On Lanczos' Conformal Trick

William O. Straub
Pasadena, California 91104
June 30, 2014

Abstract

The Weyl conformal tensor describes the distorting but volume-preserving tidal effects of gravitation on a material body. A rather complicated combination of the Riemann-Christoffel tensor, the Ricci tensor and the Ricci scalar, the Weyl tensor is used in the construction of a unique conformally-invariant Lagrangian. In 1938 Cornelius Lanczos discovered a clever simplification of the mathematics that eliminated the RC term, thus considerably reducing the complexity of the overall Lagrangian. Here we present an equivalent but simpler approach to the one Lanczos used.

Introduction

In 1918 the German mathematical physicist Hermann Weyl proposed a unification of gravitation and electromagnetism based on the invariance of physics with respect to a conformal (or scale) transformation of the metric tensor $g_{\mu\nu} \to \exp(\pi) g_{\mu\nu}$, where $\pi(x)$ is an arbitrary scalar function. A decade later Weyl's idea was recast as *gauge symmetry*, which subsequently became a cornerstone of quantum theory. More recently, the notion of conformal symmetry has been explored in numerous cosmological models, and there is increasing speculation that conformally invariant geometry may indeed underlie Nature.

Weyl's theory, which introduced a non-Riemannian geometry in an effort to embed electromagnetism into general relativity, necessarily relied upon a Lagrangian that was invariant with respect to a local rescaling of the metric tensor. Weyl believed that the scale parameter $\pi(x)$ might be related to the gauge transformation property of electromagnetism via $A_\mu \to A_\mu + \partial_\mu \pi$, and thus provide an opportunity for deriving Maxwell's equations from a purely geometric foundation. The theory failed, but it spurred a considerable amount of interest in gravitational theories based on conformal invariance. That interest has continued to this day, with many researchers contributing to the topic, now properly called *Weyl conformal gravity*.

While he did not employ it in his 1918 theory, Weyl discovered that even in ordinary Riemannian geometry there is a unique tensor quantity that is conformally invariant. Now called the *Weyl conformal tensor* $C^\lambda_{\ \nu\alpha\beta}$, its definition in four dimensions is

$$C^\lambda_{\ \nu\alpha\beta} = R^\lambda_{\ \nu\alpha\beta} + \frac{1}{2}\left(\delta^\lambda_\beta R_{\nu\alpha} - \delta^\lambda_\alpha R_{\nu\beta} + g_{\nu\alpha} R^\lambda_\beta - g_{\nu\beta} R^\lambda_\alpha\right) + \frac{1}{6}\left(\delta^\lambda_\alpha g_{\beta\nu} - \delta^\lambda_\beta g_{\alpha\nu}\right) R \tag{1}$$

where $R^\lambda_{\ \nu\alpha\beta}$ is the Riemann-Christoffel curvature tensor

$$R^\lambda_{\ \nu\alpha\beta} = \left\{ {\lambda \atop \nu\alpha} \right\}_{|\beta} - \left\{ {\lambda \atop \nu\beta} \right\}_{|\alpha} + \left\{ {\lambda \atop \sigma\beta} \right\}\left\{ {\sigma \atop \nu\alpha} \right\} - \left\{ {\lambda \atop \sigma\alpha} \right\}\left\{ {\sigma \atop \nu\beta} \right\}$$

and where $R_{\nu\beta} = R^\lambda_{\ \nu\lambda\beta}$ and $R = g^{\mu\nu} R_{\mu\nu}$ are its contracted variants (the single subscripted bar stands for ordinary partial differentiation). As can be easily verified, the quantity $C^\lambda_{\ \nu\alpha\beta}$ remains unchanged when the metric tensor is rescaled. Consequently, this tensor was considered early on as a candidate for a generalized version of Einstein's 1915 gravity theory based on this scale or conformal symmetry. The Weyl tensor leads to a unique conformal Lagrangian that can be used to build an alternative gravity theory. That Lagrangian is $\sqrt{-g}\, C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta}$ which, using (1), works out to be

$$\sqrt{-g}\, C_{\mu\nu\alpha\beta}\, C^{\mu\nu\alpha\beta} = \sqrt{-g}\left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 2R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2\right) \tag{2}$$

This quadratic quantity is of fourth order with respect to the metric tensor and its derivatives, an undesirable property that greatly complicates the solution of the associated equations of motion. But worse is its mixing of the

Riemann-Christoffel curvature tensor with its Ricci cousins, which complicates consideration of spaces that are Riemann-curved but Ricci-flat (such as the Schwarzschild metric). Nevertheless, if conformal invariance is to be demanded, the Weyl Lagrangian is the only game in town.

An early admirer (and noted investigator) of Weyl's gauge idea was the Hungarian mathematical physicist Cornelius Lanczos, who in a 1938 paper discovered a way to greatly simplify the mathematics by effectively getting rid of the troublesome $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ term in the Lagrangian. We won't reproduce his logic here, but will demonstrate an alternative approach that is equivalent and considerably easier to follow.

Approach

Following Lanczos, we can eliminate the Riemann-Christoffel term $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ if we can find coefficients for the $R_{\mu\nu}R^{\mu\nu}$ and R^2 terms differing from the ones in (2). We therefore assume that a more general conformally invariant action integral like

$$I = \int \sqrt{-g}\,\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + AR_{\mu\nu}R^{\mu\nu} + BR^2\right) d^4x \tag{3}$$

exists, where A, B are constants. We can then subtract (2) from (3) to eliminate the RC term, leaving an invariant Lagrangian consisting of just two terms.

For the infinitesimal change of scale $\delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$ in four dimensions, the variation of $\sqrt{-g}$ is simple:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu} = 2\epsilon\pi\sqrt{-g}$$

The variations of $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, $R_{\mu\nu}R^{\mu\nu}$ and R^2 are more difficult, but the calculations are greatly simplified by using the Palatini identity

$$\delta R^{\lambda}{}_{\nu\alpha\beta} = \left(\delta\begin{Bmatrix}\lambda\\ \nu\alpha\end{Bmatrix}\right)_{||\beta} - \left(\delta\begin{Bmatrix}\lambda\\ \nu\beta\end{Bmatrix}\right)_{||\alpha}$$

where, for the infinitesimal change of scale $\delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$,

$$\delta\begin{Bmatrix}\lambda\\ \nu\alpha\end{Bmatrix} = \frac{1}{2}\epsilon\delta^{\lambda}_{\nu}\pi_{|\alpha} + \frac{1}{2}\epsilon\delta^{\lambda}_{\alpha}\pi_{|\nu} - \frac{1}{2}\epsilon g_{\nu\alpha}g^{\lambda\beta}\pi_{|\beta}$$

For brevity, we will simply write down the variations we'll need:

$$\delta\sqrt{-g}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = 4\epsilon\sqrt{-g}\,g_{\alpha\beta}R^{\beta\mu\alpha\nu}\pi_{|\mu||\nu}$$

$$\delta\sqrt{-g}R_{\mu\nu}R^{\mu\nu} = \epsilon\sqrt{-g}\,(2R^{\mu\nu} + g^{\mu\nu}R)\,\pi_{|\mu||\nu}$$

$$\delta\sqrt{-g}R^2 = 6\epsilon\sqrt{-g}\,g^{\mu\nu}R\pi_{|\mu||\nu}$$

where the double subscripted bar stands for covariant differentiation. Variation of (3) thus gives

$$\delta I = \epsilon\int \sqrt{-g}\,\left(4\epsilon\sqrt{-g}\,g_{\alpha\beta}R^{\beta\mu\alpha\nu} + 2AR^{\mu\nu} + Ag^{\mu\nu}R + 6Bg^{\mu\nu}R\right)\pi_{|\mu||\nu}d^4x$$

It might seem at this point that dividing out the $\pi_{|\mu||\nu}$ term would give us a useful identity, but that would still leave the Riemann-Christoffel term. Instead, let us integrate by parts twice over the parameter π, which gives

$$\delta I = \epsilon\int \sqrt{-g}\,\left(4R^{\mu\nu}{}_{||\mu||\nu} + 2AR^{\mu\nu}{}_{||\mu||\nu} + Ag^{\mu\nu}R_{||\mu||\nu} + 6Bg^{\mu\nu}R_{||\mu||\nu}\right)\pi\, d^4x$$

or

$$\delta I = \epsilon\int \sqrt{-g}\,\left[2(A+2)R^{\mu\nu} + (A+6B)g^{\mu\nu}R\right]_{|\mu||\nu}\pi\, d^4x \tag{4}$$

Note that in (4) we have used the interesting (and useful) identity

$$g_{\alpha\beta}R^{\beta\mu\alpha\nu}{}_{||\mu||\nu} = R^{\mu\nu}{}_{||\mu||\nu}$$

The Riemann-Christoffel term is now gone, having been converted into a divergence of the Ricci tensor. Setting (4) equal to zero, we now have

$$[2(A+2)R^{\mu\nu} + (A+6B)g^{\mu\nu}R]_{||\mu||\nu} = 0$$

One obvious solution is $A = -2$, $B = 1/3$, but that just gives us back the identity (2). However, if $A \neq -2$ then we can write

$$\delta \int \sqrt{-g}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + AR_{\mu\nu}R^{\mu\nu} + BR^2\right)d^4x = 2(A+2)\epsilon \int \pi\sqrt{-g}\left(R^{\mu\nu} + \frac{A+6B}{2(A+2)}g^{\mu\nu}R\right)_{||\mu||\nu} d^4x$$

If we now set $A + 3B = -1$ we can write this as

$$\delta \int \sqrt{-g}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + AR_{\mu\nu}R^{\mu\nu} + BR^2\right)d^4x = 2(A+2)\epsilon \int \pi\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{||\mu||\nu} d^4x$$

The term on the right contains Bianchi's identity (it's also the divergence of the Einstein tensor $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$), both of which are identically zero. This is essentially what Lanczos discovered in his 1938 paper (he selected $A = -4, B = 1$). (In 1964, DeWitt discovered the same identity using a far more complicated argument.)

Finally, by subtracting (2) from (3) the Riemann-Christoffel term cancels, leaving

$$\int \sqrt{-g}\, C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}d^4x = \int \sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)d^4x \tag{5}$$

where we have dropped the meaningless $2(A+2)$ numerical coefficient. This Lagrangian is considered the "official" Lagrangian of Weyl conformal gravity theory. Its primary advantage, other than relative simplicity, is the absence of the curvature tensor, allowing for equations of motion that are consistent with $R_{\mu\nu} = 0$.

Conformal Gravity Theory

An arbitrary variation of the metric tensor in (5) now gives

$$\delta \int \sqrt{-g}\, C_{\mu\nu\alpha\beta}\, C^{\mu\nu\alpha\beta}d^4x = \int \sqrt{-g}\, W_{\mu\nu}\delta g^{\mu\nu}d^4x$$

where $W_{\mu\nu}$ is a complicated expression involving the Ricci tensor and scalar and their derivatives (in a space where matter is present, $W_{\mu\nu}$ would be proportional to the symmetric energy tensor $T_{\mu\nu}$). An exact solution for the vacuum case $W_{\mu\nu} = 0$ has been worked out in great detail by Mannheim and Kazanas using the Schwarzschild line element

$$ds^2 = e^{\nu}(dx^0)^2 - e^{\lambda}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2$$

The solution they found is

$$e^{\nu} = e^{-\lambda} = 1 - \frac{\beta(2-3\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2$$

where β, γ and k are arbitrary constants. The resemblance of the Mannheim-Kazanas solution to the ordinary Schwarzschild solution is obvious. The additional terms very possibly have application to the solution of the galactic rotation, dark matter and dark energy problems.

References

1. C. Lanczos, *A remarkable property of the Riemann-Christoffel tensor in four dimensions*, Ann. Math. **39** 4 (1938) pp. 842-850.

2. B. DeWitt, *Dynamical theory of groups and fields*, in Relativity, Groups and Topology, C. DeWitt and B. DeWitt, eds., pp. 587-822. 1964.

3. P. D. Mannheim and D. Kazanas, *Exact vacuum solution to conformal Weyl gravity and galactic rotation curves*, Astrophys. J. 342 (1989) pp. 635-638.