When they are satisfied the operator (16.10) may be replaced by the nonsingular self-adjoint operator

$$F^{\mu\nu\sigma'\tau'} \equiv \delta^2 S_g/\delta g_{\mu\nu}\delta g_{\sigma'\tau'} + \int dx'' \int dx''' R^{\mu\nu}{}_{\rho''}\, \gamma^{\rho''\lambda'''} R^{\sigma'\tau'}{}_{\lambda'''}$$

$$= \tfrac{1}{4} g^{1/2}(g^{\mu\rho}g^{\nu\lambda} + g^{\mu\lambda}g^{\nu\rho} - g^{\mu\nu}g^{\rho\lambda})\,(\delta_{(\rho\lambda)}{}^{(\sigma'\tau')}{}_{.\kappa}{}^{\kappa} - 2R_{\rho\iota\lambda\kappa}\delta^{(\iota\kappa)(\sigma'\tau')}). \qquad (16.20)$$

The computation of the final expression requires use of equations (16.9), the commutation laws for covariant differentiation, and the easily verified identities

$$\delta_{(\mu\nu)(\sigma'\tau')}{}_{.}{}^{\nu} \equiv -\tfrac{1}{2}(\delta_{\mu\sigma'.\tau'} + \delta_{\mu\tau'.\sigma'}), \qquad (16.21)$$

$$\delta_{\mu\nu'.}{}^{\mu} \equiv -\delta_{.\nu'}(x, x'). \qquad (16.22)$$

Problem 73. Verify equations (16.20), (16.21) and (16.22).

Contracted Bianchi Identities

It has been remarked in Section 3 that when an infinite dimensional invariance group is present the dynamical equations are not all independent of one another. In the present case, equation (3.14) takes the form

$$0 \equiv \int (\delta S_g/\delta g_{\nu'\sigma'}) R_{\nu'\sigma'\mu}\, dx' \equiv 2[g^{1/2}(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R) - \tfrac{1}{2}\lambda g^{1/2}g^{\mu\nu}]_{.\nu}. \qquad (16.23)$$

These identities can be obtained by contracting the Bianchi identities (13.38) and remembering that $g^{\mu\nu}$ and $g^{1/2}$ themselves have vanishing covariant derivatives.

Problem 74. Derive (16.23) from the Bianchi identities.

More Complex Lagrangians

The dynamical equations (16.7) as well as the equations for small disturbances are of the second differential order and represent covariant generalizations of field-equation types with which we are already familiar. They should therefore provide the basis for a satisfactory physical theory, making it unnecessary to look further for Lagrangians more complicated than that of Einstein's theory. Indeed, more complicated Lagrangians can only yield equations of higher differential order, giving rise to the difficulties which we have already noted in Section 10 concerning the existence of the vacuum and the positive definiteness of Hilbert space. Nevertheless, results which we shall obtain later in connection with the renormalization program of the quantum theory lead us to consider such Lagrangians.

At the next level of complexity there are essentially three different possible Lagrangians, each quadratic in the Riemann tensor:*

$$L_1 \equiv g^{1/2}R^2, \qquad (16.24)$$

$$L_2 \equiv g^{1/2}R_{\mu\nu}R^{\mu\nu}, \qquad (16.25)$$

$$L_3 \equiv g^{1/2}R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}. \qquad (16.26)$$

* A fourth possibility in principle exists, namely $L_4 \equiv \tilde{\epsilon}^{\mu\nu\sigma\tau} R_{\mu\nu\rho\lambda} R_{\sigma\tau}{}^{\rho\lambda}$. Because of parity considerations, however, this has no practical interest.

These do not lead to independent dynamical equations, however. It is found that only two adjustable parameters are needed to include these Lagrangians in the total action. Denoting the corresponding action functionals by S_1, S_2, S_3, and making use of equations (16.2) to (16.6), we easily obtain the following functional derivatives:

$$\delta S_1/\delta g_{\mu\nu} \equiv g^{1/2}(2\, g^{\mu\nu}\, R_{.\sigma}{}^{\sigma} - 2\, R_.{}^{\mu\nu} - 2\, R R^{\mu\nu} + \tfrac{1}{2}\, g^{\mu\nu}\, R^2), \tag{16.27}$$

$$\delta S_2/\delta g_{\mu\nu} \equiv g^{1/2}(g^{\mu\nu}\, R^{\sigma\tau}{}_{.\sigma\tau} + R^{\mu\nu}{}_{.\sigma}{}^{\sigma} - R^{\mu\sigma}{}_{.}{}^{\nu}{}_{\sigma} - R^{\nu\sigma}{}_{.}{}^{\mu}{}_{\sigma} - 2\, R^{\mu}{}_{\sigma}\, R^{\nu\sigma} + \tfrac{1}{2}\, g^{\mu\nu}\, R_{\sigma\tau} R^{\sigma\tau}), \tag{16.28}$$

$$\delta S_3/\delta g_{\mu\nu} \equiv g^{1/2}(2\, R^{\mu\sigma\nu\tau}{}_{.\sigma\tau} + 2\, R^{\mu\sigma\nu\tau}{}_{.\tau\sigma} - 2\, R^{\mu}{}_{\sigma\tau\rho}\, R^{\nu\sigma\tau\rho} + \tfrac{1}{2} g^{\mu\nu}\, R_{\sigma\tau\rho\lambda}\, R^{\sigma\tau\rho\lambda}). \tag{16.29}$$

With the aid of the identities

$$R^{\mu\sigma\nu\tau}{}_{.\sigma\tau} \equiv R^{\mu\nu}{}_{.\sigma}{}^{\sigma} - R^{\mu\sigma}{}_{.}{}^{\nu}{}_{\sigma}, \tag{16.30}$$

$$R^{\mu\sigma}{}_{.}{}^{\nu}{}_{\sigma} \equiv \tfrac{1}{2}\, R_.{}^{\mu\nu} + R^{\mu\sigma\nu\tau}\, R_{\sigma\tau} - R^{\mu}{}_{\sigma}\, R^{\nu\sigma}, \tag{16.31}$$

$$R^{\mu\nu}{}_{.\mu\nu} \equiv \tfrac{1}{2}\, R_{.\mu}{}^{\mu}, \tag{16.32}$$

which follow from the Bianchi identities, expressions (16.28) and (16.29) may be rewritten in the forms

$$\delta S_2/\delta g_{\mu\nu} \equiv g^{1/2}(R^{\mu\nu}{}_{.\sigma}{}^{\sigma} - R_.{}^{\mu\nu} + \tfrac{1}{2} g^{\mu\nu}\, R_{.\sigma}{}^{\sigma} - 2\, R^{\mu\sigma\nu\tau}\, R_{\sigma\tau} + \tfrac{1}{2} g^{\mu\nu}\, R_{\sigma\tau}\, R^{\sigma\tau}), \tag{16.33}$$

$$\delta S_3/\delta g_{\mu\nu} \equiv g^{1/2}(4R^{\mu\nu}{}_{.\sigma}{}^{\sigma} - 2R_.{}^{\mu\nu} - 2R^{\mu}{}_{\sigma\tau\rho}\, R^{\nu\sigma\tau\rho} + \tfrac{1}{2} g^{\mu\nu}\, R_{\sigma\tau\rho\lambda}\, R^{\sigma\tau\rho\lambda} - 4R^{\mu\sigma\nu\tau}\, R_{\sigma\tau} + 4R^{\mu}{}_{\sigma}\, R^{\nu\sigma}). \tag{16.34}$$

We then observe that

$$\delta(S_1 - 4\, S_2 + S_3)/\delta g_{\mu\nu} \equiv g^{1/2}(-2\, R^{\mu}{}_{\sigma\tau\rho}\, R^{\nu\sigma\tau\rho} + \tfrac{1}{2} g^{\mu\nu}\, R_{\sigma\tau\rho\lambda}\, R^{\sigma\tau\rho\lambda} + 4R^{\mu\sigma\nu\tau}\, R_{\sigma\tau} + 4\, R^{\mu}{}_{\sigma} R^{\nu\sigma} - 2g^{\mu\nu} R_{\sigma\tau} R^{\sigma\tau} - 2R R^{\mu\nu} + \tfrac{1}{2}\, g^{\mu\nu} R^2). \tag{16.35}$$

But this quadratic expression vanishes in four dimensions on account of the algebraic identities satisfied by the Riemann tensor. This may be verified in a straightforward but tedious manner by multiplying $R_{\sigma\tau\eta\zeta}\, R_{\iota\kappa\rho\lambda}$ by the identity

$$\sum (\pm)\, g^{\mu\nu} g^{\sigma\iota}\, g^{\tau\kappa}\, g^{\rho\eta}\, g^{\lambda\zeta} \equiv 0, \tag{16.36}$$

where the summation is over the 120 permutations of the indices ν, ι, κ, η, ζ, with the sign chosen according to the eveness or oddness of each permutation.

Therefore we need to consider only two of the three Lagrangians above, and for simplicity we choose L_1 and L_2. However, instead of introducing each with an arbitrary coefficient let us consider the case in which they appear in the particular combination $L_2 - \tfrac{1}{2} L_1$. Accordingly, we now replace the action (16.1) by

$$S_g' \equiv -\lambda \int g^{1/2}\, dx - \int g^{1/2} R\, dx - \tfrac{1}{2}\mu^{-2} \int g^{1/2}(g^{\mu\sigma} g^{\nu\tau} + g^{\mu\tau} g^{\nu\sigma} - g^{\mu\nu} g^{\sigma\tau}) R_{\mu\nu} R_{\sigma\tau}\, dx, \tag{16.37}$$

in which only a single new parameter μ appears, having the dimensions of mass when $\hbar = c = 1$. Einstein's theory is regained in the limit $\mu \to \infty$.

The dynamical equations to which the new action leads are easily found to be

$$- \mu^{-2}g^{1/2}[(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R)_{.\sigma}{}^{\sigma} - 2R^{\mu}{}_{\sigma}{}^{\nu}{}_{\tau}(R^{\sigma\tau} - \tfrac{1}{2}g^{\sigma\tau}R) + \tfrac{1}{2}g^{\mu\nu}R_{\sigma\tau}(R^{\sigma\tau} - \tfrac{1}{2}g^{\sigma\tau}R) - \mu^2(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R)] - \tfrac{1}{2}\lambda g^{1/2}g^{\mu\nu} = 0. \quad (16.38)$$

These equations have the remarkable property that when they are linearized (with $\lambda = 0$) and the supplementary conditions of Problem 14, case (c), are imposed, they reduce to

$$- \tfrac{1}{4}\,\mu^{-2}(\eta^{\mu\sigma}\eta^{\nu\tau} + \eta^{\mu\tau}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\sigma\tau})\,(\Box^2 - \mu^2)\,\Box^2\varphi_{\sigma\tau} = 0, \quad (16.39)$$

which is a generalization of the two-level mass equation of Problem 1, case (d). The same property persists in fully covariant form for the equations for small disturbances on an arbitrary background metric satisfying (16.38), provided the disturbances satisfy the conditions (16.16). We shall not verify this in detail, but merely file it for future reference in connection with the renormalization program for quantum gravidynamics.

Lagrangian for the Yang–Mills Field

Turning now to the Yang–Mills field, we choose for its action functional the covariant generalization of that provided by the Lagrangian (12.29):

$$S_A \equiv - \tfrac{1}{4}\int g^{1/2}\,F_{\alpha\mu\nu}\,F_{\alpha}{}^{\mu\nu}\,dx. \quad (16.40)$$

Here the generating group is assumed to be compact and simple, with $\mathrm{tr}(c_\alpha c_\beta) = - c^2\delta_{\alpha\beta}$, so that all group indices are written in the lower position, while the coordinate indices are raised and lowered by means of the metric tensor. With the aid of the easily verified relation

$$\delta F_{\alpha\mu\nu} = \delta A_{\alpha\nu.\mu} - \delta A_{\alpha\mu.\nu}, \quad (16.41)$$

we obtain the field equations corresponding to S_A:

$$0 = \delta S_A/\delta A_{\alpha\mu} \equiv - g^{1/2}\,F_{\alpha}{}^{\mu\nu}{}_{.\nu}. \quad (16.42)$$

The second functional derivative is given by

$$\delta^2 S_A/\delta A_{\alpha\mu}\delta A_{\beta'\nu'} \equiv g^{1/2}(\delta_{\alpha}{}^{\mu}{}_{\beta'}{}^{\nu'}{}_{.\sigma}{}^{\sigma} - \delta_{\alpha}{}^{\sigma}{}_{\beta'}{}^{\nu'}{}_{.}{}^{\mu}{}_{\sigma} + c_{\alpha\gamma\epsilon}\,F_{\gamma}{}^{\mu}{}_{\sigma}\,\delta_{\epsilon}{}^{\sigma}{}_{\beta'}{}^{\nu'}), \quad (16.43)$$

where

$$\delta_{\alpha}{}^{\mu}{}_{\beta'}{}^{\nu'} \equiv \delta_{\alpha\beta}g^{\mu\nu}\delta(x, x'). \quad (16.44)$$

Supplementary Conditions

In choosing supplementary conditions to be imposed on the small disturbances we again follow Problems 9, 14, and 15, this time covariantly generalizing part (b). Rewriting the infinitesimal transformation law (12.19) in the form

$$\delta A_{\alpha u} = \int R_{\alpha\mu\beta'}\delta\xi^{\beta'}\,dx', \quad (16.45)$$