu\alpha\beta}$ are no longer equivalent. This alone prevents a straightforward variation of the Schrödinger action with respect to $g^{\mu\nu}$ (the problem can be traced to the reduction of the quantity $g_{\mu\lambda}R^{\lambda\nu\alpha\beta}$). In Schrödinger geometry, interchange of the λ, ν indices is no longer antisymmetric, a complication that also blocks a straightforward derivation of the usual Bianchi conservation condition

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{||\nu} = 0$$

Consequently, equations of motion would at first appear to be underivable in the Schrödinger formalism.

10.1 Resolving the Riemann-Christoffel Symmetry Problem

There are three ways around this problem. One is to use the identity

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} + g_{\mu\nu||\beta||\alpha} - g_{\mu\nu||\alpha||\beta}$$

which we derived earlier, expand the non-metricity tensor using (8.3) and use this to "flip" the first two indices of the Riemann-Christoffel tensor. Another approach is to perform essentially the same operation by expanding $R_{\mu\nu\alpha\beta}$ into its Riemann form and terms involving only the metric tensor and φ_μ. Unfortunately, both of these approaches are computationally tedious.

The third approach recognizes, like Lanczos did, that what we really want to do is simply get rid of the complicated term $\sqrt{-g}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. For this reason we might as well consider flipping the indices from the outset, since in ordinary Riemannian space this results only in a change of sign in the term. Thus, we consider the Lagrangian for Schrödinger geometry to be

$$\sqrt{-g}\left(R_{\mu\nu\alpha\beta}R^{\nu\mu\alpha\beta} + AR_{\mu\nu}R^{\mu\nu} + BR^2\right)$$

where A and B are again constants to be determined. Although $R_{\mu\nu}R^{\mu\nu} \neq R_{\mu\nu}R^{\nu\mu}$, for simplicity we will assume the first form for this term.

10.2 Conformal Invariance of the Schrödinger Action

Like the approach used earlier, it is a simple matter to show that the action

$$I = \int \sqrt{-g}\left(R_{\mu\nu\alpha\beta}R^{\nu\mu\alpha\beta} + AR_{\mu\nu}R^{\nu\mu} + BR^2\right)d^4x \tag{10.2.1}$$

can be made conformally invariant by a suitable choice of the constants A, B. Per Palatini, any variation of the Riemann-Christoffel tensor can be written as

$$\delta R^\lambda_{\nu\alpha\beta} = \left(\delta\Gamma^\lambda_{\nu\alpha}\right)_{||\beta} - \left(\delta\Gamma^\lambda_{\nu\beta}\right)_{||\alpha} \tag{10.2.2}$$

A straightforward gauge variation of the curvature term in the above integral gives

$$4\int \sqrt{-g}\, g_{\mu\lambda}R^{\nu\mu\alpha\beta}\left(\delta\Gamma^\lambda_{\nu\alpha}\right)_{||\beta}\, d^4x$$

where we have taken advantage of the fact that the curvature tensor is antisymmetric in the indices α, β. Integrating once by parts, this becomes

$$-4\int \left(\sqrt{-g}\, g_{\mu\lambda}R^{\nu\mu\alpha\beta}\right)_{||\beta}\delta\Gamma^\lambda_{\nu\alpha}\, d^4x$$

By using (9.4.1) and (9.4.2), the Schroödinger identities

$$g_{\mu\nu||\alpha} = \frac{2}{3} g_{\mu\nu}\varphi_\alpha - \frac{1}{3} g_{\alpha\mu}\varphi_\nu - \frac{1}{3} g_{\nu\alpha}\varphi_\mu$$
$$g^{\mu\nu}{}_{||\alpha} = \frac{1}{3}\delta^\mu{}_\alpha\varphi^\nu + \frac{1}{3}\delta^\nu{}_\alpha\varphi^\mu - \frac{2}{3} g^{\mu\nu}\varphi_\alpha$$

and simplifying, we have, finally,

$$\delta \int \sqrt{-g}\, R_{\mu\nu\alpha\beta}R^{\nu\mu\alpha\beta}\, d^4x = 2\epsilon \int \left[-\left(\sqrt{-g}\, R^{\mu\nu}\right)_{||\nu} + \frac{1}{3}\sqrt{-g}\, (R^{\mu\nu} + R^{\nu\mu} + g^{\mu\nu}R)\, \varphi_\nu \right] \pi_{|\mu}\, d^4x$$

Variation of the terms $R_{\mu\nu}R^{\mu\nu}$ and R^2 is considerably easier to perform, and we have

$$\delta \int \sqrt{-g}\, R_{\mu\nu}R^{\mu\nu}\, d^4x = \epsilon \int \sqrt{-g} \left[-\left(\sqrt{-g}\, R^{\mu\nu}\right)_{||\nu} + \frac{1}{3}\sqrt{-g}\, (R^\mu + R^{\nu\mu})\, \varphi_\nu + \left(\sqrt{-g}\, g^{\mu\nu}R\right)_{||\nu} - \frac{5}{3}\sqrt{-g}\, g^{\mu\nu}\varphi_\nu \right] \pi_{|\mu}\, d^4x$$
$$\delta \int \sqrt{-g}\, R^2\, d^4x = 3\epsilon \int \left[\left(\sqrt{-g}\, g^{\mu\nu}R\right)_{||\nu} - 2\sqrt{-g}\, g^{\mu\nu}R\varphi_\nu \right] \pi_{|\mu}\, d^4x$$

Inserting these quantities into (10.2.1) and setting the integrand to zero, we arrive at

$$-(A+2)\left(\sqrt{-g}\, R^{\mu\nu}\right)_{||\nu} + (A+3B)\left(\sqrt{-g}\, g^{\mu\nu}R\right)_{||\nu} + \frac{1}{3}(A+2)\sqrt{-g}\, (R^{\mu\nu} + R^{\nu\mu})\, \varphi_\nu + \left(\frac{2}{3} - \frac{5}{3}A - 6B\right)\sqrt{-g}\, g^{\mu\nu} R\, \varphi_\nu = 0 \tag{10.2.3}$$

One solution is $A = -2, B = 2/3$, so that the Schrödinger action

$$I_1 = \int \sqrt{-g} \left(R_{\mu\nu\alpha\beta}R^{\nu\mu\alpha\beta} - 2R_{\mu\nu}R^{\nu\mu} + \frac{2}{3}R^2 \right) d^4x \tag{10.2.3}$$

is fully gauge invariant. However, as before we can also reject the $A = -2$ solution and write (10.2.3) as

$$\left[\sqrt{-g} \left(R^{\mu\nu} - \frac{A+3B}{A+2} g^{\mu\nu}R \right) \right]_{||\nu} = \frac{1}{3}\sqrt{-g}\, (R^{\mu\nu} + R^{\nu\mu})\, \varphi_\nu + \frac{\frac{2}{3} - \frac{5}{3}A - 6B}{A+2}\sqrt{-g}\, g^{\mu\nu}R\varphi_\nu$$

If we take $(A+3B)/(A+2) = 1/2$ or $B = -A/2 + 1/6$, we arrive at

$$\left[\sqrt{-g} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu}R \right) \right]_{||\nu} = \frac{1}{3}\sqrt{-g}\, (R^{\mu\nu} + R^{\nu\mu})\, \varphi_\nu - \frac{2}{3}\sqrt{-g}\, g^{\mu\nu}R\varphi_\nu,$$

which, when $\varphi_\nu = 0$, gives us the classical Bianchi identities. The second Schrödinger action thus appears as

$$I_2 = \int \sqrt{-g} \left(R_{\mu\nu\alpha\beta}R^{\nu\mu\alpha\beta} + AR_{\mu\nu}R^{\mu\nu} - \frac{1}{6}(A-2)R^2 \right) d^4x$$

As before, the difference $I_1 - I_2$ gives us yet another Lagrangian, or

$$I_3 = \int \sqrt{-g} \left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{6}R^2 \right) d^4x, \tag{10.2.4}$$

where we have divided out a common $A+2$ term (this quantity was also proposed by the Dutch mathematician Jan Schouten in a different context). Note that we have similarly gotten rid of the curvature term using the same Lanczos-type approach. Note also that the Schrödinger action is nearly identical to the Riemannian form in (9.1.4).

11. The Schrödinger Equations of Motion

The reduced Schrödinger action

$$\int \sqrt{-g} \left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{6}R^2 \right) d^4x$$

is fully conformally invariant while being suggestively very similar to that of the standard fourth-order action for conformal gravity in Riemannian space, differing only by a factor of two in the second term. To compare the two actions, let us write the above integral as

$$\int \sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu} - \kappa R^2\right) d^4x$$

where κ is some constant. It is a straightforward exercise to show that ordinary Riemannian variation of this action in free space with respect to $g^{\mu\nu}$ leads to

$$\begin{aligned}
&-\frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + 2R_{\mu\alpha}R_{\nu\beta}g^{\alpha\beta} + \frac{1}{2}\kappa g_{\mu\nu}R^2 - 2\kappa RR_{\mu\nu} - g_{\mu\nu}R^{\alpha\beta}{}_{||\alpha||\beta} + \kappa g_{\mu\nu}g^{\alpha\beta}R_{|\alpha||\beta} \\
&+ 2g_{\alpha\nu}R^{\alpha\beta}{}_{||\mu||\beta} + \kappa g_{\mu\nu}g^{\alpha\beta}R_{|\alpha||\beta} - 2\kappa R_{\mu||\nu} - g^{\alpha\beta}R_{\mu\nu||\alpha||\beta} = 0
\end{aligned}$$

where we have dropped the $\sqrt{-g}\,\delta g^{\mu\nu}$ term. This set of equations is difficult to solve, so to simplify we contract with $g^{\mu\nu}$ to get the scalar equations of motion, which are

$$R^{\mu\nu}{}_{||\mu||\nu} = \frac{1}{2}(6\kappa - 1)\, g^{\mu\nu}R_{|\mu||\nu}$$

Let us now set $\kappa = 1/3$. A tedious calculation shows that the Schwarzschild metric

$$ds^2 = e^{\nu}(dx^0)^2 - e^{\lambda}dr^2 - r^2 d\theta^2 - r^2\sin\theta^2 d\phi^2$$

holds, where

$$e^{\nu} = 1 + \frac{A}{r} + B + Cr + Dr^2, \quad e^{\lambda} = e^{-\nu}$$

is a solution, where A, B, C, D are constants. In 1989, Mannheim and Kazanas derived this same solution to the field equations, which appeared as

$$e^{\nu} = 1 - \frac{\beta(2-3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - \kappa r^2, \quad e^{\lambda} = e^{-\nu}$$

and β, γ, κ are constants. As was noted previously, the r^2 term in these quantities represents an acceleration for a particle under the influence of a central mass source, while the linear r term represents new territory in gravity theory. The researchers suggested that these terms may have something to do with dark matter and dark energy, as they represent a significant modification to the usual predictions of Einstein gravity.

Conversely, the Schrödinger action with $\kappa = 1/6$ gives the simpler result

$$R^{\mu\nu}{}_{||\mu||\nu} = 0$$

In view of the close similarity of the Schrödinger action with that of Mannheim and Kazanas, one may legitimately adopt a Schwarzschild-like solution in deriving the equations of motion for the case $\varphi_\mu = 0$. If we assume the Frobenius-type solution

$$e^{\nu} = 1 - \sum a_n r^n, \quad e^{\lambda} = e^{-\nu}$$

it is actually a straightforward if tedious exercise to show that the Mannheim-Kazanas and Schrödinger actions are essentially equivalent.

11.1. Summary of Schrödinger Geometry

To summarize, Schrödinger's geometry is characterized by the following:

1. There is a non-vanishing non-metricity tensor that is characterized by the cyclical property $g_{\mu\nu||\lambda} + g_{\lambda\mu||\nu} + g_{\nu\lambda||\mu} = 0$.

2. There is a Weyl-like vector in the space characterized by contraction of the above condition with $g^{\mu\nu}$, or $\varphi_\lambda = g^{\mu\nu} g_{\mu\nu||\lambda} = g^{\mu\nu}_{\ \ ||\nu} g_{\mu\lambda}$ or $\varphi^\mu = g^{\mu\nu}_{\ \ ||\nu}$. This vector is not identified with the electromagnetic four-potential nor any other physically meaningful quantity; it is simply a convenient shorthand for the cyclical property of the non-metricity tensor.

3. The geometry is based on a symmetric, non-Riemannian connection consisting of the Christoffel term and the above vector given by $\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} - \frac{1}{3}\delta^\alpha_\mu \phi_\nu - \frac{1}{3}\delta^\alpha_\nu \phi_\mu + \frac{2}{3} g_{\mu\nu} g^{\alpha\beta} \phi_\beta$

4. Under an infinitesimal conformal variation of the metric tensor $g_{\mu\nu} \to (1+\epsilon\pi) g_{\mu\nu}$ (or $\delta g_{\mu\nu} = \epsilon\pi g_{\mu\nu}, \delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$), the Schrödinger connection obeys the identities given by $\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\epsilon g_{\mu\nu} g^{\lambda\beta} \pi_\beta$ and $\delta\phi_\alpha = \frac{3}{2}\epsilon\pi_{|\alpha}$

5. Unlike Weyl's geometry, in Schrödinger's geometry the magnitudes of certain vectors need not change under parallel transport.

6. The reduced Schrödinger action
$$\int \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{6} R^2 \right) d^4x$$
is fully conformally invariant in the geometry. It is suggestively very similar to that of the standard fourth-order action for conformal gravity in Riemannian space, differing only by a factor of two in the second term. In either case, the equations of motion are essentially equivalent.

12. Conclusions

We have shown that both the Weyl and Schrödinger geometries, while conceptually and mathematically intriguing (and perhaps even beautiful), are problematic with regard to interpretation and meaning of any non-Riemannian formalism characterized by a non-metricity tensor. They are also problematic in that they are inconsistent with the Bianchi conservation condition (6.2). All of these drawbacks can be traced to the fact that several important symmetry properties of the Riemann-Christoffel tensor are invalidated in the presence of a non-vanishing non-metricity tensor. Nevertheless, if one rejects the traditional energy-momentum interpretation of the Bianchi identities, there is no reason to believe that either the Weyl or Schrödinger theories are wrong.

Although Weyl's formalism cannot account for the existence of fixed-magnitude vector quantities under parallel transport, its built-in aspect of conformal invariance displays a certain aesthetic appeal, an appeal that has persisted for nearly a century. Furthermore, Weyl's notion of conformal invariance led directly to the fundamental concept of gauge invariance in quantum theory, which has become a cornerstone of theoretical quantum physics. The Weyl formalism also provides a means of proposing a simple scale-invariant action whose equations of motion are comparable to those of the Einstein field equations. The Schrödinger formalism, which we we investigated in some detail here, provides a mjor advantage over Weyl's, namely the invariance of magnitude for certain vector quantities.

By comparison, the Schrödinger formalism appears to provide a mathematically legitimate way out of Einstein's objection to Weyl's theory, and unlike Weyl's theory it also leads to a quadratic Lagrangian that is remarkably similar to current conformal gravity theories.

It is impossible to construct a scale-invariant action in four dimensions of less than fourth order in the metric tensor and its derivatives without introducing arbitrary scalar, tensor and spinor fields. The conformal actions of Mannheim-Kazanas, Weyl and Schrödinger represent interesting and possibly physically relevant alternatives to traditional gneeral relativity, with each deserving further investigation.

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