Invisibility of the Lorentz Contraction*

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It is shown that, if the apparent directions of objects are plotted as points on a sphere surrounding the observer, the Lorentz transformation corresponds to a conformal transformation on the surface of this sphere. Thus, for sufficiently small subtended solid angle, an object will appear—optically—the same shape to all observers. A sphere will photograph with precisely the same circular outline whether stationary or in motion with respect to the camera. An object of less symmetry than a sphere, such as a meter stick, will appear, when in rapid motion with respect to an observer, to have undergone rotation, not contraction. The extent of this rotation is given by the aberration angle $(\theta-\theta')$, in which θ is the angle at which the object is seen by the observer and θ' is the angle at which the object would be seen by another observer at the same point stationary with respect to the object. Observers photographing the meter stick simultaneously from the same position will obtain precisely the same picture, except for a change in scale given by the Doppler shift ratio, irrespective of their velocity relative to the meter stick. Even if methods of measuring distance, such as stereoscopic photography, are used, the Lorentz contraction will not be visible, although correction for the finite velocity of light will reveal it to be present.

INTRODUCTION

EVER since Einstein presented his special theory of relativity[1] in 1905 there seems to have been a general belief that the Lorentz contraction should be visible to the eye. Indeed, Lorentz stated[2] in 1922 that the contraction could be photographed. Similar statements appear in other references too numerous to be mentioned, and even Einstein's first paper leaves the impression,[3] perhaps unintentionally, that the contraction due to relativistic motion should be visible. The usual statement is that moving objects "appear contracted," which is somewhat ambiguous. The special theory predicts that the contraction can be observed by a suitable experiment, and the words "observe" and "see" seem to be used interchangeably in this connection.

There is, however, a clear distinction between observing and seeing. An observation of the shape of a fast-moving object involves simultaneous measurement of the position of a number of points on the object. If done by means of light, all the quanta should leave the surface simultaneously, as determined in the observer's system, but will arrive at the observer's position at different times. Similar restrictions would apply to the use of radar as an observational method. In such observations the data received must be corrected for the finite velocity of light, using measured distances to various points of the moving object. In seeing the object, on the other hand, or photographing it, all the light quanta arrive simultaneously at the eye (or shutter), having departed from the object at various earlier times. Clearly this should make a difference between the contracted shape which is in principle observable and the actual visual appearance of a fast-moving object.

CONFORMALITY OF ABERRATION

The basic question of the visibility of the Lorentz contraction may be stated as that of the appearance of a rapidly moving object in an instantaneous photograph. The object, of known shape when at rest, is assumed to have a high uniform speed relative to the camera. The camera is assumed to be at rest in a Galilean (unaccelerated) frame of reference. Of course it would make no difference if the camera were, instead, considered to move at high speed past the stationary object, but the photograph produced must be examined at rest, so it is simpler to consider the camera as stationary. The mechanism of the camera must be such as to give it essentially instantaneous shutter speed and sharp focus over the necessary depth of field.

The questions of whether to use photographic film which lies in a plane or is curved so that all points are at the same distance from the lens (or pinhole), and whether to use a lens corrected to eliminate optical distortions, could be troublesome. To simplify matters, it is assumed that the object subtends a visual solid angle sufficiently small that these matters need not be considered. It is assumed that the camera is pointed directly at the apparent position of the object, so that the light rays strike the film in a perpendicular direction, producing an image in the center of the photographic film. The camera is assumed, also for simplicity,

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[1] A. Einstein, Ann. Physik **17**, 891 (1905).

[2] H. A. Lorentz, *Lectures on Theoretical Physics* (Macmillan and Company, Ltd., London, 1931; translated from Dutch edition of 1922), Vol. 3, p. 203.

[3] In reference 1 [English translation from *The Principle of Relativity* (Dover Publications, Inc., New York, reprinted from 1923 Methuen edition)] Einstein stated: "A rigid body which, measured in a state of rest, has the form of a sphere, therefore has in a state of motion—viewed [betrachtet] from the stationary system—the form of an ellipsoid of revolution with the axes $R(1-v^2/c^2)^{\frac{1}{2}}$, R,R. Thus, whereas the Y and Z dimensions of the sphere (and therefore of every rigid body of no matter what form) do not appear [nicht erscheinen] modified by the motion, the X dimension appears [erscheint] shortened in the ratio $1:(1-v^2/c^2)^{\frac{1}{2}}$, i.e., the greater the value of v, the greater the shortening. For $v=c$ all moving objects—viewed [betrachtet] from the "stationary" system—shrivel up into plane figures."

not to be rotating to follow the motion of the object, but this is an unessential restriction and would make no difference in the results so long as distortion of the camera due to relativistic angular motion is negligible.

With these assumptions and restrictions defined, the problem of the photographic (or visual) appearance of a rapidly moving object is not a difficult one. The optical image produced by a pinhole lens on a photographic emulsion at constant distance from the aperture is identical with the picture produced by plotting, on a spherical surface centered at the point of observation (eye or camera lens), the apparent visual directions of all points of the object as seen by observer O. For an observer O' having zero velocity relative to the object, this would clearly result in an uncontracted image. If this particular observer is located instantaneously at the same position as that of observer O, with respect to whom the object is not at rest, it is possible to calculate the apparent directions of these same points, as seen by O, from the equation for relativistic aberration.

Spherical polar coordinate angles θ and ϕ, forming an orthogonal coordinate system (θ is the polar angle and ϕ the azimuthal angle), are to be used by observer O in plotting on the spherical surface the apparent instantaneous direction of various points of the moving object. Let the object be moving at constant velocity v, relative to O, in the direction $\theta=0$. Let observer O be receiving light from some particular point of the object which appears to be in the direction (θ,ϕ). Let observer O' be instantaneously at the position of O, using the coordinate system (θ',ϕ'), and moving with velocity v relative to O in the direction $\theta=0=\theta'$. The relation between these two sets of coordinate angles is that of the aberration equation, derived[1] from the Lorentz transformation, and given by

$$\sin\theta=\frac{(1-v^2/c^2)^{\frac{1}{2}}\sin\theta'}{1-(v/c)\cos\theta'}, \qquad (1)$$

or

$$\cos\theta=\frac{\cos\theta'-v/c}{1-(v/c)\cos\theta'}. \qquad (1')$$

In these equations c is, of course, the velocity of light. The azimuthal angles are not affected by the Lorentz transformation, so that

$$\phi=\phi'. \qquad (2)$$

It may be shown that this transformation of the angles of observation is equivalent to a conformal transformation on the spherical surfaces centered on the observers. This fact and its consequences were apparently first pointed out quite recently.[4]

Consider a small rectangular area of differential extent on the surface centered on observer O, oriented along lines of constant θ and ϕ. The angles subtended by the sides of this rectangle are $d\theta$ and $\sin\theta d\phi$. As seen by observer O' the corresponding angles are $d\theta'$ and $\sin\theta' d\phi'=\sin\theta' d\phi$. Differentiation of Eq. (1) gives the simple relation

$$\frac{d\theta'}{d\theta}=\frac{\sin\theta'}{\sin\theta}=\frac{1-(v/c)\cos\theta'}{(1-v^2/c^2)^{\frac{1}{2}}}=\frac{(1-v^2/c^2)^{\frac{1}{2}}}{1+(v/c)\cos\theta}=M. \qquad (3)$$

Thus the two rectangles have identical ratios between their length and width. This, with the perpendicularity between sides which is true for both rectangles, is sufficient to establish the conformality of the transformation of angles of observation. The factor M is the magnification, the ratio between subtended angles as seen by observers O' and O, or the ratio of apparent distances of the object from the two observers. It is interesting that M is precisely the Doppler shift factor, becoming $[(1-v/c)/(1+v/c)]^{\frac{1}{2}}$ for $\theta=0=\theta'$.

The property of conformality in this sense, which is intrinsic to relativistic aberration, is sufficient to ensure that observers O and O' will obtain pictures which are identical, except for a magnification factor, over comparable regions of small subtended solid angle. Thus a spherical object will produce a perfectly round image[5] for both observers O and O', in spite of the Lorentz contraction which O may observe by suitable methods. Quite generally, objects will appear the same shape, visually, to all observers, no matter what the relative motion of object and unaccelerated observer may be. Obviously these conclusions also extend to accelerated objects. Although acceleration will in general change the shape of the object, all observers at a given point will agree as to what this shape is, as revealed in their photographs. Even accelerated observers will obtain similar photographs, provided that the cameras are not appreciably distorted by the acceleration. In this way the apparent shape of any object is invariant to the Lorentz transformation, although the "actual" shape, as given by careful measurement, will vary due to the Lorentz contraction.

Thus the Lorentz contraction is effectively invisible. Only when stereoscopic vision or photography is used, combining observations from two different locations, can any distortion of the object due to motion be seen, and even this is not the expected contraction, as will be discussed in a later section.

[4] J. Terrell, Bull. Am. Phys. Soc. Ser. II, 4, 294 (1959), and unpublished paper on *The Clock "Paradox"*, Los Alamos Document LADC-2842 (April 1957).

[5] R. Penrose, Proc. Cambridge Phil. Soc. **55**, 137 (1959), has recently proved that a sphere will be seen as having a circular outline by all observers, regardless of the relative velocity of sphere and observer. Penrose gives several proofs, of which the simplest involves the stereographic projection of a sphere centered at the point of observation onto its equatorial plane from the pole $\theta=\pi$. This transformation sends circles into circles, and aberration merely expands the plane of projection by the factor $[(c-v)/(c+v)]^{\frac{1}{2}}$. Penrose's conclusions agree with some given in this paper, although his paper deals almost exclusively with spherical objects. For this special case there is no restriction as to subtended visual angle. For finite subtended angle the surface of a moving sphere would appear somewhat distorted, although its outline would be precisely circular.

APPEARANCE OF MOVING METER STICK

At this point it may be objected that a meter stick in motion past the observer in such a way that it is moving parallel to its length, and is momentarily seen by the observer at its point of closest approach, will surely be seen as contracted. This case, probably the first to come to mind, is illustrated in Fig. 1 for the case $v/c=0.8$. Two meter sticks, S and S', are shown here in such positions as to be seen instantaneously by observer O at 90°. Meter stick S is stationary with respect to observer O; meter stick S' is moving with velocity v in the direction $\theta=0°$; both meter sticks are aligned along the direction $\theta=0°$. At the earlier time when the light which reaches observer O left S', both ends of the front face of the meter stick were at the same distance from O, so that he does indeed see them as they were at simultaneous earlier times, and the length of the meter stick S' appears contracted by comparison with S, which was at the same distance. However, at the still earlier time when light left the back side of the meter stick, stick S' was displaced farther to the left. This results in the visibility of the left-hand end of S', if it is assumed to be a physical stick having three dimensions. Thus the meter stick gives the appearance of having undergone rotation rather than contraction.

Consider how this situation appears to observer O', who is also moving with velocity v, with respect to O, in the direction $\theta=0=\theta'$. To O' both meter sticks will appear to be in the direction $\theta'=\cos^{-1}(v/c)$. Stick S' will appear stationary and turned through the angle $(\theta-\theta')$ with respect to his line of sight. Stick S will appear to be moving at high speed v to the left, but will not appear contracted. Because the right-hand side of S was much farther away from O' at the time light seen by O' left it than was the left-hand side when light left it, the time lags increase the apparent length of S in such a way that its contracted length appears quite normal (in two dimensions, as in the photograph).

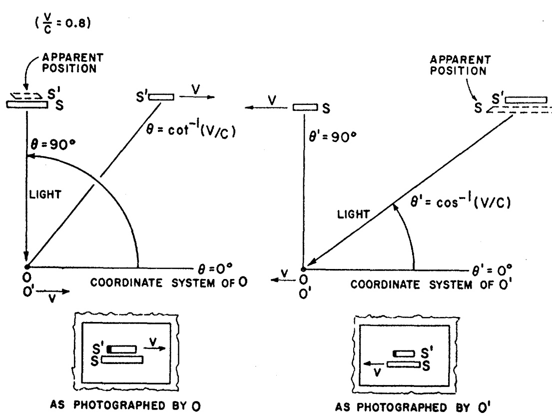

FIG. 1. Two meter sticks, S and S', as seen by observers O and O', who are located momentarily at the same point. In the coordinate system of observer O, O', and S' are moving to the right with velocity v, while S is stationary.

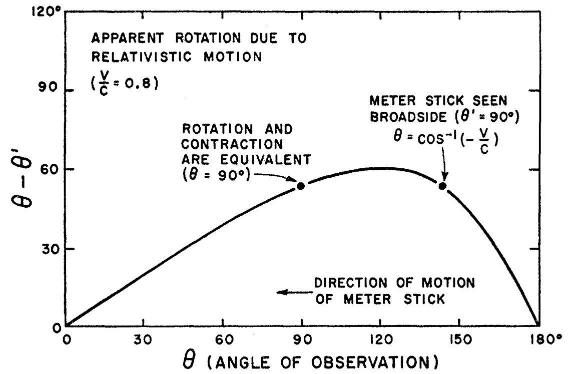

FIG. 2. Apparent rotation from known orientation as seen for relativistic motion of a meter stick with respect to an observer. The meter stick is assumed to be moving in the direction $\theta=0°$ and to be oriented along its direction of motion.

In fact, as has been shown, both observers O and O' see the same things, except for the apparent distance. Thus the photographs taken by O and O', shown in Fig. 1, are identical, or could be made identical by the use of an enlarger. It is probable that observers O and O' will put different interpretations on what they see, but the conformality of aberration ensures that, at least over small solid angles, each will see precisely what the other sees. No Lorentz contractions will be visible, and all objects will appear normal.

APPARENT ROTATION DUE TO RELATIVISTIC MOTION

It is apparent from the discussion above that objects in rapid motion appear visually to have undergone a rotation of extent $(\theta-\theta')$, the aberration angle, from their "true" or known orientations. The angle θ is the angle at which the object appears to be, with the coordinate system chosen so that the object is moving past the observer O (considered stationary) in the direction $\theta=0$. The angle θ' is the apparent direction of the object as perceived by another observer O', located at the same position at the same time, to whom the object appears stationary. The angles θ and θ' are related by the aberration equation, Eq. (1).

The dependence of the apparent rotation on the angle of observation is shown in Fig. 2 for the case $v/c=0.8$. For $\theta=0$ and $\theta=\pi$, the apparent rotation is zero. Two other angles are of special interest. For $\theta=\pi/2$ the rotation is such that $\cos(\theta-\theta')=(1-v^2/c^2)^{\frac{1}{2}}$, and a linear object which was oriented in the direction $\theta=0$, at the earlier time when light left it, will appear contracted by the rotation just to the extent of the Lorentz contraction. This does not constitute a proof of the visibility of the contraction, as this relation does not hold for other orientations, angles of observation, and shapes, and since the appearance of the object is normal at all times. The apparent rotation would, to observer O', be a real rotation. The other angle of interest is that for which $\cos\theta=-v/c$; for this angle $\theta'=\pi/2$, and the

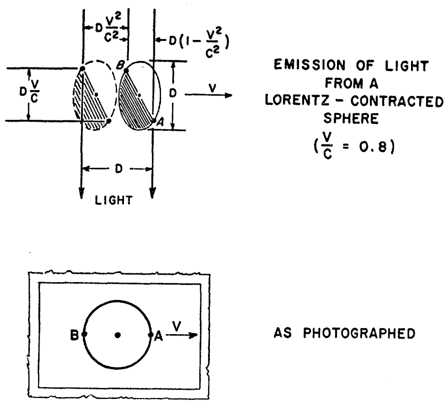

FIG. 3. Mechanism by which a Lorentz-contracted moving sphere produces a round photographic image. The shaded area is the visible portion of the spherical surface, with A and B the farthest visible points along the direction of motion. The dashed ellipse represents the earlier position of the sphere when the light which will arrive at the camera simultaneously with light from A left B.

object, if linear and oriented along $\theta=0$, will then be seen broadside, with no view of the ends.

Thus a meter stick which is traveling, and oriented, in the direction $\theta=0$ will appear to observer O to be rotating about its line of motion in such a way as to appear broadside at $\theta=\cos^{-1}(-v/c)$, and to present a view of its rear end from that time on. For $\theta=\pi/2$ the rotation will foreshorten the length to the same extent as the Lorentz contraction, and for a meter stick traveling nearly at the speed of light little will be seen at this angle of apparent closest approach, or at most angles, except the rear end.

For an object of rotational symmetry, such as a sphere, no possibility of confusing rotation and contraction exists. Thus a sphere will always produce a round photographic image, no matter what its unaccelerated motion. The mechanism by which this occurs is shown in Fig. 3. A Lorentz-contracted sphere is assumed to be moving to the right with velocity v relative to the observer; for the purpose of this figure $v/c=0.8$. The sphere is to be viewed at $\theta=\pi/2$. The uncontracted diameter of the sphere is D, giving a contracted diameter of $D(1-v^2/c^2)^{\frac{1}{2}}$. However, the farthest visible points on the sphere, A and B, as measured along the direction of motion, are not this far apart. This corresponds to the visual effect of apparent rotation. As plotted on the uncontracted sphere, the visible area is tilted from its position for $v=0$ by $(\theta-\theta')$; here $\theta=\pi/2$ so that $\cos(\theta-\theta')=(1-v^2/c^2)^{\frac{1}{2}}$. Thus the distance between the farthest visible points is reduced to $D(1-v^2/c^2)$ as measured along the direction of motion. As measured along the line of sight, perpendicular to the motion, this distance is Dv/c. Thus the light which reaches the observer from B must leave B at a time Dv/c^2 earlier than the light that leaves A in order to arrive simultaneously with the light from A. During this time the sphere moves a distance Dv^2/c^2, so that the distance between A and B appears to be D, as seen or photographed by the observer. Thus the sphere appears uncontracted in the observer's photograph.

Physically, the reason that A is the farthest visible point is that light leaving points beyond A on the spherical surface will be intercepted by the motion of the sphere. Similarly, point B is visible, though on the far side of the sphere, because light emitted from this point will not be stopped by the sphere, which moves out of the path of the light.

STEREOSCOPIC VISION

If stereoscopic vision or photography is to be considered, the situation becomes more complicated. Simultaneous observations of direction of a given object from two observation points constitute a valid means of measuring distance to the object. Thus, with stereoscopic vision, all points will appear to be at the proper distance even with relativistic speeds. However, what is seen at a given time is the situation which existed at an earlier time, and not all parts of the object are seen at the same earlier time. This produces curious visual distortions of the sort shown in Fig. 1 at the apparent positions of S and S', constituting shear and contraction or elongation, depending on the situation. For instance, an object coming directly toward the observer is seen in three dimensions to be elongated along its direction of motion by the ratio $[(1+v/c)/(1-v/c)]^{\frac{1}{2}}=M(180°)$, and incidentally appears farther away, by the same ratio, than if the observer had the same velocity as the object.

At other angles of observation the situation is less simple to describe. In general, if an observer sees two points on a stationary object which are at precisely the same visual angle but at different distances, another observer at the same point but moving in a different reference frame will see the two points as M times farther away and M times farther apart. Here M is given by Eq. (3). In general, this results in apparent shear of the object, as seen with stereoscopic vision. Precisely the same effects would occur with the apparent perspective of the object, even with nonstereoscopic vision, if the object were near enough to make perspective noticeable.

CONCLUSIONS

It has been shown that the Lorentz transformation is conformal in the angles of observation, so that the photograph obtained by an observer depends only on the place and time of taking the picture and is independent of the relative motion of observer and object photographed. This statement must be restricted to small solid angles in the same way that conformal transformations preserve shapes only for differential areas. Thus the visual appearance of an object is invariant (except for Doppler shifts of frequency), not depending on its (unaccelerated) motion. Effectively, then, the Lorentz contraction is invisible. Any hopes of seeing the contraction in a rapidly moving space vehicle or astronomical body must be discarded.

Although apparent distortion due to rapid motion can be seen by means of steroscopic vision or photography, it is not of the same type as one might expect from the Lorentz contraction.

None of the statements here should be construed as casting any doubt on either the observability or the reality of the Lorentz contraction, as all the results given are derived from the special theory of relativity.

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Active Gravitational Mass*

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Tolman states that ". . .disordered radiation in the interior of a fluid sphere contributes roughly speaking twice as much to the gravitational field of the sphere as the same amount of energy in the form of matter." The gravitational pull exerted by a system on a distant test particle might therefore at first sight be expected to increase if within the system a pair of oppositely charged electrons annihilate to produce radiation. This apparent paradox is analyzed here in the case where gravitational effects internal to the system are unimportant. It is shown that tensions in the wall of the container compensate the effect mentioned by Tolman so that the net gravitational pull exerted by the system does not change.

I. INTRODUCTION

IN Newtonian mechanics the equivalence of active and passive gravitational mass, that is of mass as a quantity which gives rise to, and as a quantity acted upon by, gravitational fields, is made obvious in the form of the familiar equation for the gravitational potential ϕ, namely $\nabla\phi=4\pi\rho$, where ρ is the density of inertial mass.

However, in relativity theory where the field equations take the form $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=T_{\mu\nu}$, the inference can sometimes not be drawn so easily. Here not only does the source term include stresses and momenta as well as energy, but the equations are nonlinear. The question presents itself, therefore, to what extent are the distant gravitational fields as calculated by classical and special relativity theory the same as those calculated using general relativity?

The following statement by Tolman suggests that there are important differences: ". . .disordered radiation in the interior of a fluid sphere contributes roughly speaking twice as much to the gravitational field of the sphere as the same amount of energy in the form of matter."[1]

Such a result would seem to lead to certain paradoxes. Consider the conversion of a gamma ray, enclosed in a box, into mass, say an electron-positron pair. This transformation might be thought to halve the contribution of the mass energy to distant gravitational fields.

However, we shall show here that the active gravitational mass of a system is made up of the energy of the walls and other material plus the energy of radiation, divided by c^2, without the added factor of two, provided that the gravitational fields internal to the system are weak.

II. ENCLOSED RADIATION

Tolman's argument is based upon an expression for the distant gravitational field which involves only the classical stress-energy tensor $T_{\mu\nu}$. The reasoning applies to a wide class of cases roughly describable as quasi-static. Included in such cases are those in which the matter is confined to some limited region. This region is considered to be small as compared to the distance at which its gravitational field is to be measured. Moreover, within this region the behavior of the system is not significantly influenced by its own gravitational field. When these conditions are satisfied, and when the distant metric field is expressed in a form,

$$ds^2=-(1+2m^*/r)(dx^2+dy^2+dz^2)+(1-2m^*/r)dt^2, \quad (1)$$

which reveals the mass of the system, $m=(c^2/G)m^*$, or its energy $E=mc^2=(c^4/G)m^*$, then Tolman's arguments[2] give for the energy of the system the value

$$mc^2=(c^4/G)m^*=\int(T_4{}^4-T_1{}^1-T_2{}^2-T_3{}^3)(-g)^{\frac{1}{2}}d^3x. \quad (2)$$

Since the electromagnetic stress-energy tensor has zero trace, it follows that $T_4{}^4$ equals $-(T_1{}^1+T_2{}^2+T_3{}^3)$. Therefore according to (2), Tolman argues, the system

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[1] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Clarendon Press, Oxford, 1934), p. 272.

[2] See Tolman, reference 1, p. 235, Eq. (92.3).