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Long range forces and broken symmetries

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There are reasons for believing that the gravitational constant varies with time. Such a variation would force one to modify Einstein's theory of gravitation. It is proposed that the modification should consist in the revival of Weyl's geometry, in which lengths are non-integrable when carried around closed loops, the lack of integrability being connected with the electromagnetic field. A new action principle is set up, much simpler than Weyl's, but requiring a scalar field function to describe the gravitational field, in addition to the $g_{\mu\nu}$. The vacuum field equations are worked out and also the equations of motion for a particle.

An important feature of Weyl's geometry is that it leads to a breaking of the C and T symmetries, with no breaking of P or of CT. The breaking does not show itself up with the simpler kinds of charged particles, but requires a more complicated kind of term in the action integral for the particle.

1. The long-range forces

Long-range forces are those that fall off inversely proportional to the square of the distance between the interacting bodies, as distinct from short-range forces, that fall off exponentially. There are two known long-range forces, the gravitational and the electromagnetic.

The gravitational field is very well explained by Einstein's theory, which accounts for it in terms of the curvature of space. This has led people to believe that the electromagnetic field should also be ascribed to some property of space, instead of being merely something embedded in space, and thus it would require one to set up a more general space than the Riemannian space which underlies Einstein's theory. The more general space would then account for both the gravitational and the electromagnetic fields and would provide a unification of the long-range forces.

Soon after the appearance of Einstein's theory, a solution of the problem was proposed by Weyl. The curvature of space required by Einstein's theory can be discussed in terms of the notion of the parallel displacement of a vector, the transport of a vector around a closed loop by parallel displacement resulting in the final direction of the vector differing from its initial direction. Weyl's generalization was to suppose that the final vector has a different length as well as a different direction, which is a very natural generalization of Riemannian space.

With Weyl's geometry there is no absolute way of comparing elements of length at two different points, unless the points are infinitely close together. The comparison can be made only with respect to a path joining the two points, and different paths will lead to different results for the ratio of the two elements of length. In order to have a mathematical theory of lengths one must set up arbitrarily a standard of length at each point, and then refer any length that turns up in the description of space at some point to the local standard at that point. One then has a definite value for the length of a vector at any point, but this value changes when one changes the local standard of length. The standard of length can, of course, be changed by a factor that varies from place to place.

Consider a vector of length l situated at a point with coordinates x^μ ($\mu = 0, 1, 2, 3$). Suppose it is transported by parallel displacement to the point $x^\mu + \delta x^\mu$. Its change in length δl will be proportional to l and to the δx^μ, so it will be of the form

$$\delta l = l\kappa_\mu \delta x^\mu. \tag{1.1}$$

We have some coefficients κ_μ appearing here. They are further field quantities, occurring in the theory together with the Einstein $g_{\mu\nu}$ and just as fundamental.

Let us suppose the standards of length are changed so that lengths get multiplied by the factor $\lambda(x)$, depending on the x's. Then l gets changed to $l' = l\lambda(x)$ and $l + \delta l$ gets changed to

$$l' + \delta l' = (l + \delta l)\,\lambda(x + \delta x) = (l + \delta l)\,\lambda(x) + l\lambda_{,\mu}\,\delta x^\mu,$$

with neglect of second-order quantities, and with $\lambda_{,\mu}$ denoting $\partial\lambda/\partial x^\mu$. We get

$$\delta l' = \lambda\,\delta l + l\lambda_{,\mu}\delta x^\mu = \lambda l(\kappa_\mu + \phi_{,\mu})\,\delta x^\mu,$$

where
$$\phi = \log \lambda. \tag{1.2}$$

Thus
$$\delta l' = l'\kappa'_\mu\,\delta x^\mu,$$

with
$$\kappa'_\mu = \kappa_\mu + \phi_{,\mu}. \tag{1.3}$$

If our vector is transported by parallel displacement round a small closed loop, the total change in its length will be

$$\delta l = lF_{\mu\nu}\,\delta S^{\mu\nu},$$

where
$$F_{\mu\nu} = \kappa_{\mu,\nu} - \kappa_{\nu,\mu}, \tag{1.4}$$

and $\delta S^{\mu\nu}$ denotes the element of area enclosed by the small loop. This change of course is unaffected by the transformation (1.3) arising from a change in the standards of length.

It will be seen that the new field quantities κ_μ appearing in Weyl's theory may be taken to be the electromagnetic potentials. They have all the desired properties. They are subject to the transformations (1.3), which correspond to no change in the geometry, but a change only in the choice of the artificial standards of length. The derived quantities $F_{\mu\nu}$ have a geometrical meaning independent of the standards of length and correspond to physically significant quantities, the electric and magnetic field. Thus Weyl's geometry provides just what one needs for describing both the gravitational and electromagnetic fields in geometrical terms.

In spite of these beautiful features of the theory, it was not acceptable to physicists, because it clashes with the quantum theory. Quantum phenomena provide an absolute standard of length. An atomic clock measures time in an absolute way, and if one takes the velocity of light to be unity one gets an absolute standard of distance. There is thus no need for the arbitrary metric standards of Weyl's theory. One may suppose that under parallel displacement a vector keeps the same length with respect to these atomic standards and then the variation δl of equation (1.1) does not arise.

Weyl's geometry was rejected by physicists, even by Weyl himself, but some of the terminology that it introduced was retained. The arbitrary standard of length at each point was called a *gauge*. Transformations of this standard were called gauge transformations. In Weyl's geometry they gave rise to transformations (1.3) of the potentials. These transformations of the potentials were still called gauge transformations after the rejection of Weyl's theory, when they no longer had any geometrical meaning.

With the rejection of Weyl's theory physicists returned to the view that the gravitational field influences the geometry of space, but the electromagnetic field is only something immersed in the space established by the gravitational field. This view provided a satisfactory working basis. But many physicists remained fascinated by the problem of finding a geometrical interpretation of the electromagnetic field and so obtaining a unified field theory. Many theories on these lines have been proposed, but they are all complicated and rather artificial, and are not generally accepted. Weyl's theory remains as the outstanding one, unrivalled by its simplicity and beauty.

2. The Large Numbers hypothesis

From the constants of Nature one can construct some dimensionless numbers. The important ones are hc/e^2, which is about 137, and the ratio of the mass of the proton to that of the electron, M/m, which is about 1840. There is no explanation for these numbers, but physicists believe that with increasing knowledge an explanation will some day be found.

Another dimensionless number is provided by the ratio of the electric to the gravitational force between an electron and a proton, namely e^2/GMm. This has a value about 2×10^{39}, quite a different order from the previous ones. One wonders how it could ever be explained.

The recession of the spiral nebulae provides an age for the Universe of about 2×10^{10} years. If one expresses it in terms of some atomic unit, say e^2/mc^3, one gets a number about 7×10^{39}, which is comparable with the previous large number. It is hard to believe that this is just a coincidence. One suspects that there is some connexion between the two numbers, which will get explained when we have more knowledge of cosmology and of atomic theory.

One can set up a general hypothesis, which we may call 'the Large Numbers hypothesis', that all dimensionless numbers of this order that turn up in nature are connected. (See the author's paper (Dirac 1938).) There is one other such number appearing in an elementary way, namely the square root of the number of nucleons in the Universe, which should be included in the scheme.

Now one of these large numbers is the epoch t, the present time reckoned from the time of creation as zero, and this increases with the passage of time. The Large Numbers hypothesis now requires that they shall all increase, in proportion to the epoch, so as to maintain the connexion between them. One can infer that the gravitational constant G, measured in atomic units, must be decreasing in proportion to t^{-1}.

Now Einstein's theory of gravitation requires that G shall be constant; in fact with a suitable choice of units it is 1. Thus Einstein's theory of gravitation is irreconcilable with the Large Numbers hypothesis.

The Large Numbers hypothesis is a speculation, not an established fact. It can become established only by direct observation of the variation of G. The effect is not too small to be beyond the capabilities of present-day techniques. Shapiro's (1968) measurements of the distances of the planets by radar are extremely accurate, and if G is really varying to the required extent, it should show up in his observations in a few years time.

In the present paper it will be assumed that the Large Numbers hypothesis is correct and the question will be faced as to how Einstein's theory is to be modified to agree with it.

A simple way of effecting a reconciliation is to suppose that the Einstein equations refer to an interval ds_E connecting two neighbouring points which is not the same as the interval ds_A measured by atomic apparatus. By taking the ratio of ds_E to ds_A to vary with the epoch we get G varying with the epoch. The ratio is sufficiently nearly constant for the modification in Einstein's theory to be very small.

With the introduction of two metrices ds_E and ds_A, we see that the objection to Weyl's theory discussed in the preceding section falls away. One can apply Weyl's geometry to ds_E, supposing that it is non-integrable when transported by parallel displacement, so that we must refer it to an arbitrary metric gauge to get a definite value for it. Then ds_E gets altered when we make a transformation (1.3) of the potentials. On the other hand ds_A is referred to atomic units and does not depend on an arbitrary metric gauge and is not affected by a transformation (1.3) of the potentials.

The measurements ordinarily made by physicists in the laboratory use apparatus which is fixed by the atomic properties of matter, so the measurements will refer to the metric ds_A. The metric ds_E cannot be measured directly, but it shows itself up through its influence on the equations of motion. It forms the basis of all dynamical theory, whether the theory is the accurate one of Einstein or the Newtonian approximation. The relation of the two metrics is exemplified by radar observations of the planets. Here a distance which is determined by equations of motion is measured by atomic apparatus.

With the two metrics playing their respective roles there is no longer any basic objection to Weyl's theory. In the present paper it will be revived and built up anew. The equations will all refer, of course, to the metric ds_E.

3. Co-covariant differentiation

The main features of Weyl's geometry will be briefly reviewed here. Details may be found in Weyl's book (1921) or in Eddington's book (1922).

We deal with transformations of the metric gauge under which any length, such as $\mathrm{d}s$, gets multiplied by a factor λ depending on its position in space, $\mathrm{d}s' = \lambda\,\mathrm{d}s$. A localized quantity Y may get transformed according to the law $Y' = \lambda^n Y$. Then Y is said to be of power n. If Y is a tensor and transforms in this way, it is called a co-tensor. If n is zero it is called an in-tensor. It is then invariant under gauge transformations.

The equation
$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$
shows that $g_{\mu\nu}$ is a co-tensor of power 2, as the $\mathrm{d}x^\mu$ are not affected by a gauge transformation. It follows that $g^{\mu\nu}$ is a co-tensor of power -2. We shall write $\surd(-g)$ simply as $\surd$, for brevity. It is of power 4.

A tensor T with various suffixes (here left understood) has a covariant derivative $T_{:\mu}$. In general if T is a co-tensor, $T_{:\mu}$ is not a co-tensor. However, there is a modified covariant derivative, called the co-covariant derivative and written $T_{*\mu}$, which is a co-tensor.

Let us first take a scalar S of power n. Its covariant derivative $S_{:\mu}$ is the same as its ordinary derivative $S_{,\mu}$ and we may write either of these simply as S_μ. Under a change of gauge it transforms to
$$S'_\mu = (\lambda^n S)_{,\mu} = \lambda^n S_\mu + n\lambda^{n-1}\lambda_\mu S$$
$$= \lambda^n\{S_\mu + n(\kappa'_\mu - \kappa_\mu)\,S\}$$
from (1.2) and (1.3). Thus
$$(S_\mu - n\kappa_\mu S)' = \lambda^n(S_\mu - n\kappa_\mu S),$$
so $S_\mu - n\kappa_\mu S$ is a co-vector of power n. We define it to be the co-covariant derivative of S,
$$S_{*\mu} = S_\mu - n\kappa_\mu S. \tag{3.1}$$

To get the co-covariant derivative of co-vectors and co-tensors we need a modified Christoffel symbol $^*\Gamma^\alpha_{\mu\nu}$, defined in terms of the usual one $\Gamma^\alpha_{\mu\nu}$ by
$$^*\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - g^\alpha_\mu \kappa_\nu - g^\alpha_\nu \kappa_\mu + g_{\mu\nu}\kappa^\alpha. \tag{3.2}$$
It is easily verified that $^*\Gamma^\alpha_{\mu\nu}$ is invariant under gauge transformations (see Eddington, equation 86.3).

Let A_μ be a co-vector of power n and form
$$A_{\mu,\nu} - {}^*\Gamma^\alpha_{\mu\nu} A_\alpha.$$
It is evidently a tensor, since it differs from the covariant derivative $A_{\mu:\nu}$ by a tensor. It transforms under a gauge transformation to
$$(A_{\mu,\nu} - {}^*\Gamma^\alpha_{\mu\nu} A_\alpha)' = \lambda^n A_{\mu,\nu} + n\lambda^{n-1}\lambda_\nu A_\mu - {}^*\Gamma^\alpha_{\mu\nu}\lambda^n A_\alpha$$
$$= \lambda^n\{A_{\mu,\nu} + n(\kappa'_\nu - \kappa_\nu)\,A_\mu - {}^*\Gamma^\alpha_{\mu\nu} A_\alpha\}.$$

Thus $$(A_{\mu,\nu}-n\kappa_\nu A_\mu - {}^*\Gamma^\alpha_{\mu\nu}A_\alpha)' = \lambda^n(A_{\mu,\nu}-n\kappa_\nu A_\mu - {}^*\Gamma^\alpha_{\mu\nu}A_\alpha).$$

We therefore take $$A_{\mu*\nu} = A_{\mu,\nu}-n\kappa_\nu A_\mu - {}^*\Gamma^\alpha_{\mu\nu}A_\alpha$$

as the co-covariant derivative of A_μ. It may be written, according to (3.2),

$$A_{\mu*\nu} = A_{\mu:\nu}-(n-1)\kappa_\nu A_\mu + \kappa_\mu A_\nu - g_{\mu\nu}\kappa^\alpha A_\alpha. \tag{3.3}$$

Similarly, for a vector B^μ of power n we find

$$B^\mu{}_{*\nu} = B^\mu{}_{:\nu}-(n+1)\kappa_\nu B^\mu + \kappa^\mu B_\nu - g^\mu_\nu \kappa_\alpha B^\alpha. \tag{3.4}$$

For a co-tensor with various suffixes upstairs and downstairs, we can form its co-covariant derivative by the same rules, bringing in the appropriate κ terms for each suffix. Note that the co-covariant derivative always has the same power as the original quantity.

One easily sees that the product law holds for any two co-tensors T and U,

$$(TU)_{*\sigma} = T_{*\sigma}U + TU_{*\sigma}.$$

Also $$g_{\mu\nu*\sigma} = 0, \quad g^{\mu\nu}{}_{*\sigma} = 0.$$

This means we can raise and lower suffixes freely in a co-tensor before carrying out co-covariant differentiation. Thus we may raise the μ in formula (3.3). The result is formula (3.4) with A^μ replacing B^μ and $n-2$ replacing n.

The potentials κ_μ do not form a co-vector as they have the wrong transformation laws (1.3). But the $F_{\mu\nu}$ defined by (1.4) are unaffected by gauge transformations, so they form an in-tensor.

We get the co-covariant divergence of a co-vector B^μ by putting $\nu = \mu$ in (3.4). Thus

$$B^\mu{}_{*\mu} = B^\mu{}_{:\mu}-(n+4)\kappa_\mu B^\mu. \tag{3.5}$$

Note that if $n=-4$, the co-covariant divergence is the same as the ordinary covariant divergence.

4. Second co-covariant derivatives

For a scalar S of power n, we have

$$S_{*\mu*\nu} = S_{*\mu:\nu}-(n-1)\kappa_\nu S_{*\mu}+\kappa_\mu S_{*\nu} - g_{\mu\nu}\kappa^\sigma S_{*\sigma}.$$

Substituting $S_{*\mu} = S_\mu - n\kappa_\mu S$, we get

$$S_{*\mu*\nu} = S_{\mu:\nu} - n\kappa_{\mu:\nu}S - n\kappa_\mu S_\nu - n\kappa_\nu(S_\mu - n\kappa_\mu S) + \kappa_\nu S_{*\mu} + \kappa_\mu S_{*\nu} - g_{\mu\nu}\kappa^\sigma S_{*\sigma}.$$

Now $S_{\mu:\nu} = S_{\nu:\mu}$, so

$$\begin{aligned} S_{*\mu*\nu} - S_{*\nu*\mu} &= -n(\kappa_{\mu:\nu}-\kappa_{\nu:\mu})S \\ &= -nF_{\mu\nu}S. \end{aligned} \tag{4.1}$$

Let A_μ be a co-vector of power n. We have

$$A_{\mu*\nu*\sigma} = A_{\mu*\nu:\sigma} - n\kappa_\sigma A_{\mu*\nu} + (g^\rho_\mu\kappa_\sigma + g^\rho_\sigma\kappa_\mu - g_{\mu\sigma}\kappa^\rho)A_{\rho*\nu} + (g^\rho_\nu\kappa_\sigma + g^\rho_\sigma\kappa_\nu - g_{\sigma\nu}\kappa^\rho)A_{\mu*\rho}.$$

A straightforward but lengthy calculation leads to the result

$$A_{\mu*\nu*\sigma} - A_{\mu*\sigma*\nu} = {}^*\mathscr{B}_{\mu\nu\sigma\rho}A^\rho - (n-1)F_{\nu\sigma}A_\mu, \tag{4.2}$$

where

$$\begin{aligned} {}^*\mathscr{B}_{\mu\nu\sigma\rho} = B_{\mu\nu\sigma\rho} + g_{\rho\nu}(\kappa_{\mu:\sigma} + \kappa_\mu\kappa_\sigma) + g_{\mu\sigma}(\kappa_{\rho:\nu} + \kappa_\rho\kappa_\nu) - g_{\rho\sigma}(\kappa_{\mu:\nu} + \kappa_\mu\kappa_\nu) \\ - g_{\mu\nu}(\kappa_{\rho:\sigma} + \kappa_\rho\kappa_\sigma) + (g_{\rho\sigma}g_{\mu\nu} - g_{\rho\nu}g_{\mu\sigma})\kappa^\alpha\kappa_\alpha. \end{aligned} \tag{4.3}$$

We may consider ${}^*\mathscr{B}_{\mu\nu\sigma\rho}$ as the generalized Riemann–Christoffel tensor. But it does not have the usual symmetries for a Riemann–Christoffel tensor. However, we can put

$${}^*\mathscr{B}_{\mu\nu\sigma\rho} = {}^*B_{\mu\nu\sigma\rho} + \tfrac{1}{2}(g_{\rho\nu}F_{\mu\sigma} + g_{\mu\sigma}F_{\rho\nu} - g_{\rho\sigma}F_{\mu\nu} - g_{\mu\nu}F_{\rho\sigma}), \tag{4.4}$$

and then ${}^*B_{\mu\nu\sigma\rho}$ has all the symmetries, namely

$${}^*B_{\mu\nu\sigma\rho} = -{}^*B_{\mu\sigma\nu\rho} = -{}^*B_{\rho\nu\sigma\mu} = {}^*B_{\nu\mu\rho\sigma},$$

and also

$${}^*B_{\mu\nu\sigma\rho} + {}^*B_{\mu\sigma\rho\nu} + {}^*B_{\mu\rho\nu\sigma} = 0.$$

Thus it would be appropriate to call ${}^*B_{\mu\nu\sigma\rho}$ the Riemann–Christoffel tensor for Weyl space. It is a co-tensor of power 2.

The contracted Riemann–Christoffel tensor is

$${}^*R_{\mu\nu} = {}^*B_{\mu\nu\sigma}{}^{\sigma} = R_{\mu\nu} - \kappa_{\mu:\nu} - \kappa_{\nu:\mu} - g_{\mu\nu}\kappa^\sigma{}_{:\sigma} - 2\kappa_\mu\kappa_\nu + 2g_{\mu\nu}\kappa^\sigma\kappa_\sigma. \tag{4.5}$$

It is an in-tensor. A further contraction gives the total curvature

$${}^*R = {}^*R^\sigma_\sigma = R - 6\kappa^\sigma{}_{:\sigma} + 6\kappa^\sigma\kappa_\sigma. \tag{4.6}$$

It is a co-scalar of power -2.

5. The action principle

We get field equations from an action principle with an action that is in-invariant. It is thus of the form

$$I = \int \Omega \sqrt{}\, \mathrm{d}^4x, \tag{5.1}$$

where Ω must be a co-scalar of power -4, to compensate for $\sqrt{}$ having the power 4.

The usual contribution to Ω from the electromagnetic field is $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. It is of power -4, since it can be written $F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\rho}$ and each of the F factors is of power zero and each of the g factors is of power -2. Thus it is suitable to be included in the present theory.

We need also a gravitational term in the action. In the usual Einstein theory we have the term $-R$ in Ω. In the present theory R should be replaced by the co-scalar ${}^*R$. However, this is of power -2 and will not do. Weyl proposed instead $({}^*R)^2$, which has the correct power -4, but is too complicated to be satisfactory. Weyl was able to develop his theory only by specializing it to one particular gauge, thereby ruining its most attractive feature.

One can bring the gravitational field into the theory by taking an action principle in which the equation ${}^*R = 0$ is assumed as a constraint. The power of ${}^*R$ then does not matter. One takes the constraint into account by adding to Ω the term $\gamma{}^*R$, where γ is a Lagrangian multiplier. γ then appears as a new field function. It is a co-scalar, and it must be assumed to be of power -2 to make $\gamma{}^*R$ of power -4.

We are led to a scalar–tensor theory of gravitation. The scalar field function is necessary if one is to avoid the great complications that would arise with Weyl's action principle. Such theories of gravitation have been considered previously, in particular by Fierz (1956) and by Jordan (1959), for the purpose of obtaining a varying gravitational constant, and by Brans & Dicke (1961), working from Mach's principle.

Once one has found the need for a scalar field function, one can very well add on to the action further terms containing its derivatives, like the previous authors did. It is convenient to put $\gamma = -\beta^2$ and consider β as the basic field variable. It is a co-scalar of power -1. We can then add to Ω the term $k\beta^{*\sigma}\beta_{*\sigma}$, which is a co-scalar of power -4, k being an arbitrary number. Without substantially increasing the complication we can add to Ω the further term $c\beta^4$, with c another arbitrary number. We then get the total

$$I = \int\{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \beta^2\,{}^*R + k\beta^{*\mu}\beta_{*\mu} + c\beta^4\}\sqrt{}\,\mathrm{d}^4x, \tag{5.2}$$

which may be taken as the action for the vacuum.

We have $\beta^{*\mu}\beta_{*\mu} = (\beta^\mu + \beta\kappa^\mu)(\beta_\mu + \beta\kappa_\mu)$. With the help of (4.6) we now get

$$-\beta^2\,{}^*R + k\beta^{*\mu}\beta_{*\mu} = -\beta^2R + k\beta^\mu\beta_\mu + (k-6)\,\beta^2\kappa^\mu\kappa_\mu + 6(\beta^2\kappa^\mu)_{:\mu} + (2k-12)\,\beta\kappa^\mu\beta_\mu.$$

The term involving $(\beta^2\kappa^\mu)_{:\mu}$ can be discarded, since its contribution to the action density is a perfect differential, namely

$$(\beta^2\kappa^\mu)_{:\mu}\sqrt{} = (\beta^2\kappa^\mu\sqrt{})_{,\mu}.$$

To get the simplest equations for the vacuum we now choose $k = 6$, so that (5.2) becomes

$$I = \int\{\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \beta^2R + 6\beta^\mu\beta_\mu + c\beta^4\}\sqrt{}\,\mathrm{d}^4x. \tag{5.3}$$

It will be observed that the action for the vacuum no longer involves the potentials κ_μ explicitly, but only through the field quantities $F_{\mu\nu}$. Thus the action is invariant under transformations $\kappa_\mu \to \kappa_\mu + \phi_{,\mu}$, and the equations of motion that follow from the action principle will be unaffected by such transformations. Such transformations should thus be considered as having no physical significance.

We now have the following transformations that can be applied to our equations:

(1) We may make any transformation of the coordinates.

(2) We may make any transformation of the metric gauge combined with the appropriate transformation of potentials $\kappa_\mu \to \kappa_\mu + \phi_{,\mu}$.

(3) In the vacuum we may make a transformation of the potentials $\kappa_\mu \to \kappa_\mu + \phi_{,\mu}$ without changing the metric gauge, or alternatively we may transform the metric gauge without changing the potentials.

It is important to note that the transformations (3) can in general be applied only where there is no matter. We shall see later that there are some simple kinds of matter for which the transformations (3) are still possible. But in general where matter is present there will be other terms in the action which may very well involve the potentials explicitly and so make the transformations (3) impossible.

The general equations of the theory are valid with any choice of the metric gauge. For practical calculations one chooses a gauge that is convenient for the work in hand. There are three choices of gauge that are of general interest.

The natural gauge. This involves taking $\kappa_\mu = 0$ when $F_{\mu\nu} = 0$. The natural gauge is not well defined when $F_{\mu\nu} \neq 0$. It requires only that κ_μ must be of the same order of magnitude as $F_{\mu\nu}$.

The Einstein gauge. This involves taking $\beta = 1$. It gives Einstein's gravitational equations when the electromagnetic quantities vanish.

The atomic gauge. This is the metric gauge that is measured by atomic apparatus.

All three gauges are liable to be different. But for the vacuum and for certain specially simple kinds of matter, one can apply a transformation of type 3 so as to make the natural gauge coincide with the Einstein gauge, or alternatively with the atomic gauge. There is no way of making the Einstein gauge coincide with the atomic gauge. It is just the difference of these two gauges that leads to the variation of the gravitational constant expressed in atomic units.

6. Field equations for the vacuum

We make small variations in all our field quantities $g_{\mu\nu}$, κ_μ, β and calculate the change in the action integral (5.3) and put it equal to zero.

Let us write

$$\delta I = \int (\tfrac{1}{2} P^{\mu\nu} \delta g_{\mu\nu} + Q^\mu \delta\kappa_\mu + S\,\delta\beta)\sqrt{}\,d^4x. \tag{6.1}$$

We shall drop the term $c\beta^4\sqrt{}$ from the action density, as it is probably not of importance except for questions of cosmology. The variation of the electromagnetic term yields (see Eddington, 1922, p. 188)

$$\delta(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\sqrt{}) = \tfrac{1}{2}E^{\mu\nu}\sqrt{}\,\delta g_{\mu\nu} - J^\mu\sqrt{}\,\delta\kappa_\mu, \tag{6.2}$$

with neglect of a perfect differential. Here $E^{\mu\nu}$ is the electromagnetic stress tensor

$$E^{\mu\nu} = \tfrac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu\alpha}F^{\nu}{}_{\alpha} \tag{6.3}$$

and J^μ is the charge–current vector

$$J^\mu = F^{\mu\nu}{}_{:\nu} = \sqrt{}^{-1}(F^{\mu\nu}\sqrt{})_{,\nu}. \tag{6.4}$$

Again

$$\begin{aligned}\delta(\beta^\mu\beta_\mu\sqrt{}) &= \beta^\mu\sqrt{}\,\delta\beta_\mu + \beta_\mu\sqrt{}\,\delta\beta^\mu + \beta^\mu\beta_\mu\,\delta\sqrt{} \\ &= 2\beta^\mu\sqrt{}\,\delta\beta_\mu + \beta_\mu\beta_\nu\sqrt{}\,\delta g^{\mu\nu} + \beta^\sigma\beta_\sigma\,\delta\sqrt{} \\ &= -2(\beta^\mu\sqrt{})_{,\mu}\,\delta\beta - \beta^\mu\beta^\nu\sqrt{}\,\delta g_{\mu\nu} + \tfrac{1}{2}\beta^\sigma\beta_\sigma g^{\mu\nu}\sqrt{}\,\delta g_{\mu\nu} \\ &= -2\beta^\mu{}_{:\mu}\sqrt{}\,\delta\beta - (\beta^\mu\beta^\nu - \tfrac{1}{2}g^{\mu\nu}\beta^\sigma\beta_\sigma)\sqrt{}\,\delta g_{\mu\nu},\end{aligned} \tag{6.5}$$

with neglect of a perfect differential.

From standard work in Einstein's theory one has (see Eddington, equations 60.3 and 58.52):

$$\delta(R\surd) = R_{\mu\nu}\delta(g^{\mu\nu}\surd) - \{g^{\mu\nu}\surd\delta(\Gamma^{\rho}_{\mu\nu} - g^{\rho}_{\mu}\Gamma^{\sigma}_{\nu\sigma})\}_{,\rho}.$$

Thus $$\delta(\beta^2 R\surd) = \beta^2 R_{\mu\nu}\delta(g^{\mu\nu}\surd) + 2\beta\beta_{\rho}g^{\mu\nu}\surd\delta(\Gamma^{\rho}_{\mu\nu} - g^{\rho}_{\mu}\Gamma^{\sigma}_{\nu\sigma}) + 2\beta R\surd\delta\beta,$$

with neglect of a perfect differential. A straightforward calculation now gives, with neglect of a further perfect differential,

$$\delta(\beta^2 R\surd) = \{\beta^2(\tfrac{1}{2}g^{\mu\nu}R - R^{\mu\nu}) + 2g^{\mu\nu}(\beta\beta^{\rho})_{:\rho} - 2(\beta\beta^{\mu})^{:\nu}\}\surd\delta g_{\mu\nu} + 2\beta R\surd\delta\beta. \quad (6.6)$$

Collecting together the terms in (6.2) minus (6.6) plus 6 times (6.5) and comparing with (6.1), we get

$$P^{\mu\nu} = E^{\mu\nu} + \beta^2(2R^{\mu\nu} - g^{\mu\nu}R) - 4g^{\mu\nu}\beta\beta^{\rho}{}_{:\rho} + 4\beta\beta^{\mu:\nu} + 2g^{\mu\nu}\beta^{\sigma}\beta_{\sigma} - 8\beta^{\mu}\beta^{\nu}, \quad (6.7)$$

$$Q^{\mu} = -J^{\mu}, \quad (6.8)$$

$$S = -2\beta R - 12\beta^{\mu}{}_{:\mu}. \quad (6.9)$$

The field equations for the vacuum are

$$P^{\mu\nu} = 0, \quad Q^{\mu} = 0, \quad S = 0. \quad (6.10)$$

It should be noted that these equations are not all independent. We have

$$P^{\sigma}_{\sigma} = -2\beta^2 R - 12\beta\beta^{\sigma}{}_{:\sigma} = \beta S, \quad (6.11)$$

so the S equation is a consequence of the P equations.

If one omits the electromagnetic term from the action, it becomes the same as the action of Brans & Dicke (1961), except that the latter allows an arbitrary value for the constant k. With k differing from 6, the vacuum equations are independent, and thus Brans & Dicke have one more vacuum field equation than the present theory. See their equation (13). In the absence of matter their $T = 0$, and for $k \neq 6$ their $(3 + 2\omega)$ does not vanish, so the extra equation is $\Box(\beta^2) = 0$.

7. The conservation laws

The action integral has two general invariance properties, namely, it is invariant under any transformation of the coordinate system and under any transformation of gauge. Each of these leads to a conservation law connecting the quantities $P^{\mu\nu}$, Q^{μ}, S defined by (6.1).

Consider first a transformation of coordinates, the point with coordinates x^{μ} being shifted to $x^{\mu} + b^{\mu}$, with b^{μ} infinitesimal. Following the method of Eddington, § 61, we get

$$-\delta g_{\mu\nu} = g_{\mu\sigma}b^{\sigma}{}_{,\nu} + g_{\nu\sigma}b^{\sigma}{}_{,\mu} + g_{\mu\nu,\sigma}b^{\sigma},$$

$$-\delta\beta = \beta_{\sigma}b^{\sigma},$$

$$-\delta\kappa_{\mu} = \kappa_{\sigma}b^{\sigma}{}_{,\mu} + \kappa_{\mu,\sigma}b^{\sigma}.$$

Substituting these variations into (6.1), we get

$$\begin{aligned}\delta I &= -\int\{\tfrac{1}{2}P^{\mu\nu}(g_{\mu\sigma}b^{\sigma}{}_{,\nu}+g_{\nu\sigma}b^{\sigma}{}_{,\mu}+g_{\mu\nu,\sigma}b^{\sigma})+Q^{\mu}(\kappa_{\sigma}b^{\sigma}{}_{,\mu}+\kappa_{\mu,\sigma}b^{\sigma})+S\beta_{\sigma}b^{\sigma}\}\surd \mathrm{d}^4x \\ &= \int\{(P^{\mu}_{\sigma}\surd)_{,\mu}-\tfrac{1}{2}P^{\mu\nu}g_{\mu\nu,\sigma}\surd+(Q^{\mu}\kappa_{\sigma}\surd)_{,\mu}-Q^{\mu}\kappa_{\mu,\sigma}\surd-S\beta_{\sigma}\surd\}\,b^{\sigma}\,\mathrm{d}^4x.\end{aligned}$$

This δI vanishes for arbitrary b^{σ}, so we can put the coefficient of b^{σ} equal to zero. With the help of the formulas

$$(P^{\mu}_{\sigma}\surd)_{,\mu}-\tfrac{1}{2}P^{\mu\nu}g_{\mu\nu,\sigma}\surd = P^{\mu}_{\sigma:\mu}\surd$$

(see Eddington, equation 51.51), and

$$(Q^{\mu}\kappa_{\sigma}\surd)_{,\mu} = \kappa_{\sigma}Q^{\mu}{}_{:\mu}\surd+\kappa_{\sigma,\mu}Q^{\mu}\surd,$$

this reduces to

$$P^{\mu}_{\sigma:\mu}+\kappa_{\sigma}Q^{\mu}{}_{:\mu}+F_{\sigma\mu}Q^{\mu}-S\beta_{\sigma} = 0. \tag{7.1}$$

Let us now make a small transformation of gauge, $\mathrm{d}s' = (1+\lambda)\,\mathrm{d}s$, with λ infinitesimal. We have

$$\begin{aligned}\delta g_{\mu\nu} &= 2\lambda g_{\mu\nu} \\ \delta\beta &= -\lambda\beta \\ \delta\kappa_{\mu} &= \{\log(1+\lambda)\}_{,\mu} = \lambda_{\mu}.\end{aligned}$$

Substituting these variations into (6.1), we get

$$\begin{aligned}\delta I &= \int\{P^{\mu\nu}\lambda g_{\mu\nu}+Q^{\mu}\lambda_{\mu}-S\lambda\beta\}\surd \mathrm{d}^4x \\ &= \int\{P^{\mu}_{\mu}\surd-(Q^{\mu}\surd)_{,\mu}-S\beta\surd\}\,\lambda\,\mathrm{d}^4x.\end{aligned}$$

Putting the coefficient of λ here equal to zero, we get

$$P^{\mu}_{\mu}-Q^{\mu}{}_{:\mu}-S\beta = 0. \tag{7.2}$$

Equations (7.1), (7.2) are the conservation laws. For the vacuum, we see that (7.2) is the same as (6.11), since $Q^{\mu}{}_{:\mu} = 0$ from (6.8). For the vacuum, (7.1) reduces to

$$P^{\mu}_{\sigma:\mu}+F_{\sigma\mu}Q^{\mu}-\beta^{-1}\beta_{\sigma}P^{\mu}_{\mu} = 0, \tag{7.3}$$

which may be considered as the generalization of the Bianci identities.

The conservation laws (7.1), (7.2) hold more generally than for the vacuum, namely, whenever the action integral can be constructed from the field variables $g_{\mu\nu}$, κ_{μ}, β alone. Thus they would hold with the term $k\beta^{*\mu}\beta_{*\mu}$ in Ω, with arbitrary k. If more field variables are needed to describe any matter that is present, there will be similar conservation laws with extra terms arising from the extra variables.

8. Motion of a particle

Let the coordinates of the particle be z^{μ}, functions of the proper time s measured along its world-line. Put $\mathrm{d}z^{\mu}/\mathrm{d}s = v^{\mu}$, the velocity vector. We have $v_{\mu}v^{\mu} = 1$, and v^{μ} is a co-vector of power -1.

We add to the action the further terms I_1+I_2, where

$$I_1 = -m\int\beta\,\mathrm{d}s, \tag{8.1}$$

$$I_2 = e\int\beta^{-1}\beta_{*\mu}v^\mu\,\mathrm{d}s, \tag{8.2}$$

m and e being constants. One easily sees that these further terms are in-invariants. We have

$$I_2 = e\int(\beta^{-1}\beta_\mu+\kappa_\mu)\,v^\mu\,\mathrm{d}s = e\int\{(\mathrm{d}/\mathrm{d}s)\,(\log\beta)+\kappa_\mu v^\mu\}\,\mathrm{d}s,$$

and the first term of the integrand here contributes nothing to the action principle. Thus

$$I_2 = e\int\kappa_\mu v^\mu\,\mathrm{d}s. \tag{8.3}$$

Note that I_2 is unchanged when one puts $\kappa_\mu+\phi_{,\mu}$ for κ_μ, since the extra term is $e\int(\mathrm{d}\phi/\mathrm{d}s)\,\mathrm{d}s$. So for a particle with the action I_1+I_2, the transformations (3) of § 5 are still possible.

Now make variations δz^μ in the z^μ. We have (see Eddington, equation 28.1):

$$2\mathrm{d}s\,\delta(\mathrm{d}s) = 2g_{\mu\nu}\,\mathrm{d}z^\mu\,\mathrm{d}(\delta z^\nu)+g_{\mu\nu,\sigma}\,\mathrm{d}z^\mu\,\mathrm{d}z^\nu\,\delta z^\sigma, \tag{8.4}$$

or

$$\delta(\mathrm{d}s) = \{g_{\mu\nu}v^\mu\,\mathrm{d}(\delta z^\nu)/\mathrm{d}s+\tfrac{1}{2}g_{\mu\nu,\sigma}v^\mu v^\nu\,\delta z^\sigma\}\,\mathrm{d}s. \tag{8.5}$$

Hence

$$\begin{aligned}\delta I_1 &= -m\int\{\beta(g_{\mu\nu}v^\mu\,\mathrm{d}(\delta z^\nu)/\mathrm{d}s+\tfrac{1}{2}g_{\mu\nu,\sigma}v^\mu v^\nu\,\delta z^\sigma)+\beta_\sigma\,\delta z^\sigma\}\,\mathrm{d}s\\ &= m\int\{\mathrm{d}(\beta g_{\mu\sigma}v^\mu)/\mathrm{d}s-\tfrac{1}{2}\beta g_{\mu\nu,\sigma}v^\mu v^\nu-\beta_\sigma\}\,\delta z^\sigma\,\mathrm{d}s\\ &= m\int\{g_{\mu\sigma}\,\mathrm{d}(\beta v^\mu)/\mathrm{d}s+\beta\Gamma_{\sigma\mu\nu}v^\mu v^\nu-\beta_\sigma\}\,\delta z^\sigma\,\mathrm{d}s.\end{aligned} \tag{8.6}$$

Also

$$\begin{aligned}\delta I_2 &= e\int\{\kappa_\mu\,\mathrm{d}(\delta z^\mu)/\mathrm{d}s+v^\mu\kappa_{\mu,\sigma}\,\delta z^\sigma\}\,\mathrm{d}s\\ &= e\int\{-\mathrm{d}\kappa_\sigma/\mathrm{d}s+v^\mu\kappa_{\mu,\sigma}\}\,\delta z^\sigma\,\mathrm{d}s\\ &= e\int v^\mu F_{\mu\sigma}\,\delta z^\sigma\,\mathrm{d}s.\end{aligned} \tag{8.7}$$

Adding (8.6) and (8.7) and equating to zero the total coefficient of δz^σ, we get

$$m\{g_{\mu\sigma}\,\mathrm{d}(\beta v^\mu)/\mathrm{d}s+\beta\Gamma_{\sigma\mu\nu}v^\mu v^\nu-\beta_\sigma\} = -ev^\mu F_{\mu\sigma},$$

or

$$m\{\mathrm{d}(\beta v^\mu)/\mathrm{d}s+\Gamma^\mu_{\rho\sigma}v^\rho v^\sigma-\beta^\mu\} = eF^{\mu\nu}v_\nu. \tag{8.8}$$

This is the equation of motion for a particle of mass m and charge e. If $e=0$, the trajectory may be called an in-geodesic.

We may work with the Einstein gauge, and then for the case $e=0$ we get the usual geodesic equation. It follows that the motion of the perihelion of Mercury is the same with the present theory as with the Einstein theory. Brans & Dicke (1961) assumed the usual geodesic equation still applies with varying β. They thus obtained a different motion for the perihelion of Mercury and were led to postulate an oblateness of the Sun.

9. Influence of the particle on the field

To make the theory of the particle complete, we should take into account also the variation of I_1 and I_2 produced by variations $\delta g_{\mu\nu}$, $\delta\kappa_\mu$, $\delta\beta$ at a particular field point, as well as the variations of these field quantities at the point z^μ arising from shifting z^μ. There is then a further term $\mathrm{d}z^\mu \mathrm{d}z^\nu \delta g_{\mu\nu}$ to be included in (8.4), leading to the further term $\frac{1}{2}v^\mu v^\nu \delta g_{\mu\nu} \mathrm{d}s$ to be included in (8.5). We then get

$$(\delta I_1)_{\text{extra}} = -m\int\{\tfrac{1}{2}\beta v^\mu v^\nu \delta g_{\mu\nu} + \delta\beta\}\,\mathrm{d}s.$$

We would like to express this extra bit of action in the form (6.1), say

$$(\delta I_1)_{\text{extra}} = \int(\tfrac{1}{2}P_1^{\mu\nu}\delta g_{\mu\nu} + S_1\delta\beta)\sqrt{}\mathrm{d}^4x.$$

We can do this with the help of the 4-dimensional δ-function $\delta_4(x-z)$ connecting a general field point x^μ with the particle point z^μ. We get

$$P_1^{\mu\nu}\sqrt{} = -m\int\beta v^\mu v^\nu \delta_4(x-z)\,\mathrm{d}s, \tag{9.1}$$

$$S_1\sqrt{} = -m\int\delta_4(x-z)\,\mathrm{d}s. \tag{9.2}$$

Similarly, we have

$$(\delta I_2)_{\text{extra}} = e\int v^\mu \delta\kappa_\mu\,\mathrm{d}s = \int Q_2^\mu \delta\kappa_\mu \sqrt{}\mathrm{d}^4x,$$

with

$$Q_2^\mu\sqrt{} = e\int v^\mu \delta_4(x-z)\,\mathrm{d}s. \tag{9.3}$$

The vacuum field equations (6.10) get modified to

$$P^{\mu\nu} + P_1^{\mu\nu} = 0, \quad Q^\mu + Q_2^\mu = 0, \quad S + S_1 = 0. \tag{9.4}$$

The particle gives rise to singularities in the field functions, expressible in terms of $\delta_4(x-z)$. The equations (9.4) should determine the singularities, but there may be technical difficulties in the application of the equations. We can avoid such difficulties by replacing the point particle by a particle with a finite size.

The equations of motion that we have obtained for a point particle may be applied to a dust of particles all moving along together and having e/m varying only slightly from one particle to a neighbouring one, so that we can have neighbouring particles keeping almost the same velocity. We can then pass from the dust to a continuous fluid, and there will be a definite velocity vector v^μ at each point of space.

Let us label each dust particle by three parameters λ_1, λ_2, λ_3. The z^μ then become functions of four variables λ_1, λ_2, λ_3, s, and m and e are functions of the three λ's. Any field point x^μ can now be specified by the four variables λ_1, λ_2, λ_3, s.

On passing to the continuous fluid, we shall have a mass density

$$\rho\sqrt{} = \int m_\lambda\,\mathrm{d}^3\lambda\int\delta_4(x-z)\,\mathrm{d}s = \int m_\lambda\,\delta_4(x-z)\frac{\partial(\lambda_1\lambda_2\lambda_3 s)}{\partial(x^1x^2x^3x^0)}\mathrm{d}^4x = m_\lambda\frac{\partial(\lambda_1\lambda_2\lambda_3 s)}{\partial(x^1x^2x^3x^0)}. \tag{9.5}$$

Similarly, we shall have a charge density

$$\sigma\surd = \int e_\lambda\, \mathrm{d}^3\lambda \int \delta_4(x-z)\,\mathrm{d}s = e_\lambda \frac{\partial(\lambda_1\lambda_2\lambda_3 s)}{\partial(x^1x^2x^3x^0)}. \tag{9.6}$$

ρ and σ are co-scalars of power -3.

To get the expressions for $P_1^{\mu\nu}$, S_1, Q_2^μ for a continuous distribution of matter, we must multiply the right-hand sides of (9.1), (9.2) and (9.3) by $\mathrm{d}^3\lambda$ and integrate over the three λ's. The results are

$$\left.\begin{aligned} P_1^{\mu\nu} &= -\beta v^\mu v^\nu \rho, \\ S_1 &= -\rho, \\ Q_2^\mu &= v^\mu \sigma. \end{aligned}\right\} \tag{9.7}$$

These values substituted in (9.4) will then give the field equations for a continuous distribution of matter. We may multiply both sides of equation (8.8) by $\delta_4(x-z)\,\mathrm{d}^3\lambda\,\mathrm{d}s$ and integrate. The result is

$$\rho\{(\beta v^\mu)_{,\nu} v^\nu + \Gamma^\mu_{\rho\sigma} v^\rho v^\sigma - \beta^\mu\} = \sigma F^{\mu\nu} v_\nu. \tag{9.8}$$

This is the equation of motion for a continuous distribution of matter.

We may suppose m_λ and e_λ vanish except in a small domain of the λ's. We then have matter existing only in a small tube. It would represent a particle of finite size. Its motion will be determined by the equations (9.4), (9.7), (9.8), which do not involve singularities.

10. Symmetry breaking

The Weyl interpretation of the electromagnetic field as influencing the geometry of space and not merely as something immersed in a Riemannian space has a striking consequence – symmetry breaking. Consider a charged particle and take a field point P close to its world-line. For simplicity, suppose the coordinate system to be chosen so that the particle is momentarily at rest.

Now take an element of length l at P and suppose it to be shifted by parallel displacement into the future, by an amount δx^0. From the fundamental formula, it will change by

$$\delta l = l\kappa_0\,\delta x^0.$$

Here κ_0 will consist mainly of the Coulomb potential arising from the charged particle. Suppose the sign of the charge is such that l increases when it is shifted into the future. With the opposite sign of the charge it will decrease. Now there is no symmetry between a quantity increasing and the same quantity decreasing. Consequently there is no symmetry between positive and negative charge.

If l increases when it is shifted into the future, it decreases when it is shifted into the past. So there is no symmetry between future and past. But if one changes the sign of the charge and also interchanges future and past, one gets back to the original situation.

Atomic physicists have introduced the operators P for changing the parity, C for charge conjugation and T for time reversal. In elementary theories all these symmetries are preserved. The present theory does not provide any breaking of the P symmetry. However, it does break the C and T symmetries, while preserving their product CT.

Experimentally, all three symmetries are observed to be broken, but the product PCT is conserved, so far as is known. It would seem that the breaking of the P symmetry must be ascribed to the short range forces. However, the breaking of C and of T, with preservation of CT, is caused by the long-range forces, if they are handled in accordance with Weyl's geometry. It would seem that this symmetry breaking arises from the interaction of the gravitational and electromagnetic fields. The effect must be small, because gravitational effects are always small in the atomic domain.

The symmetry breaking applies in the first place to the metric gauge, and the metric gauge is not directly observable. It shows itself up only in equations of motion. Thus the symmetry breaking is not an effect that can be directly measured in the geometry, but it will appear only in equations of motion.

For the vacuum, and for certain simple kinds of matter, such as the particle of § 8 and the continuous matter of § 9, one can apply transformations of the type 3 of § 5, and then transformations of the metric gauge can be made independently of transformations of the potentials. Under these circumstances symmetry breaking will not show up. It will show up only when a more complicated kind of matter is present, for which transformations of the metric gauge are coupled to transformations of the potentials.

As an example of this kind of matter, we may suppose a particle for which there is a term in the action

$$I_3 = a\int \beta^{-3}\beta^{*\mu}\beta_{*\mu}\,\mathrm{d}s, \tag{10.1}$$

where a is a number. This is an in-invariant. It can be written

$$I_3 = a\int (\beta^{-3}\beta^{\mu}\beta_{\mu} + 2\beta^{-2}\beta^{\mu}\kappa_{\mu} + \beta^{-1}\kappa^{\mu}\kappa_{\mu})\,\mathrm{d}s.$$

The middle term in the integrand here changes sign when one changes the sign of the potentials, so it is symmetry breaking. Time reversal changes the sign of β^{μ}, and this middle term is invariant if one changes the sign of both β^{μ} and κ_{μ}, so it does not break CT.

Of course one may use the Einstein gauge $\beta = 1$, and then this middle term vanishes. But symmetry breaking has not then vanished from the theory. Before one puts $\beta = 1$ one must obtain the field equation from the variation of β, namely $S = 0$, and this field equation will still show symmetry breaking.

The breaking of the C and T symmetries is a rare event experimentally. It has been observed only for one particle, the K meson. According to the present theory we should expect it to be a rare event, because it does not occur for simple particles consisting of just a mass carrying a charge. There must be some further complication in the particle, such as an action term like (10.1).

11. Conclusion

The foregoing theory involves a drastic revision of our ideas of space and time and the question arises: Why should one believe in it? Even if the variation of the gravitational constant is confirmed by experiment, it still will not prove that Weyl's geometry is needed to explain the electromagnetic field.

There is one strong reason in support of the theory. It appears as one of the fundamental principles of Nature that the equations expressing basic laws should be invariant under the widest possible group of transformations. The confidence that one feels in Einstein's theory arises because its equations are invariant under a very wide group, the group of transformations of curvilinear coordinates in Riemannian space. Of course even if space were flat, the equations of physics could still be expressed in terms of curvilinear coordinates and would still be invariant under transformations of the coordinates, but there would then exist preferred systems of coordinates, the rectilinear ones, and it would be only the group of transformations of the preferred coordinates that would be physically significant. The wider group of transformations of curvilinear coordinates would then be just a mathematical extension, of no importance for the discussion of physical laws.

The passage to Weyl's geometry is a further step in the direction of widening the group of transformations underlying physical laws. One now has to consider transformations of gauge as well as transformations of curvilinear coordinates and one has to take one's physical laws to be invariant under all these transformations, which imposes stringent conditions on them. There is no preferred gauge that one can restrict oneself to for dealing with physical laws and then consider the other gauges as just a mathematical extension that the physicist can ignore. The discussion of § 5 did suggest certain gauges that could be specially convenient, but there is more than one of them, and we should need to have a unique preferred gauge if we did not have a theory with invariance under all transformations of gauge.

The theory that has been set up here is intended as the basis of classical mechanics. The equations are not immediately applicable to the quantum theory. The whole problem of the passage from classical to quantum mechanics will have to be considered anew and will have to be adjusted to fit the two metrics ds_E and ds_A.

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