A Note on Weyl Conformal Gravity

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Here I just wanted to clarify a neat trick that the late Cornelius Lanczos of Purdue University used to simplify the equations of quadratic conformal gravity. His 1938 paper is rather difficult to follow, while a related formula presented in 1964 by the late British mathematical physicist Bryce DeWitt is no easier. To be honest I could never really understand either approach, but the one I outline in the following should make Lanczos' trick a bit clearer.

Hermann Weyl's 1918 gauge theory, which introduced a non-Riemannian geometry in an effort to embed electromagnetism into general relativity, necessarily relied upon a Lagrangian that was invariant with respect to the metric *gauge transformation* $g^{\mu\nu} \to e^{-\pi} g^{\mu\nu}$, where $\pi(x)$ is an arbitrary function of position. Weyl believed that the scalar parameter $\pi(x)$ might be related to the gauge transformation of electromagnetism, $A_\mu \to A_\mu + \partial_\mu \pi$, and thus provide an opportunity for deriving Maxwell's equations from a purely geometric foundation. This theory failed, but it spurred a considerable amount of interest in gravitational theories based on what would now be called *conformal invariance* or *scale invariance* (rather than gauge invariance). That interest has continued up to this day, and over the past ninety years many researchers have contributed to the topic, now properly called *Weyl conformal gravity*.

It was Weyl himself who realized in 1919 that even in ordinary Riemannian geometry there is a *unique* tensor quantity that is conformally invariant. Now called the *Weyl conformal tensor* $C^\lambda_{\ \nu\alpha\beta}$, its definition in four dimensions is

$$C^\lambda_{\ \nu\alpha\beta} = R^\lambda_{\ \nu\alpha\beta} + \frac{1}{2}\left[\delta^\lambda_\beta\, R_{\nu\alpha} - \delta^\lambda_\alpha\, R_{\nu\beta} + g_{\nu\alpha}\, R^\lambda_{\ \beta} - g_{\nu\beta}\, R^\lambda_{\ \alpha}\right] + \frac{1}{6}\left[\delta^\lambda_\alpha\, g_{\beta\nu} - \delta^\lambda_\beta\, g_{\alpha\nu}\right] R \tag{1}$$

where $R^\lambda_{\ \nu\alpha\beta}$ is the Riemann-Christoffel curvature tensor

$$R^\lambda_{\ \nu\alpha\beta} = \begin{Bmatrix}\lambda \\ \nu\alpha\end{Bmatrix}_{|\beta} - \begin{Bmatrix}\lambda \\ \nu\beta\end{Bmatrix}_{|\alpha} + \begin{Bmatrix}\lambda \\ \sigma\beta\end{Bmatrix}\begin{Bmatrix}\sigma \\ \nu\alpha\end{Bmatrix} - \begin{Bmatrix}\lambda \\ \sigma\alpha\end{Bmatrix}\begin{Bmatrix}\sigma \\ \nu\beta\end{Bmatrix}$$

and $R_{\nu\beta} = R^\lambda_{\ \nu\lambda\beta}$ and $R = g^{\mu\nu} R_{\mu\nu}$ are its contracted variants (the single subscripted bar stands for partial differentiation). The quantity $C^\lambda_{\ \nu\alpha\beta}$ remains unchanged when the metric tensor is rescaled via $g^{\mu\nu} \to e^{-\pi} g^{\mu\nu}$. Consequently, this tensor was considered very early on as a candidate for a generalized version of Einstein's 1916 gravity theory. The Weyl tensor similarly offers a unique conformal Lagrangian that can be used to build an alternative gravity theory. This Lagrangian is $\sqrt{-g}\, C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta}$ which, using (1), works out to be

$$\sqrt{-g}\, C_{\mu\nu\alpha\beta}\, C^{\mu\nu\alpha\beta} = \sqrt{-g}\left[R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2\right] \tag{2}$$

Unfortunately, this quadratic quantity is of fourth order with respect to the metric tensor and its derivatives, an undesirable property that greatly complicates the solution of the associated equations of motion. It also mixes the Riemann-Christoffel curvature tensor with its Ricci cousins, complicating consideration of spaces that are Riemann-curved but Ricci-flat (such as the Schwarzschild metric). Nevertheless, if conformal invariance is to be demanded, the Weyl Lagrangian is the only game in town.

An early admirer (and principle investigator) of Weyl's gauge ideas was Cornelius Lanczos, who in a 1938 paper discovered a way to greatly simplify the mathematics by getting rid of the troublesome $R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$ term. We won't reproduce his logic here, but we'll demonstrate an alternative approach that's equivalent and much easier to follow.

The variational expressions for the Weyl Lagrangian terms $R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$, $R_{\mu\nu} R^{\mu\nu}$and R^2 with respect to an arbitrary variation $\delta g^{\mu\nu}$ are not difficult to derive, but it is much easier to work out the terms for the infinitesimal conformal variation $g^{\mu\nu} \to e^{-\epsilon\pi} g^{\mu\nu}$= $(1 - \epsilon\pi) g^{\mu\nu}$ or $\delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$. In that case we have

$$\begin{aligned}
\delta\sqrt{-g}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} &= 4\epsilon\pi\sqrt{-g}g_{\alpha\sigma}R^{\alpha\beta\sigma\lambda}{}_{||\beta||\lambda} = 4\epsilon\pi\sqrt{-g}R^{\mu\nu}{}_{||\mu||\nu} \\
\delta\sqrt{-g}R_{\mu\nu}R^{\mu\nu} &= 2\epsilon\pi\sqrt{-g}R^{\mu\nu}{}_{||\mu||\nu} + \epsilon\pi\sqrt{-g}g^{\mu\nu}R_{|\mu||\nu} \\
\delta\sqrt{-g}R^2 &= 6\epsilon\pi\sqrt{-g}g^{\mu\nu}R_{|\mu||\nu}
\end{aligned}$$

where the double subscript notation refers to covariant differentiation. We now ask for some combination of these four terms that vanishes either identically or under integration (by the divergence theorem). We consider the combination

$$\delta\sqrt{-g}\left[R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + A\,R_{\mu\nu}R^{\mu\nu} + B\,R^2\right]$$

where A and B are constants. Using the above formulas, this can be written as

$$\delta\sqrt{-g}\left[R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + A\,R_{\mu\nu}R^{\mu\nu} + B\,R^2\right] = 2(A+2)\epsilon\pi\sqrt{-g}R^{\mu\nu}{}_{||\mu||\nu} + (A+6B)\epsilon\pi\sqrt{-g}g^{\mu\nu}R_{|\mu||\nu}$$

Picking $A = -2$, $B = 1/3$ works, but that just gives us the identity (2). However, if $A \neq -2$ then we can write

$$\delta\sqrt{-g}\left[R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + A\,R_{\mu\nu}R^{\mu\nu} + B\,R^2\right] = 2(A+2)\epsilon\pi\sqrt{-g}\left[R^{\mu\nu} + \frac{A+6B}{2(A+2)}g^{\mu\nu}R\right]_{||\mu||\nu} \tag{3}$$

If we now set $A + 3B = -1$ we can write this as

$$\delta\sqrt{-g}\left[R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + A\,R_{\mu\nu}R^{\mu\nu} + B\,R^2\right] = 2(A+2)\epsilon\pi\sqrt{-g}\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right]_{||\mu||\nu}$$

The term on the right is one form of the Bianchi identities (it's also the divergence of the Einstein tensor $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$), which is identically zero. Lanczos' identity (which uses $A = -4$, $B = 1$) thus vanishes as a divergence under the integral. It also vanishes simply because it's algebraically zero, as DeWitt discovered.

By subtracting Lanczos' variational identity from the Weyl conformal Lagrangian (2), we can rid ourselves of the Riemann term. This gives

$$\int \sqrt{-g}\,C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}d^4x = \int \sqrt{-g}\,\left[R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right]d^4x \tag{4}$$

where we have dropped a meaningless numerical coefficient. This Lagrangian is considered to be the "official" Lagrangian of Weyl conformal gravity theory. Its primary advantage, other than relative simplicity, is the absence of the curvature tensor, allowing for equations of motion that are consistent with $R_{\mu\nu} = 0$.

A general variation of the metric tensor in (4) now produces

$$\delta\int \sqrt{-g}\,C_{\mu\nu\alpha\beta}\,C^{\mu\nu\alpha\beta}d^4x = \int \sqrt{-g}\,W_{\mu\nu}\delta g^{\mu\nu}d^4x$$

where $W_{\mu\nu}$ is a complicated expression involving the Ricci tensor and scalar and their derivatives. In a space where matter is present, $W_{\mu\nu}$ is proportional to the symmetric energy tensor $T_{\mu\nu}$. An exact solution for the vacuum case $W_{\mu\nu} = 0$ has been worked out in great detail by Mannheim and Kazanas using the Schwarzschild line element

$$ds^2 = e^{\nu}(dx^0)^2 - e^{\lambda}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2$$

The solution they found is

$$e^{\nu} = e^{-\lambda} = 1 - \frac{\beta(2-3\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2$$

where β, γ and k are arbitrary constants. The resemblance of the Mannheim-Kazanas solution to the ordinary Schwarzschild solution is obvious. The additional terms very possibly have application to the solution of the galactic rotation, dark matter and dark energy problems.

References

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