THE SPIN CONNECTION IN WEYL SPACE

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"The use of general connections means asking for trouble."

Abraham Pais,
Subtle is the Lord: The Science and the Life of Albert Einstein

In addition to his seminal 1929 exposition on quantum mechanical gauge invariance[2], Hermann Weyl demonstrated how the concept of a spinor (essentially a flat-space two-component quantity with non-tensor-like transformation properties) could be carried over to the curved space of general relativity. Prior to Weyl's paper, spinors were recognized primarily as mathematical objects that transformed in the space of $SU(2)$, but in 1928 Dirac showed that spinors were fundamental to the quantum mechanical description of spin–1/2 particles (electrons). However, the spacetime stage that Dirac's spinors operated in was still Lorentzian. Because spinors are neither scalars nor vectors, at that time it was unclear how spinors behaved in curved spaces. Weyl's paper provided a means for this description using tetrads (vierbeins) as the necessary link between Lorentzian space and curved Riemannian space.

Weyl's elucidation of spinor behavior in curved space and his development of the so-called spin connection $\omega^a{}_{b\lambda}$ and the associated spin vector $\omega_\lambda = \omega_{ab\lambda}\sigma^{ab}$ was noteworthy, but his primary purpose was to demonstrate the profound connection between quantum mechanical gauge invariance and the electromagnetic field. Weyl's 1929 paper served to complete his earlier (1918) theory[1] in which Weyl attempted to derive electrodynamics from the geometrical structure of a generalized Riemannian manifold via a scale-invariant transformation of the metric tensor. This attempt failed, but the manifold he discovered (known as *Weyl space*), is still a subject of interest in theoretical physics.

Although Weyl's paper reflected upon his earlier effort, it is obvious from the 1929 paper that he had moved on, and consequently he did not address spinor descriptions in a curved Weyl space. In the following elementary discussion we pick up on this topic and consider the modifications that such a space forces upon the spin connection and its associated algebra. In particular, we shall consider the issue of *metricity* for Lorentz and coordinate vector spaces and how Weyl's geometry affects metricity in these spaces. There is no physics in this at all, mostly just index juggling, but it serves as an object lesson in the hazards of dealing with non-metric-compatible manifolds.

1. Basic Tetrad Formalism

It is always possible to find a coordinate system in which the space is Lorentzian at a given point. Einstein demonstrated this with his famous thought experiment involving a passenger on an elevator. Unless the elevator is equipped with windows, the passenger cannot know whether she is in a stationary elevator in Earth's gravitational field or if her elevator is being uniformly accelerated somewhere out in space. If she stands in one place in the elevator, then her coordinates are sufficiently *local* to the extent that slight variations in the Earth's radial gravitational field cannot be detected. However, if her elevator is sufficiently large, she could move around and discover that the gravity field is convergent (that is, points to the center of the Earth) and gets either weaker or stronger as she climbs up and down the elevator's walls.

If a *locally-flat coordinate system* can be found even in a strong gravitational field, then there must be a way to express the Lorentzian metric $\eta_{\mu\nu}$ with the curved-space metric $g_{\mu\nu}(x)$. In four-dimensional spacetime, one uses quantities called tetrads e^a_μ (or *vierbeins*, which is German for "four legs") to link the two metrics:

$$g_{\mu\nu}(x) = e^a_\mu(x)\, e^b_\nu(x)\, \eta_{ab} \tag{1.1}$$

A tetrad is a rather odd little fellow having one foot in flat space and the other in curved space. To distinguish the two with regard to tetrad notation, we will utilize Latin indices (a, b, c, etc.) for the Lorentz index and Greek indices for the curved-space part. Thus, you can think of a tetrad as a tensor quantity whose curved-space part transforms just like a coordinate vector:

$$\hat{e}^a_\mu = \frac{\partial x^\lambda}{\partial \hat{x}^\mu} e^a_\lambda$$

From (1.1), it is easy to see that we can also write

$$\eta_{ab} = e_a^{\ \mu}(x)\, e_b^{\ \nu}(x)\, g_{\mu\nu}(x)$$

provided we make the requirement that

$$e_a^{\ \mu}(x)\, e^b_{\ \mu}(x) = \delta^b_{\ a} \ \text{ and } \ e_a^{\ \mu}(x)\, e^a_{\ \nu}(x) = \delta^\mu_{\ \nu}\,, \quad \text{etc.} \tag{1.2}$$

Because each tetrad index runs from 0 to 3, there are a total of 16 components in the tetrad. Taken as a matrix, we can consider $e_a^{\ \mu}$ (with an upper Greek index) to be the tetrad inverse. Consequently, $e_a^{\ \mu} = |e|^{-1} M$, where M is the transpose of the tetrad cofactor matrix.

Tetrads in mixed Lorentz/curved-space quantities serve essentially as "index-exchange operators" because the tetrads themselves are always mixed. This characteristic will be used often in what follows.

2. Parallel Transfer and Covariant Differentiation

Consider the change in a given contravariant vector field $\xi^\mu(x)$ from point to point in some manifold. Provided ξ^μ is not a constant field, the vector $\xi^\mu(x+dx)$ at an infinitesimally-near point will differ to first order from $\xi^\mu(x)$ according to

$$\begin{aligned} \xi^\mu(x+dx) &= \xi^\mu(x) + \partial_\alpha \xi^\mu\, dx^\alpha \quad \text{or} \\ d\xi^\mu &= \partial_\alpha \xi^\mu\, dx^\alpha \end{aligned}$$

where $d\xi^\mu = \xi^\mu(x+dx) - \xi^\mu(x)$ is the total change in the vector. The quantity $d\xi^\mu$ cannot be a tensor because, as it is obtained by taking the difference between two vectors at different points in space, it is not a coordinate-independent quantity. More importantly, the partial differential $\partial_\alpha \xi^\mu$ is not a tensor. This is something of a disaster, as there seems to be no way to define vector differentiation in a covariant sense. To see this more clearly, consider how ξ^μ is transformed by a change of coordinates to the system x':

$$\xi'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}\, \xi^\nu(x) \tag{2.1}$$

The total change in $\xi'^\mu(x')$ at the neighboring point is therefore

$$d\xi'^\mu(x') = \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\alpha} \xi^\nu(x)\, dx^\alpha + \frac{\partial x'^\mu}{\partial x^\nu}\, d\xi^\nu(x) \tag{2.2}$$

Thus, $d\xi^\mu$ does not transform properly because of the second-order differential term.

Clearly, we need a prescription for vector and tensor differentiation that obeys standard tensor transformation laws. To do this, we need to be able to compute the difference $\xi^\mu(x+dx) - \xi^\mu(x)$ *at the same point*. As odd as this sounds, it is in fact possible using the concept known as *parallel transfer*.

The figure on the following page shows a vector $\xi^\mu(x)$ located at an arbitrary point x on a given curve λ. Treated as a vector field, it will have a slightly different orientation at the infinitesimally-near point $x + dx$ located elsewhere on the curve. The vector difference is given by $d\xi^\mu = \xi^\mu(x+dx) - \xi^\mu(x)$ which, as explained above, has no intrinsic geometrical significance because the vectors are separated. On the other hand, if the vector ξ^μ represented a constant vector field (that is, $d\xi^\mu = 0$), then the separation clearly would not matter anymore, as the vectors at x and $x + dx$ would be (trivially) identical. Under the change of coordinates (2.1), however, the new vector ξ'^μ would clearly be a function of its coordinates, and (2.2) would become

$$\begin{aligned} d\xi'^\mu(x') &= \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\alpha} \xi^\nu dx^\alpha \\ &= \frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\alpha} \frac{\partial x^\nu}{\partial x'^\lambda} \frac{\partial x^\alpha}{\partial x'^\beta} \xi'^\lambda\, dx'^\beta \end{aligned} \tag{2.3}$$

where in the last line we have transformed everything into the primed coordinate system. Eq.(2.3), which represents a transformed constant vector field, can be more succinctly written as

$$d\xi'^\mu = \Gamma'^\mu_{\lambda\beta}(x')\, \xi'^\lambda dx'^\beta \tag{2.4}$$

where

$$\Gamma'^{\mu}_{\lambda\beta} = \frac{\partial^2 x'^{\mu}}{\partial x^{\nu}\partial x^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\lambda}} \frac{\partial x^{\alpha}}{\partial x'^{\beta}}$$

The (non-tensor) quantities $\Gamma'^{\mu}_{\lambda\beta}$ are called *coefficients of affine connection* because they affinely (linearly) relate the change in a vector with the vector itself and the transport distance dx. With the use of (2.1), it can easily be shown that the connections transform according to

$$\Gamma'^{\mu}_{\lambda\beta}(x') = \frac{\partial x^{\nu}}{\partial x'^{\lambda}} \frac{\partial x^{\sigma}}{\partial x'^{\beta}} \frac{\partial x'^{\mu}}{\partial x^{\beta}} \Gamma^{\beta}_{\nu\sigma}(x) + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu}\partial x^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\lambda}} \frac{\partial x^{\alpha}}{\partial x'^{\beta}}$$

We now consider (2.4) to be the quantity which, when added to a vector field at x, effectively resurrects the original vector at the point $x + dx$. Taken as such, we will rename (2.4) as

$$\delta\xi^{\mu} = \Gamma^{\mu}_{\lambda\beta}\, \xi^{\lambda} dx^{\beta} \tag{2.5}$$

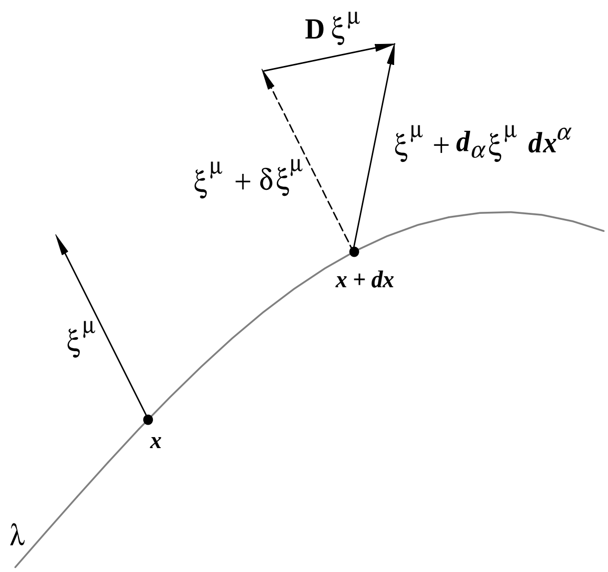
Thus, $\delta\xi^{\mu}$ is a non-vector quantity which, when added to $\xi^{\mu}(x)$, produces a parallel copy of the vector at the neighboring point on the curve. The vector quantity $\xi^{\mu}(x) + \delta\xi^{\mu}(x)$ therefore represents the parallel-transferred vector which can now be compared with the vector $\xi^{\mu}(x) + \partial_{\beta}\xi^{\mu}$ in a truly covariant sense, since they both occur at the same spacetime point. It can easily be shown that the difference

$$\partial_{\beta}\xi^{\mu} dx^{\beta} - \delta\xi^{\mu} = \left[\partial_{\beta}\xi^{\mu} - \Gamma^{\mu}_{\lambda\beta}\, \xi^{\lambda}\right] dx^{\beta}$$

is in fact a tensor. We define $\partial_{\beta}[\;\dot{}\;]dx^{\beta} - \delta$ to be the *covariant derivative operator* D. The associated covariant derivative of ξ^{μ} is then defined by

$$D_{\beta}\xi^{\mu} = \partial_{\beta}\xi^{\mu} - \Gamma^{\mu}_{\lambda\beta}\, \xi^{\lambda} \tag{2.6}$$

so that $D/dx^{\beta} = D_{\beta}$. The covariant derivative is of paramount importance in differential geometry and Einstein's theory of general relativity, where the coefficients of affine connection account for the presence of gravitational fields.



Eq (2.6) defines the covariant derivative of a contravariant vector, and the same process can be developed for covariant vectors. Consider a scalar field $\eta(x)$ defined by the invariant product $\eta = \xi^{\mu}\varphi_{\mu}$; partial differentiation of this with respect to x^{α} gives

$$\begin{aligned} \partial_{\alpha}\eta &= \partial_{\alpha}\xi^{\mu}\varphi_{\mu} + \xi^{\mu}\partial_{\alpha}\varphi_{\mu} \\ &= D_{\alpha}\xi^{\mu}\varphi_{\mu} + \Gamma^{\mu}_{\lambda\alpha}\, \xi^{\lambda}\varphi_{\mu} + \xi^{\mu}\partial_{\alpha}\varphi_{\mu} \end{aligned}$$

Using the fact that the difference $\partial_\alpha \eta - D_\alpha \xi^\mu \varphi_\mu$ is a covariant vector, it is obvious that the quantity $\partial_\alpha \varphi_\mu + \Gamma^\lambda_{\mu\alpha}\varphi_\lambda$ is a covariant tensor. We therefore define the derivative of a covariant vector as

$$D_\nu \xi_\mu = \partial_\nu \xi_\mu + \Gamma^\lambda_{\mu\nu}\,\xi_\lambda$$

The main difference between this and (2.6) is the changed sign in the connection term. The application of these rules to mixed and unmixed tensors of any rank should be obvious.

How does parallel transfer affect non-vector quantities? Obviously, the key difference between a vector like ξ^μ and a tensor like $g^{\mu\nu}$ lies in the fact that vectors have specific orientations or directions at every spacetime point, whereas tensors do not. Consequently, a parallel-transferred tensor follows the standard partial differentiation rule ($\delta g^{\mu\nu} = \partial_\alpha g^{\mu\nu} dx^\alpha$, etc.).

Now that we have defined covariant differentiation for contravariant and covariant vectors and tensors, it is natural to ask what the quantity $\delta\xi_\mu$ represents. To answer this, consider the definition of vector magnitude given by $l^2 = \xi^\mu \xi_\mu$. Then parallel transfer gives

$$2l\,\delta l = \Gamma^\mu_{\lambda\alpha}\,\xi_\mu \xi^\lambda dx^\alpha + \xi^\mu\,\delta\xi_\mu$$

As it is reasonable to assume that vector length should not change under parallel transfer, rearrangement of the above expression for $\delta l = 0$ shows that $\delta\xi_\mu = -\Gamma^\lambda_{\mu\alpha}\xi_\lambda dx^\alpha$. Later we will consider the case where δl does not vanish.

The notion of parallel transfer described here is necessarily very elementary, but it is given primarily as a reminder of how the concept of covariant differentiation is derived. It should not be surprising that the notion of parallel transfer of spinor fields can also be derived along similar lines (and involving a different kind of connection term), and we will do just that shortly. For now it will suffice to note that covariant differentiation of tensor fields will be denoted by $D_\mu(\Gamma)$, where the argument is a reminder that we are dealing with coefficients of affine connection.

Because the second-order differential term in (2.3) in symmetric with regard to the lower indices, the affine coefficient is also symmetric. Over the years there have been efforts by many physicists (notably Einstein) to develop theories involving non-symmetric coefficients, but we will not consider them here.

It should be noted that the precise makeup of the connections is, up to now, completely arbitrary. The connections can presumably include terms involving gravitational and electromagnetic fields (and other fields as well), but all efforts to define the connections outside of the metric tensor $g_{\mu\nu}$ and its first and second derivatives have failed. Weyl was the first researcher to seriously attempt the embedding of electrodynamics into gravity via a generalized connection term, but this attempt also failed. To date, gravity is the only theory that has been successfully encoded into the connection.

Lastly, it should be remarked that most authors use a different sign convention for the affine connection than the one used here. It really makes no difference but, for those of you who may be a tad confused, just replace $\Gamma^\mu_{\lambda\beta}$ with $-\Gamma^\mu_{\lambda\beta}$ and everything should make sense

3. Metric Compatibility and Weyl's 1918 Gauge Theory

By way of review, Riemannian space is characterized by two conditions: symmetry of the connection coefficients ($\Gamma^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha}$) and the vanishing of the metric covariant derivative, $D_\lambda(\Gamma) g_{\mu\nu} = D_\lambda(\Gamma) g^{\mu\nu} = 0$. The latter condition enforces the invariance of vector length or magnitude, and also identifies the connection with the Christoffel symbol:

$$\begin{aligned} \Gamma^\lambda_{\alpha\beta} &= -\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}, \quad \text{where} \qquad (3.1) \\ \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} &= \frac{1}{2} g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) \end{aligned}$$

Riemannian space is therefore called *metric compatible*, and the statement $D_\lambda(\Gamma) g_{\mu\nu} = 0$ is called *metricity.* As a consequence, Riemannian space preserves the lengths of vectors under parallel transfer. To see this, we parallel-transfer vector length given by $l^2 = g_{\mu\nu} A^\mu A^v$, giving, after some rearrangement,

$$\begin{aligned} 2l\,\delta l &= \partial_\alpha g_{\mu\nu} A^\mu A^\nu dx^\alpha + g_{\mu\nu}\delta A^\mu A^\nu + g_{\mu\nu} A^\mu \delta A^\nu \\ &= D_\alpha(\Gamma) g_{\mu\nu}\, A^\mu A^\nu dx^\alpha \qquad (3.2) \end{aligned}$$

which of course vanishes in Riemannian space.

Weyl noted that, by relaxing the requirement of constant vector magnitude, he could generalize the Riemannian manifold. He assumed that the physical change in vector length under parallel transfer, if not identically zero, should be proportional to its original length, and he wrote

$$\delta l = \phi_\alpha dx^\alpha l$$

where $\phi_\alpha(x)$ is a new field that Weyl subsequently identified with the electromagnetic four-potential. Inserting this identity into (3.2), Weyl noted that his new spacetime geometry (now called a *Weyl space*) was no longer metric compatible:

$$D_\alpha(\Gamma) g_{\mu\nu} = 2 g_{\mu\nu} \phi_\alpha \tag{3.3}$$

Weyl went on to derive an expression for the connection in terms of the metric and the ϕ–field, and noted that the connection was invariant with respect to a local rescaling of the metric tensor, $g_{\mu\nu} \to \lambda(x) g_{\mu\nu}$, where λ is an arbitrary function of the coordinates. Weyl gave this rescaling the name *Eichinvarianz*, meaning "gauge invariance." While Weyl was ultimately forced to abandon his hope that this non-Riemannian geometry would unify gravitation and electrodynamics, he subsequently demonstrated that gauge invariance, suitably applied to the quantum field Ψ, provided the hoped-for connection with electrodynamics via gauge-invariant Lagrangians. Today, the gauge invariance principle, Weyl's brainchild, is one of the most seminal concepts in modern theoretical physics.

4. Spinors in Curved Space

Recall that Dirac's equation in an electromagnetic field is

$$i\gamma^\mu (\partial_\mu - \frac{ie}{\hbar c} A_\mu)\psi - \frac{mc}{\hbar}\psi = 0$$

where A_μ is the electromagnetic 4-potential in some given set of units. The presence of an electromagnetic field mandates the transformation of the simple partial differential operator from ∂_μ to $\partial_\mu - ie/\hbar c\, A_\mu$ (which is called the covariant derivative for spinor quantities). This change is also necessary if the Dirac-Maxwell Lagrangian is to be gauge invariant. One might reasonably expect that a similar form holds for the covariant derivative of a spinor in curved space. Indeed, Weyl had the insight to recognize this identification holds as a general principle, and he expressed the curved-space derivative of the Dirac spinor as

$$\begin{aligned} \partial_\mu \psi &\to D_\mu \psi \\ &= (\partial_\mu + \Gamma_\mu)\psi \end{aligned} \tag{4.1}$$

where $\Gamma_\mu(x)$ is some 4×4 matrix that makes the Dirac equation valid for curved space.

As for the transformation properties of $\psi(x)$ itself, consider the change in ψ that results from an infinitesimal pure displacement:

$$\begin{aligned} \psi(x + dx) &= \psi(x) + \partial_\mu \psi(x)\, dx^\mu \quad \text{or} \\ d\psi &= \partial_\mu \psi(x)\, dx^\mu \end{aligned}$$

where $d\psi = \psi(x + dx) - \psi(x)$.We can then demand that the total change in ψ in a curved space under parallel transfer be

$$\begin{aligned} \psi(x + dx) &= \psi(x) + \Gamma_\mu \psi(x)\, dx^\mu \quad \text{or} \\ D\psi &= \Gamma_\mu\, \psi(x)\, dx^\mu \quad \text{and} \\ D\psi^\dagger &= \psi^\dagger\, \Gamma^\dagger_\mu(x)\, dx^\mu \end{aligned} \tag{4.2}$$

This is our starting point for the derivation of the field Γ_μ.

5. Coordinate and Lorentz Vectors in Curved Space

In order to derive $\Gamma_\mu(x)$, we will utilize the equivalence of vectors expressed in what is called the coordinate form (or *C-form*) $V^\mu(x)$ and the Lorentz form (*L-form*) $V^a(x)$, where

$$V^a = e^a_{\ \mu}\, V^\mu \tag{5.1}$$

and vice versa. The magnitude or length L of these vectors is the same for both forms:

$$L^2 = \eta_{ab} V^a V^b = g_{\mu\nu}(x) V^\mu V^\nu$$

The use of L-vectors presents an immediate problem: how do they transform under parallel displacement? In similarity with (2.4), we assume the existence of an L-form connection term such that

$$D(\omega) V^a = \omega^a{}_{b\lambda} V^b dx^\lambda$$

where $\omega^a{}_{b\lambda}(x)$ is called the *spin connection*. Although the spin connection and the metric connection $\Gamma^\alpha_{\mu\nu}$ can be viewed as different versions of the same quantity, we shall see that the spin connection has different symmetry properties with respect to its indices.

The notion of covariant differentiation can also be specified for L-forms. We define the covariant derivative of the vector V^a as

$$D_\lambda(\omega) V^a = \partial_\lambda V^a - \omega^a{}_{b\lambda} V^b$$

Furthermore, we will define the *total covariant derivative* of a "mixed" tensor as

$$D_\lambda(\omega + \Gamma) T^a_{\ \beta} = \partial_\lambda T^a_{\ \beta} + \Gamma^\mu_{\beta\lambda} T^a_{\ \mu} - \omega^a{}_{b\lambda} T^b_{\ \beta}$$

Of particular interest is the total covariant derivative of the Lorentz metric η_{ab}:

$$\begin{aligned} D_\lambda(\omega + \Gamma)\eta_{ab} &= D_\lambda(\omega)\eta_{ab} \\ &= \eta_{as}\omega^s{}_{b\lambda} + \eta_{sb}\omega^s{}_{a\lambda} \\ &= \omega_{ab\lambda} + \omega_{ba\lambda} \end{aligned}$$

If we make the reasonable demand that the Lorentz metric be constant under parallel transfer, its covariant derivative should vanish; the lower-index spin connection must then be antisymmetric in its first two indices:

$$\omega_{ab\lambda} = -\omega_{ba\lambda} \tag{5.2}$$

6. The Tetrad Postulate

Let us parallel-transfer the vector relation expressed in 5.1:

$$\begin{aligned} D(\omega + \Gamma) V^a &= V^\mu \, D(\omega + \Gamma) e^a_{\ \mu} + e^a_{\ \mu} \, D(\omega + \Gamma) V^\mu \\ \omega^a{}_{b\lambda} V^b dx^\lambda &= V^\mu \, \partial_\lambda e^a_{\ \mu} \, dx^\lambda + e^a_{\ \mu} \, \Gamma^\mu_{\alpha\lambda} V^\alpha dx^\lambda \end{aligned}$$

(Remember that the tetrad is not a vector, so it transfers via the partial derivative.) Relabeling indices, we get

$$e^s_{\ \mu} \omega^a{}_{s\lambda} V^\mu dx^\lambda = V^\mu \, \partial_\lambda e^a_{\ \mu} \, dx^\lambda + e^{\ \mu}_{\alpha} \, \Gamma^\alpha_{\mu\lambda} V^\mu dx^\lambda$$

Dropping the common $V^\mu dx^\lambda$ term, we have

$$e^s_{\ \mu} \omega^a{}_{s\lambda} = \partial_\lambda e^a_{\ \mu} + e^{\ \mu}_{\alpha} \, \Gamma^\alpha_{\mu\lambda} \tag{6.1}$$

However, this is just the statement that the total covariant derivative of the tetrad vanishes:

$$D_\lambda(\omega + \Gamma) e^a_{\ \mu} = 0 \tag{6.2}$$

This important result is known as the *tetrad postulate*.

It will now be instructive to show from the tetrad postulate the explicit relationship between the two connections $\Gamma^\alpha_{\mu\lambda}$ and $\omega^a{}_{s\lambda}$. Using (1.2) and (6.1), it is easy to see that

$$\Gamma^\lambda_{\mu\nu} = -e_a^{\ \lambda} \, \partial_\nu e^a_{\ \mu} + e_a^{\ \lambda} \, e^s_{\ \mu} \, \omega^a{}_{s\nu} \tag{6.3}$$

This is a most interesting result – the connection separates into two terms, one representing the Christoffel term (2.1) and another involving spin. From this, we see already that the spin connection cannot be completely metrical. (The apparent economy of notation in expressing the Christoffel symbol in terms of tetrads is illusory, as the inverse tetrad e_a^{λ} is a rather messy expression.)

Because the connection is symmetric in its lower indices, the expression (6.3) is problematic. For one thing, the corresponding Christoffel term $e_a^{\lambda}\,\partial_\nu e^a_{\mu}$ is obviously not symmetric in μ and ν (nor can it be symmetrized). For another, we see that the spin term can absorb the tetrad indices to become $\omega^{\lambda}_{\ \mu\nu}$, implying that the this quantity is also symmetric. However, we will show later that the spin connection has no such symmetry. There is also a difficulty with the contracted connection; clearly, contraction can only occur between λ and μ, not ν:

$$\begin{aligned}\Gamma^{\lambda}_{\lambda\nu} &= -e_a^{\lambda}\,\partial_\nu e^a_{\ \lambda} + e_a^{\lambda}\,e^s_{\ \lambda}\,\omega^a_{\ s\nu} \\ &= -e_a^{\lambda}\,\partial_\nu e^a_{\ \lambda} + \omega_\nu \end{aligned} \tag{6.4}$$

where $e_a^{\lambda}\,\partial_\nu e^a_{\lambda} = \partial_\nu \log\sqrt{-g}$ and $\omega_\nu = \omega^{\lambda}_{\ \lambda\nu}$. As we will see, ω_ν is identically zero in a metric-compatible space. Evidently, a non-metric-compatible space must accommodate a contracted spin connection. We will see that this is in fact the case in Weyl space, but there are dark clouds ahead.

7. Derivation of Γ_μ

Consider the Dirac scalar quantity

$$\begin{aligned} I &= \overline{\psi}\psi \\ &= \psi^\dagger\gamma^0\psi \end{aligned}$$

where γ^0 is the time-coordinate gamma matrix in either the Weyl or Dirac representation,

$$\begin{aligned}\psi^0 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad \text{(Weyl representation)} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad \text{(Dirac representation)} \end{aligned}$$

We assume that the Dirac matrices are expressible in either C-form or L-form, even though they are not vector quantities:

$$\gamma^\mu(x) = e_a^{\mu}\,\gamma^a$$

In doing so, we must allow for the possibility that the C-form gamma matrices can be functions of the coordinates. Consequently, we will write γ^0 as $\gamma^0(x)$, despite the fact that we have no idea what these coordinate-dependent matrices might look like. Under parallel transfer of the scalar I, we then have

$$\begin{aligned} DI &= D\psi^\dagger(x)\,\gamma^0(x)\,\psi(x) + \psi^\dagger(x)\,D\gamma^0(x)\,\psi(x) + \psi^\dagger(x)\,\gamma^0(x)\,D\psi(x) \\ &= \psi^\dagger\,\Gamma^\dagger_\mu\,\gamma^0\psi dx^\mu + \psi^\dagger\partial_\mu\gamma^0\psi dx^\mu + \psi^\dagger\,\gamma^0\Gamma_\mu\psi dx^\mu = 0 \end{aligned}$$

The terms $\psi^\dagger$ and $\psi\,dx^\mu$ bracket each of these quantities; dropping them, we thus have the requirement

$$\Gamma^\dagger_\mu\,\gamma^0 + \partial_\mu\gamma^0 + \gamma^0\Gamma_\mu = 0 \tag{7.1}$$

Now, the quantity $\overline{\psi}\gamma^\lambda\psi$ is a coordinate vector; let us call it $V^\lambda(x)$. We assume that this vector is parallel-transferred in the same manner as it is in general relativity, which is

$$\delta V^\lambda = \Gamma^\lambda_{\alpha\mu}(x)\,V^\alpha dx^\mu$$

Similarly, we shall assume that $V^a = \psi\gamma^a\psi$ is an L-vector. Parallel-transferring the identity $V^\lambda = \overline{\psi}\gamma^\lambda\psi$ then gives

$$\begin{aligned} DV^\lambda &= D\psi^\dagger\,\gamma^0\gamma^\lambda\,\psi + \psi^\dagger\,D\gamma^0\gamma^\lambda\,\psi + \psi^\dagger\,\gamma^0 D\gamma^\lambda\,\psi + \psi^\dagger\,\gamma^0\gamma^\lambda\,D\psi \\ &= \psi^\dagger\,\Gamma^\dagger_\mu\,\gamma^0\gamma^\lambda\psi dx^\mu + \psi^\dagger(\partial_\mu\gamma^0)\gamma^\lambda\psi dx^\mu + \psi^\dagger\gamma^0\partial_\mu\gamma^\lambda\psi dx^\mu + \psi^\dagger\,\gamma^0\gamma^\lambda\Gamma_\mu\psi dx^\mu \\ &= \Gamma^\lambda_{\alpha\mu}\,V^\alpha dx^\mu \\ &= \psi^\dagger\,\Gamma^\lambda_{\ \alpha\mu}\,\gamma^\alpha\psi dx^\mu \end{aligned} \tag{7.2}$$

Cancelling common terms, we have

$$\Gamma_\mu^\dagger\,\gamma^0\gamma^\lambda + (\partial_\mu\gamma^0)\gamma^\lambda + \gamma^0\partial_\mu\gamma^\lambda + \gamma^0\gamma^\lambda\Gamma_\mu = \Gamma^\lambda_{\ \alpha\mu}\,\gamma^\alpha$$

Using (7.1), we can get rid of the $\Gamma_\mu^\dagger\,\gamma^0$ term and thus arrive at the the elegant identity

$$D_\mu(\Gamma)\gamma^\lambda = \Gamma_\mu\gamma^\lambda - \gamma^\lambda\Gamma_\mu \tag{7.3}$$

where $D_\mu(\Gamma)\gamma^\lambda = \partial_\mu\gamma^\lambda - \gamma^\alpha\Gamma^\lambda_{\alpha\mu}$. Note that the $\partial_\mu\gamma^0$ term vanished when we inserted (7.1) into (7.2).

Similarly, for the L-form $V^a = \overline{\psi}\gamma^a\psi$, we have the differential quantity

$$\begin{aligned} DV^a &= D\psi^\dagger\,\gamma^0\gamma^a\,\psi + \psi^\dagger\,D\gamma^0\gamma^a\,\psi + \psi^\dagger\,\gamma^0\gamma^a\,D\psi \\ &= \psi^\dagger\,\Gamma_\mu^\dagger\,\gamma^0\gamma^a\psi dx^\mu + \psi^\dagger(\partial_\mu\gamma^0)\gamma^a\psi dx^\mu + \psi^\dagger\,\gamma^0\gamma^a\Gamma_\mu\psi dx^\mu \\ &= \omega^a_{\ b\mu}V^b dx^\mu = \psi^\dagger\omega^a_{\ b\mu}\,\gamma^b\psi dx^\mu \end{aligned}$$

(Note that $d\gamma^a = 0$ because γ^a are the constant Dirac matrices.) Again, the spinor terms $\psi^\dagger$ and ψdx^μ bracket the other quantities, and we're left with

$$\omega^a_{\ b\mu}\,\gamma^b = \gamma^a\Gamma_\mu - \Gamma_\mu\,\gamma^a \tag{7.4}$$

By using the tetrad identity $\gamma^b = e^b_{\ \mu}\,\gamma^\mu$, it is easy to show from this and (6.3) that

$$\omega^a_{\ b\mu} = e_b^{\ \lambda}\,D_\mu(\Gamma)e^a_{\ \lambda}$$

Let us now consider the length of some L-vector V^a, which is

$$l^2 = \eta_{ab}V^aV^b \tag{7.5}$$

Under parallel transfer, the total change in the length is then

$$\begin{aligned} 2l\,Dl &= \eta_{ab}V^aDV^b + \eta_{ab}V^bDV^a \\ &= \eta_{ab}V^a\omega^b_{\ s\mu}V^sdx^\mu + \eta_{ab}V^b\omega^a_{\ s\mu}V^sdx^\mu \\ &= (\omega_{ab\mu} + \omega_{ba\mu})V^aV^bdx^\mu \end{aligned}$$

(note that we have lowered the upper index on $\omega^a_{\ s\mu}$ with the L-metric). Since dl vanishes in Riemannian space, we see confirmation that the lower-indexed connection $\omega_{ab\mu}$ is antisymmetric with respect to its L-indices.

Now consider (7.4), whose lower-index form is

$$\omega_{ab\mu}\,\gamma^b = \gamma_a\Gamma_\mu - \Gamma_\mu\,\gamma_a$$

Left-multiplying by γ^a and rearranging, we get

$$4\Gamma_\mu = \omega_{ba\mu}\,\gamma^a\gamma^b + \gamma_a\Gamma_\mu\gamma^a$$

where we have used the fact that $\gamma^a\gamma_a = 4$. Unfortunately, Γ_μ is also present on the rhs of this expression. In order to solve for it, let's pre- and post-multiply both sides by γ_a and γ^a, respectively. This gives

$$4\gamma_a\Gamma_\mu\gamma^a = \omega_{ba\mu}\,\gamma_c\gamma^a\gamma^b\gamma^c + \gamma_b\gamma_a\Gamma_\mu\gamma^a\gamma^b$$

where care has been taken in labeling the indices. Evaluation of the term $\omega_{ba\mu}\,\gamma_c\gamma^a\gamma^b\gamma^c$ requires a trick: we have to move the γ^c term over to its partner γ_c, where they can cancel each other (remember that $\gamma_c\gamma^c = 4$). We do this in two steps, first by writing $\gamma^b\gamma^c = 2\eta^{bc} - \gamma^c\gamma^b$ and then $\gamma^a\gamma^c = 2\eta^{ac} - \gamma^c\gamma^a$. This gives

$$\begin{aligned} \omega_{ba\mu}\,\gamma_c\gamma^a\gamma^b\gamma^c &= (\omega_{ba\mu} + \omega_{ab\mu})(\gamma^a\gamma^b + \gamma^b\gamma^a) \\ &= 2\eta^{ab}(\omega_{ba\mu} + \omega_{ab\mu}) \\ &= 0 \end{aligned}$$

We now have

$$\begin{aligned} 4\gamma_a \Gamma_\mu \gamma^a &= \gamma_b \gamma_a \Gamma_\mu \gamma^a \gamma^b \quad \text{or} \\ \gamma_a \Gamma_\mu \gamma^a &= \frac{1}{4} \gamma_b \gamma_a \Gamma_\mu \gamma^a \gamma^b \end{aligned}$$

If we now pre- and post-multiply this expression again by γ_a and γ^a , we get

$$\gamma_a \Gamma_\mu \gamma^a = \frac{1}{16} \gamma_c \gamma_b \gamma_a \Gamma_\mu \gamma^a \gamma^b \gamma^c$$

After k such iterations, we have

$$\gamma_a \Gamma_\mu \gamma^a = \frac{1}{4^{k+1}} \left[...\gamma_d \gamma_c \gamma_b \gamma_a \Gamma_\mu \gamma^a \gamma^b \gamma^c \gamma^d ... \right]$$

Provided the term in brackets remains finite, we will have $\gamma_a \Gamma_\mu \gamma^a = 0$ as $k \to \infty$. The sought-after identity for the spin vector is then

$$\begin{aligned} \Gamma_\mu &= \frac{1}{4} \omega_{ba\mu} \, \gamma^a \gamma^b \\ &= \frac{1}{8} \omega_{ba\mu} \, \sigma^{ab} \end{aligned} \tag{7.6}$$

where $\sigma^{ab} = \gamma^a \gamma^b - \gamma^b \gamma^a$.

8. The Spin Connection in Weyl Space

Since metricity is not preserved in Weyl space, we expect that some changes will have to be made to the above findings. However, the tetrad postulate remains valid in Weyl space, and we can easily determine the total covariant derivative of $g_{\mu\nu} = e^a_\mu \, e^b_\nu \, \eta_{ab}$:

$$\begin{aligned} D_\lambda(\Gamma + \omega) g_{\mu\nu} &= D_\lambda(\Gamma) g_{\mu\nu} \\ &= D_\lambda(\Gamma + \omega) e^a_\mu \, e^b_\nu \, \eta_{ab} \\ &= e^a_\mu \, e^b_\nu D_\lambda(\omega) \eta_{ab} \\ &= e^a_\mu \, e^b_\nu (n_{ac} \omega^c{}_{b\lambda} + n_{cb} \omega^c{}_{a\lambda}) \\ &= e^a_\mu \, e^b_\nu (\omega_{ab\lambda} + \omega_{ba\lambda}) \end{aligned}$$

But now we have the Weyl identity $D_\lambda(\Gamma) g_{\mu\nu} = 2 g_{\mu\nu} \phi_\lambda$, so that

$$2 g_{\mu\nu} \phi_\lambda = e^a_\mu \, e^b_\nu (\omega_{ab\lambda} + \omega_{ba\lambda})$$

Contraction with $g^{\mu\nu}$ leads to

$$\begin{aligned} 2n\phi_\lambda &= g^{\mu\nu} e^a_\mu \, e^b_\nu (\omega_{ab\lambda} + \omega_{ba\lambda}) \\ &= \eta^{ab} (\omega_{ab\lambda} + \omega_{ba\lambda}) \\ &\equiv 2\omega_\lambda \end{aligned}$$

Thus, in a Weyl space we have $\omega_\mu = 4\phi_\mu$, again indicating that the non-metricity inherent in Weyl space is connected with a non-zero contracted spin connection. In a non-trivial Weyl space this quantity cannot vanish, so the usual antisymmetry property for $\omega_{ab\lambda}$ (5.2) must be modified. What can that property be?

To answer that question, consider the definition for the invariant line element,

$$ds^2 = \eta_{ab} dx^a dx^b$$

The associated unit L-vector $U^a = dx^a/ds$ cannot change length under parallel transfer, as its length is unity, so we must have

$$D(\omega) \eta_{ab} U^a U^b = (\eta_{as} \omega^s{}_{b\lambda} + \eta_{sb} \omega^s{}_{a\lambda}) U^a U^b dx^\lambda + \eta_{ab} U^s \omega^a{}_{s\lambda} U^b dx^\lambda + \eta_{ab} U^a U^s \omega^b{}_{s\lambda} dx^\lambda = 0$$

or

$$\begin{aligned} 0 &= \omega_{ab\lambda} U^a U^b U^\lambda \\ &= \omega_{abc} U^a U^b U^c \end{aligned}$$

where we have used the identity $U^c = e^c_{\ \lambda} U^\lambda$. The lower-case spin connection ω_{abc} cannot be identically zero, so it must satisfy the peculiar cyclic symmetry condition

$$\omega_{abc} + \omega_{cab} + \omega_{bca} = 0 \tag{8.1}$$

Contraction with η^{ab} gives

$$\eta^{ab}(\omega_{abc} + \omega_{cab} + \omega_{bca}) = \omega_c + \eta^{ab}\omega_{cab} + \eta^{ab}\omega_{bca} = 0$$

which shows that ω_{abc} must be antisymmetric in the "wrong" index pair, $\omega_{abc} = -\omega_{acb}$. Because of this, the spin vector Γ_μ that we derived earlier must also be wrong, as the procedure we used to arrive at (7.6) no longer holds. The same calculation that we did to derive (7.6) leads to

$$\Gamma_\mu = \lim_{k\to\infty} \left\{ \frac{1}{4}\omega_{ba\mu}\,\gamma^a\gamma^b + \frac{1}{4}k\omega_\mu + \frac{1}{4^{k+1}}\left[...\gamma_d\gamma_c\gamma_b\gamma_a\Gamma_\mu\gamma^a\gamma^b\gamma^c\gamma^d...\right]\right\} \tag{8.2}$$

which blows up unless ω_μ is identically zero. Thus, in Weyl space there can be no spinor covariant derivative!

In view of (8.1). it is interesting to note that a similar cyclic symmetry property holds for the C-form unit vector U^μ, which can be derived from parallel-transfer in the analogous C-form space:

$$D(\Gamma)g_{\mu\nu}U^\mu U^\nu = [D_\lambda(\Gamma)g_{\mu\nu}]U^\mu U^\nu U^\lambda = 0$$

From this we see that either the metric covariant derivative is identically zero, or it satisfies the peculiar cyclic symmetry condition

$$D_\lambda(\Gamma)g_{\mu\nu} + D_\nu(\Gamma)g_{\lambda\mu} + D_\mu(\Gamma)g_{\nu\lambda} = 0 \tag{8.3}$$

Note, however, that Weyl's identity for the metric covariant derivative, $D_\lambda(\Gamma)g_{\mu\nu} = 2g_{\mu\nu}\phi_\lambda$, does not satisfy this condition. This not only throws doubt on Weyl's 1918 theory but on the concept of a Weyl space itself.

9. Alternatives to Weyl Space

The basis for Weyl's 1918 theory is $D_\alpha g_{\mu\nu} = 2g_{\mu\nu}\phi_\alpha$, which results from his assumption that vector magnitude parallel-transfers according to $\delta l = \phi_\alpha dx^\alpha l$. Using this definition, it is easy to show that the connection in Weyl space is

$$\Gamma^\lambda_{\mu\nu} = -\begin{Bmatrix}\lambda \\ \mu\nu\end{Bmatrix} + \delta^\lambda_\mu\,\phi_\nu + \delta^\lambda_\nu\,\phi_\mu - g_{\mu\nu}g^{\lambda\beta}\phi_\beta$$

Weyl's primary reason for believing that this connection was physically meaningful lies in the fact that it is invariant with respect to a change of scale in the metric tensor, $g_{\mu\nu} \to \lambda(x)g_{\mu\nu}$. Since the equations of electrodynamics are invariant with respect to a regauging of the electromagnetic four-potential, Weyl believed his connection might be used to unify gravitation with electrodynamics. In view of the difficulties outlined above regarding non-metricity, it is not surprising that Weyl's attempt failed, although historically the failure was attributed to physical, rather than mathematical, arguments.

But could not Weyl's theory be generalized to produce a consistent, non-metric-compatible theory? Consider the following modification to the Weyl connection

$$\Gamma^\lambda_{\mu\nu} = -\begin{Bmatrix}\lambda \\ \mu\nu\end{Bmatrix} + a\,\delta^\lambda_\mu\,\phi_\nu + b\,\delta^\lambda_\nu\,\phi_\mu + c\,g_{\mu\nu}g^{\lambda\beta}\phi_\beta$$

where a, b and c are arbitrary constants. To maintain symmetry in the connection indices, we must have $a = b$. We now note that this connection must satisfy the cyclic symmetry operation shown in (8.3). It can be shown without difficulty that this requires $c = -2a$, so the the "revised" Weyl connection is

$$\Gamma^\lambda_{\mu\nu} = -\begin{Bmatrix}\lambda \\ \mu\nu\end{Bmatrix} + a\,\delta^\lambda_\mu\,\phi_\nu + a\,\delta^\lambda_\nu\,\phi_\mu - 2a\,g_{\mu\nu}g^{\lambda\beta}\phi_\beta$$

Similarly, it is easy to show that the parallel transfer formulas for vectors are now

$$\begin{aligned}\delta\xi^{\lambda} &= -\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}\xi^{\mu}dx^{\nu} + a\,\xi^{\lambda}\phi_{\nu}dx^{\nu} + a\,\delta^{\lambda}_{\nu}\xi^{\mu}\phi_{\mu}\,dx^{\nu} - 2a\,g_{\beta\nu}g^{\mu\lambda}\xi^{\beta}\phi_{\mu}\,dx^{\nu}\\ \delta\xi_{\lambda} &= \begin{Bmatrix}\mu\\ \lambda\nu\end{Bmatrix}\xi_{\mu}dx^{\nu} + a\,\xi_{\lambda}\phi_{\nu}dx^{\nu} + a\,g_{\lambda\nu}g^{\mu\beta}\xi_{\mu}\phi_{\beta}\,dx^{\nu} - 2a\,\xi_{\nu}\phi_{\lambda}dx^{\nu}\end{aligned}$$

These expressions are consistent with the parallel-transfer formula for the length of the vector ξ^{λ}, which is

$$\begin{aligned}2l\delta l &= D_{\nu}g_{\mu\alpha}\xi^{\mu}\xi^{\alpha}dx^{\nu}\\ &= 2al^2\phi_{\nu}dx^{\nu} - 2a\xi_{\nu}\xi^{\mu}\phi_{\mu}dx^{\nu}\end{aligned} \tag{9.1}$$

where

$$\begin{aligned}D_{\nu}g_{\mu\alpha} &= 2ag_{\mu\alpha}\phi_{\nu} - ag_{\mu\nu}\phi_{\alpha} - ag_{\alpha\nu}\phi_{\mu}\\ D_{\nu}g^{\mu\alpha} &= -2ag^{\mu\alpha}\phi_{\nu} + a\delta^{\mu}_{\nu}g^{\alpha\beta}\phi_{\beta} + a\delta^{\alpha}_{\nu}g^{\mu\beta}\phi_{\beta}\end{aligned}$$

Furthermore, by contracting $D_{\nu}g_{\mu\alpha}$ with $g^{\mu\alpha}$ we have

$$\phi_{\nu} = \frac{1}{6a}g^{\mu\alpha}D_{\nu}g_{\mu\alpha}$$

Thus, if ϕ_{ν} is a given external field, it induces a non-zero metric covariant derivative consistent with this identity.

This development seems to be the most reasonable generalization of Weyl space possible. But there are major problems with it that cannot be overcome. While the revised Weyl space preserves the tetrad postulate, it no longer admits a gauge-invariant connection. This is no great loss, as we know Weyl's 1918 theory didn't work, anyway. More serious is the fact that the contracted spin connection ω_{μ} in the revised space is identical to the corresponding Weyl vector derived earlier, so (8.2) still blows up. But the main problem concerns the revised Weyl ϕ-field itself which, by (9.1), is obviously transfer-invariant. The vector ϕ_{μ} thus appears as a constant vector field on the revised Weyl manifold, and its hoped-for association with the electromagnetic 4-potential seems remote. In this context, a constant vector field is of little theoretical or physical interest.

10. Comments

In his 1918 theory, Weyl attempted to derive electromagnetism from generalized Riemannian geometry. This attempt was based on the premise of variable vector length under physical transplantation in spacetime, which requires that Weyl's manifold be non-metric-compatible. Einstein was the first to object to Weyl's theory on purely physical grounds, and in spite of heroic efforts to salvage the theory, Weyl ultimately gave up on the idea that spacetime is gauge-invariant. However, Weyl subsequently applied the concept of gauge invariance to quantum theory, where it worked brilliantly. The principle of quantum-mechanical gauge invariance is today considered one of the irreducible pillars of modern theoretical physics.

In view of the foregoing, however, it is somewhat surprising that the concept of a Weyl space continues to crop up in the literature. Admittedly, these appearances are motivated primarily by mathematical and not physical considerations. Nevertheless, the problems inherent in Weyl space in particular and non-metric-compatible spaces in general serve to emphasize the validity of Pais' warning.

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