Differential Forms for Physics Students

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This is the writer's poison-pen letter addressed to *differential forms*, also known as *exterior calculus*. Having avoided them for years, at the urging of a colleague I decided to learn the formalism to see for myself if it's of any practical use for physics students. The answer is: no, differential forms really have no practical use, but there is one aspect of the formalism that makes them of some interest to students, which is that they point to a profound connection between general relativity, electromagnetism and quantum physics. This connection, which is difficult to see without the formalism, is provided by the *Cartan structure equations*, which all physics students should at least be aware of.

The key idea of differential forms is that they dispense with the usual indices of tensor analysis, thus making them valid in any coordinate system. However, certain indices invariably go along for the ride, and when it comes to actually calculating something truly useful (like the Schwarzschild metric) the student finds that the usual indices are necessary after all. Consequently, outside of some aesthetic appeal, differential forms can be safely omitted from the standard student curriculum.

In this paper I derive the Cartan structure equations in the most elementary manner possible. The mathematical formalism is not difficult, but the rules associated with the antisymmetry of differential forms can be exasperating to deal with, especially since most textbooks on the subject are overly mathematical and don't bother to adequately explain them. While it is assumed that the student has some familiarity with the notions of parallel transport and covariant derivatives, it's just basic algebra from there on out.

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1. Preliminaries

The student is expected to be familiar with the notion of a *metric*, a symmetric tensor that determines the length or magnitude of a vector. In a flat (Minkowski) space, vector length is given by

$$L^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$$

where $\eta_{\mu\nu}$ is the Minkowski tensor which, in four dimensions, is represented by the constant matrix

$$\eta_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The length of a vector in a curved space (in which gravitational, electromagnetic or other fields are present) is determined by

$$L^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

where $g_{\mu\nu}(x)$ is in general a non-constant tensor whose components appear in the more involved symmetric matrix

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & g_{11} & g_{12} & g_{13} \\ g_{02} & g_{12} & g_{22} & g_{23} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{bmatrix}$$

In a flat space $g_{\mu\nu}$ may or may not degenerate into the Minkowski metric, depending on which coordinate system is being employed.

In addition to the metric, there is another 4×4 matrix that determines the rotations (and boosts) of vector quantities whose components are usually expressed as $\Lambda^{\mu}{}_{\nu}$. They play an essential role in the calculus of differential forms, where they are normally written as $\omega^{\mu}{}_{\nu}$ (the notation we will be using). For pure rotations in a flat space, the matrix is antisymmetric in its off-diagonal elements $\omega^{i}{}_{j}$ (where $i, j = 1, 2, 3$), a property that will be useful later on.

1.1. Some Notation

We will be using quantities labeled by Latin and/or Greek indices, each spanning any dimension (although $n = 4$ is the most common). We will designate a vector in a flat (Minkowski) space as a *Lorentz vector* wearing a Latin index (V^a or V_a), the kind normally encountered in undergraduate physics courses. It acts like a scalar under a change of coordinates, but is changed by Lorentz transformations according to

$$V'^a = \Lambda^a{}_b V^b, \quad V'_a = \Lambda^b{}_a V_b$$

Conversely, we have the *world vector* or *coordinate vector* having a Greek index (V^μ or V_μ), which changes under a transformation of coordinates:

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu, \quad V'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu$$

Lorentz vectors thus live in a flat space devoid of gravitational, electromagnetic and other fields, while coordinate vectors live in a curved space.

Because we will be dealing with quantities that are antisymmetric with respect to an interchange in world indices, it is convenient to adopt a notation in which the lower-case indices of various differential form terms and their derivatives appear as close to these indices as possible. We therefore adopt the notation of Adler et al., which uses a single subscripted bar to denote partial differentiation,

$$\partial_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu} = A_{\nu|\mu}, \tag{1.1.1}$$

while covariant differentiation is denoted by a double subscripted bar, as in

$$A_{\mu||\nu} = A_{\mu|\nu} - A_\lambda \Gamma^\lambda_{\mu\nu} \tag{1.1.2}$$

where the Γ quantities are the Levi-Civita (or world) connection coefficients, symmetric in their lower indices.

1.2 Tetrads

Einstein's *principle of equivalence* states that at the local level a material body cannot distinguish between a gravitational field and an acceleration. Consequently, there must be a way to *transform away* a gravitational field at a point, so that the space appears locally flat. This is accomplished with the use of *tetrads* or *vierbeins* $e^a{}_\mu(x)$, which mix Lorentz and world indices. When applied to the curved-space metric $g_{\mu\nu}$, they allow a Minkowski frame to be set up at that point:

$$\eta_{ab} = g_{\mu\nu}(x)\, e^\mu{}_a(x)\, e^\nu{}_b(x) \tag{1.2.1}$$

Tetrads work the other way, too:

$$g_{\mu\nu}(x) = \eta_{ab}\, e^a{}_\mu \eta^b{}_\nu \tag{1.2.2}$$

Tetrads can be viewed as 4×4 spacetime-dependent matrices that have one "leg" (*Bein* in German) in flat space and the other in curved space. They obey the identities

$$e^\mu{}_a\, e^b{}_\mu = \delta^b{}_a, \quad e^a{}_\nu\, e^\mu{}_a = \delta^\mu{}_\nu \tag{1.2.3}$$

where the deltas are Kronecker deltas.

Tetrads can also change a Lorentz vector into a world vector and vice versa:

$$V^a = V^\mu\, e^a{}_\mu, \quad V^\mu = V^b e^\mu{}_b, \tag{1.2.4}$$

which also works for the lower-index cases

$$V_a = V_\mu \, e^\mu{}_a\,, \quad V_\mu = V_a \, e^a{}_\mu \tag{1.2.5}$$

In general, a tetrad is specified with the Lorentz index on top ($e^a{}_\mu$), so that the "flipped" tetrad is the matrix inverse, $e^\mu{}_a = (e^a{}_\mu)^{-1}$.

2. Review of Parallel Transport

If we move a vector in a flat space from one point to another, the vector's direction can either remain the same (as it would in a Cartesian coordinate system) or rotate (as it would in polar or spherical coordinates). However, a vector in a gravitational field is generally obliged to *always* change direction, since we cannot establish a locally-flat frame at two separate points (even if the distance between them is infinitesimal). This presents a problem when the issue of vector differentiation is considered: how can we compare the original vector and its transported version in a covariant manner, since the derivative of the vector depends on our ability to compare the vector at two different points? Remember that to be strictly covariant, we need to compare the vectors at the *same* point, which seems an impossible task.

The answer was worked out in the early 20th century by Elie Cartan and Hermann Weyl, who assumed the existence of a quantity which, when added to the transported vector at the new point, provides a "parallel" version of the original vector. This is the basis of *parallel transport* in differential geometry, which effectively allows the original and transported vector to be compared at the same point. In a curved space, we denote the infinitesimal difference between the original and transported vectors as

$$\delta V^\mu = -\Gamma^\mu_{\alpha\beta} \, V^\alpha dx^\beta \tag{2.1}$$

That is, the difference is proportional to the original vector and the distance it has been transported, with the connection acting as the proportionality term. The notion of parallel transport then provides a means for a covariant version of vector differentiation:

$$V^\mu{}_{||\alpha} = V^\mu{}_{|\alpha} + \Gamma^\mu_{\alpha\beta} \, V^\alpha dx^\beta \tag{2.2}$$

Similarly, for lower-indexed vectors we have

$$V_{\mu||\alpha} = V_{\mu|\alpha} - V_\lambda \, \Gamma^\lambda_{\mu\alpha} \tag{2.3}$$

The notion of *covariant differentiation* is easily extended from vectors to tensors.

In the absence of a gravitational field, the connection term Γ vanishes whenever $g_{\mu\nu} \to \eta_{\mu\nu}$. However, a vector can still rotate when transported (depending on the coordinate system), so we need a different way to express covariant differentiation for Lorentz vectors. In proceeding with this idea, from this point on we'll utilize Latin indices exclusively for all Lorentz quantities and Greek indices for all world quantities.)

In a flat space, all we have left to work with is the set of Lorentz transformations $\Lambda^a{}_b$, which we now identify with $\omega^a{}_b$. Parallel transport of a Lorentz vector is then given by

$$\delta V^a = -\omega^a{}_{b\lambda} \, V^b \, dx^\lambda \quad \text{and} \quad \delta V_a = \omega^b{}_{a\lambda} \, V_b \, dx^\lambda \tag{2.4}$$

where $\omega^a{}_{b\lambda}$ is a new kind of connection coefficient associated with flat-space parallel transport called the *spin connection*. Its resemblance to the Lorentz rotation matrix is not a coincidence, as we we see shortly. Thus, for Lorentz vectors we have the covariant derivatives

$$V^a{}_{||\mu} = V^a{}_{|\mu} + V_b \, \omega^b{}_{a\mu} \quad \text{and} \quad V_{a||\mu} = V_{a|\mu} - V_b \, \omega^b{}_{a\mu} \tag{2.5}$$

with similar definitions for Lorentz tensors.

3. Metricity

In Riemannian geometry the covariant derivative of the metric tensor $g_{\mu\nu}$ vanishes, a condition called *metricity*. We thus have the important identities

$$g_{\mu\nu||\lambda} = g^{\mu\nu}{}_{||\lambda} = 0$$

As is shown in any text on general relativity, metricity provides for a unique definition of the world connection coefficient:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\, g^{\lambda\beta}\left(g_{\mu\beta|\nu} + g_{\beta\nu|\mu} - g_{\mu\nu|\beta}\right)$$

Consequently, the connection vanishes in a flat space where the metric tensor is constant.

If we make the reasonable assumption that the covariant derivative of the Minkowski metric tensor ηab also vanishes, then we have the important expression

$$\eta_{ab||\lambda} = \eta_{ab|\lambda} - \omega^{c}{}_{a\lambda}\,\eta_{cb} - \omega^{c}{}_{b\lambda}\,\eta_{ac} = 0$$

Since $\eta_{ab|\lambda} = 0$, this reduces to

$$\omega^{c}{}_{a\lambda}\,\eta_{cb} + \omega^{c}{}_{b\lambda}\,\eta_{ac} = 0 \tag{3.1}$$

or

$$\omega_{ba\lambda} + \omega_{ab\lambda} = 0 \tag{3.2}$$

If we strip off the λ term, we recover the antisymmetry condition for the Lorentz rotation matrix:

$$\omega_{ba} + \omega_{ab} = 0$$

At this point the student may rightly question how we could so cavalierly drop the λ index from (3.1). The answer lies in the nature of ω_{ab}, which is related to a quantity known as a *differential 1-form*. That's the main topic of this paper, and we address it in the next section.

4. Differential Forms

In ordinary multivariable calculus, the student learns to treat integration arguments like $dxdy$ and $dxdydz$ with no regard to their ordering, so that $dxdy = dydx$, etc. But since $dxdy$ represents an infinitesimal element of area, it is necessarily *directional* (in this simple case, along the z-axis), so it is easy to imagine that if the normal vector associated with $dxdy$ points along the z-axis, then the normal vector of $dydx$ should point along the negative z-axis, so that $dxdy = -dydx$. The same reasoning holds for the infinitesimal volume element $dxdydz$ and elements of higher order, although the associated normal vectors are more difficult to visualize.

This odd behavior belongs to a class of mathematical quantities known as *Grassmann variables*, in which products change sign under reordering (e.g., $AB = -BA$). Consequently, in the calculus of differential forms the sign of a form is critically dependent on the ordering of the element string $dx^{\mu}dx^{\nu}\ldots$ associated with the form. We restrict this antisymmetry behavior to the differential elements dx^{μ} themselves (sometimes called *ur-vectors* in view of their fundamental nature), while the ordering of the underlying product terms is immaterial. These elements will always wear world indices; while elements such as dx^{a} are perfectly legitimate, we will not be using them. The reason for this is that differentiation in this paper will always be associated with a lower-case world index, making world indices for the vectors dx a necessity.

(The literature on differential forms often uses the *wedge symbol* $\wedge$ to denote exchange antisymmetry, as in $dx^{\mu}\wedge dx^{\nu} = -dx^{\nu}\wedge dx^{\mu}$. Per Zee's text, we will omit that symbol here, since the context of the arithmetic will always be evident.)

A differential 1-form is defined by

$$A = A_{\mu}\, dx^{\mu} \tag{4.1}$$

where summation over the index ($\mu = 0, 1, 2\ldots$) is always assumed. Since A is itself a kind of differential, it is not uncommon to see it used in integral form,

$$\int A = \int A_{\mu}\, dx^{\mu}$$

but in this elementary paper we will not be using it.

Similarly, there are differential 2-forms, which we define as

$$A = \frac{1}{2} A_{\mu\nu}\, dx^\mu dx^\nu = -\frac{1}{2} A_{\mu\nu}\, dx^\nu dx^\mu \tag{4.2}$$

An arbitrary p-form is defined as

$$A = \frac{1}{p!} A_{\mu_1\mu_2\ldots\mu_p}\, dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p} \tag{4.3}$$

For example, a 3-form is written as

$$A = \frac{1}{6} A_{\mu\nu\alpha}\, dx^\mu dx^\nu dx^\alpha$$

Note that the underlying terms in differential forms all obey strict antisymmetrization rules. If, for example, $A_{\mu\nu} = A_{\nu\mu}$, then the form vanishes identically because the symmetric terms cancel in pairs against the antisymmetric $dx^\mu dx^\nu$ term.

Sums of differing forms are allowed, provided they are both of the same p-form:

$$A + B = B + A = \frac{1}{p!} \left(A_{\mu_1\mu_2\ldots\mu_p} + B_{\mu_1\mu_2\ldots\mu_p} \right) dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p} \tag{4.4}$$

Lastly, please note that x^μ and dx^μ are not considered to be forms.

4.1. Products of Forms

Meanwhile, products can be expressed for differing forms (say, a p-form and a q-form) but, somewhat like matrices, they do not necessarily commute. Consider the product *AB*, where A is p-form and B is a q-form:

$$AB = \frac{1}{p!q!} \left(A_{\mu_1\mu_2\ldots\mu_p} B_{\nu_1\nu_2\ldots\nu_q} \right) dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p} dx^{\nu_1} dx^{\nu_2} \ldots dx^{\nu_q} \tag{4.1.1}$$

At this point it's important to realize that the terms $A_{\mu_1\mu_2\ldots\mu_p}$ and $B_{\nu_1\nu_2\ldots\nu_q}$ are just numbers, so they commute. This allows us to write (4) equivalently as

$$AB = \frac{1}{p!q!} \left(B_{\nu_1\nu_2\ldots\nu_q} A_{\mu_1\mu_2\ldots\mu_p} \right) dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p} dx^{\nu_1} dx^{\nu_2} \ldots dx^{\nu_q} \tag{4.1.2}$$

The trick now is to move all of the dx^{ν_i} terms to the left, past all of the dx^{μ_k} terms, so they will sit together. This will leave the B form in the lead, followed by the A_μ and dx^μ terms as A on the right of B. If p is an even number, then each of the dx^{ν_i} terms can be moved past the dx^{μ_k} terms without changing the overall sign. For example, if $p = 2$, then for any dx^i we have

$$dx^1 dx^2 dx^i = -dx^1 dx^i dx^2 = +dx^i dx^1 dx^2$$

so that (4.1.2) is just $AB = BA$. On the other hand, if p is an odd number, an overall minus sign will result. For example, if $p = 3$ then

$$dx^1 dx^2 dx^3 dx^i = -dx^1 dx^2 dx^i dx^3 = +dx^1 dx^i dx^2 dx^3 = -dx^i dx^1 dx^2 dx^3$$

However, if q is also an even number, this minus sign will cancel. For example, if $q = 2$ we will have

$$dx^1dx^2dx^3dx^{\nu_1}dx^{\nu_2} = -dx^1dx^2dx^{\nu_1}dx^3dx^{\nu_2} = +dx^1dx^{\nu_1}dx^2dx^3dx^{\nu_2} = -dx^{\nu_1}dx^1dx^2dx^3dx^{\nu_2}$$
$$= +dx^{\nu_1}dx^1dx^2dx^{\nu_2}dx^3 = -dx^{\nu_1}dx^1dx^{\nu_2}dx^2dx^3 = +dx^{\nu_1}dx^{\nu_2}dx^1dx^2dx^3$$

(isn't this fun?) so that again we have $AB = BA$. Similarly, if q is also an odd number we'll end up with $AB = -BA$. To summarize, if either p or q is an even number then we'll always have $AB = BA$. It is only when *both* p and q are odd numbers that have $AB = -BA$. This product behavior can be expressed simply as

$$AB = (-1)^{pq} BA \tag{4.1.3}$$

It is easy for the student to get confused by all this interchanging, but practice should resolve any difficulties.

Lastly, let us consider a simple misconception. Consider the product of two 1-forms:

$$AB = A_\mu B_\nu dx^\mu dx^\nu = -A_\mu B_\nu dx^\nu dx^\mu, \quad \text{or} \quad AB = -A_\mu B dx^\mu$$

where we restored the 1-form B, now sandwiched between A_μ and its associated dx^μ. It is tempting to now assume that $Bdx^\mu = -dx^\mu B$, but the student must not make this basic mistake for differential products in general. When interchanging terms in a differential product, everything must be broken down into the underlying terms, or there will be problems.

Notationally, that's pretty much it. All that remains now is for us to consider derivatives of forms, which will lead us into their gravitational and electromagnetic representations.

5. The Exterior Derivative

Consider the total differential operator d operating on an arbitrary function f:

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu$$

which is valid in any coordinate system. This can also be expressed as

$$d = dx^\mu \frac{\partial}{\partial x^\mu} \tag{5.1}$$

Thus, the total differential operator looks exactly like a differential 1-form, and it indeed is. It's conventionally known as the *exterior derivative*, but the student can safely go on calling it the total derivative. We will shortly see that there exists a covariant version known as the *exterior covariant derivative*.

Let us take the derivative of the 1-form $A = A_\mu dx^\mu$:

$$dA = dx^\mu \frac{\partial A}{\partial x^\mu} = dx^\mu \frac{\partial}{\partial x^\mu} \left(A_\nu dx^\nu\right) = A_{\nu|\mu} dx^\mu dx^\nu \tag{5.2}$$

Please note two things about this expression, which apply to the exterior derivative of *any* form: we didn't bother to differentiate dx^ν, and we placed the dx^μ term directly *after* the derivative of A_ν. The reason for both is purely conventional, although there's an important (and for the case of $d\,dx^\nu$, amusing) reason why double derivatives in the formalism are always set to zero.

Since the $dx^\mu dx^\nu$ term in (5.2) is antisymmetric, we can also write dA as

$$dA = \frac{1}{2}\left(A_{\nu|\mu} - A_{\mu|\nu}\right) dx^\mu dx^\nu \tag{5.3}$$

which we recognize as a 2-form. Let us now differentiate an arbitrary 2-form B, where we have

$$dB = \frac{1}{2} B_{\mu\nu|\lambda} dx^\lambda dx^\mu dx^\nu$$

By a cyclic permutation of indices in three copies of this equation (and taking into account the antisymmetry of the dx terms), this can be written as

$$dB = \frac{1}{3!}\left(B_{\mu\nu|\lambda} + B_{\lambda\mu|\nu} + B_{\nu\lambda|\mu}\right) dx^\lambda dx^\mu dx^\nu$$

which is a 3-form. In general, differentiation of a p-form yields a $(p+1)$-form.

What happens when we take twice the exterior derivative of a form? For the 1-form in (5.3), we have

$$d\,dA = d^2 A = A_{\nu|\mu|\lambda}\, dx^\lambda dx^\mu dx^\nu$$

From elementary calculus we know that the above derivative is symmetric in μ, λ; the antisymmetry of the corresponding $dx^\lambda dx^\mu$ term cancels this in pairs, so we have the very important identity $d\,dA = 0$ or, in minimal notation,

$$d\,d = 0 \tag{5.4}$$

which is an identity that holds for any differential form. As noted earlier, it is amusing to assume that the term $d\,dx^\nu$ must also vanish for the same reason, even though dx^ν is not a form!

Also of great importance is the exterior derivative of form products, ie., $d\,(AB)$. As compared with the exchange behavior of products derived earlier, the derivative problem is actually much easier to deal with. Consider the product of a p-form A with any other kind of form B. We have

$$d\,(AB) = d\left(A_{\mu_1\mu_2\ldots\mu_p} B_{\nu_1\nu_2\ldots}\right) dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p}\, dx^{\nu_1} dx^{\nu_2} \ldots$$

or

$$d\,(AB) = \left(A_{\mu_1\mu_2\ldots\mu_p|\lambda}\, B_{\nu_1\nu_2\ldots} + A_{\mu_1\mu_2\ldots\mu_p}\, B_{\nu_1\nu_2\ldots|\lambda}\right) dx^\lambda dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p} dx^{\nu_1} dx^{\nu_2} \ldots$$

The first term will always be just $(dA)B$. The second term,

$$A_{\mu_1\mu_2\ldots\mu_p}\, B_{\nu_1\nu_2\ldots|\lambda}\, dx^\lambda dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p} dx^{\nu_1} dx^{\nu_2} \ldots$$

can be written as

$$B_{\nu_1\nu_2\ldots|\lambda}\, A_{\mu_1\mu_2\ldots\mu_p}\, dx^\lambda dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p} dx^{\nu_1} dx^{\nu_2} \ldots$$

As we did before, we must now drag all the dx^{ν_i} terms to the left, past the dx^{μ_k} terms, and marry them up with the B terms. If p is an even number, there will be no sign change, and we'll have $d(AB) = (dA)B + A(dB)$. But if p is odd, there will be an overall minus sign, giving $d(AB) = (dA)B - A(dB)$. This rule can be summarized with

$$d\,(AB) = (dA)\,B + (-1)^p A\,(dB) \tag{5.5}$$

Note that this holds for any form B, including $B = A$!

6. The Tetrad Postulate and the First Cartan Structure Equation

With the completion of the above discussions, we can now begin the derivation of the Cartan structure equations. The first of these equations involves the exterior derivative of the tetrad.

It was noted earlier that the Lorentz and world versions of vectors can be connected using

$$V^a = e^a_{\;\mu} V^\mu$$

Let us parallel transport the vectors on both sides of this expression. The tetrad is not a vector, so it "transports" via the ordinary partial derivative:

$$\delta V^a = e^a_{\;\mu|\lambda}\, V^\mu dx^\lambda + e^a_{\;\mu} \delta V^\mu$$

Using the identities provided earlier, we then have

$$-V^b \omega^a_{\;b\lambda} dx^\lambda = e^a_{\;\mu|\lambda}\, V^\mu dx^\lambda - e^a_{\;\mu} \Gamma^\mu_{\nu\lambda}\, V^\nu dx^\lambda$$

Using (1.2.4), we can write this as

$$-e^b_{\ \nu} V^\nu \omega^a_{\ b\lambda} dx^\lambda = e^a_{\ \nu|\lambda} V^\nu dx^\lambda - e^a_{\ \mu} \Gamma^\mu_{\nu\lambda} V^\nu dx^\lambda$$

Dividing out the common V^b and dx^λ terms, we're left with

$$-e^b_{\ \nu} \omega^a_{\ b\lambda} = e^a_{\ \nu|\lambda} - e^a_{\ \mu} \Gamma^\mu_{\nu\lambda}$$

or

$$e^a_{\ \nu|\lambda} - e^a_{\ \mu} \Gamma^\mu_{\nu\lambda} + e^b_{\ \nu} \omega^a_{\ b\lambda} = 0 \tag{6.1}$$

The tetrad is not a vector, but it is a mixed tensor, and the above expression shows that its covariant derivative vanishes:

$$e^a_{\ \mu||\lambda} = 0 \tag{6.2}$$

The vanishing of the covariant derivative of the tetrad is known as the *tetrad postulate*. It might have been expected, given the metricity condition.

Let us now take a leap of faith and assume, as Cartan did, that the tetrad is a 1-form, which we write in condensed notation as

$$e^a = e^a_{\ \nu} dx^\nu \tag{6.3}$$

Its exterior derivative is given by

$$d\, e^a = e^a_{\ \nu|\mu} dx^\mu dx^\nu$$

We can use the tetrad postulate to get rid of the partial derivative, leaving us with

$$d\, e^a = -e^b_{\ \nu} \omega^a_{\ b\mu} + e^a_{\ \lambda} \Gamma^\lambda_{\mu\nu}$$

Let us now convert this to differential form notation:

$$d\, e^a = \left(-e^b_{\ \nu} \omega^a_{\ b\mu} + e^a_{\ \lambda} \Gamma^\lambda_{\mu\nu}\right) dx^\mu dx^\nu$$

The world connection is symmetric, so it drops out by virtue of the antisymmetry of $dx^\mu dx^\nu$, leaving

$$d\, e^a = -e^b_{\ \nu} \omega^a_{\ b\mu} dx^\mu dx^\nu \tag{6.4}$$

Let us now take another leap of faith, and assume that the spin connection is itself a 1-form,

$$\omega^a_{\ b} = \omega^a_{\ b\mu}\, dx^\mu \tag{6.5}$$

confirming our earlier assertion that the "stripped" version ($\omega^a_{\ b}$) of the $\omega^a_{\ b\mu}$ is related to the Lorentz rotation elements.

So, we finally have the *first Cartan structure equation* (6.4), which we can also express in condensed notation as

$$de^a = -\,\omega^a_{\ b}\, e^b \tag{6.6}$$

which is often written simply as $de = -\omega e$. The above equation is not of much utility in itself, but is used to derive the second Cartan equation, which expresses the gravitational equations of general relativity in differential form.

Before moving on, let us note that the first Cartan equation $de = -\omega e$ can also be written as $(d + \omega)\, e = 0$, strongly suggesting that the operator $d + \omega$ is the covariant form of the exterior derivative, making the *covariant exterior derivative* of the tetrad equal to zero, which the student has probably already surmised. This is in fact the case, and we'll employ it in the next section, but using a somewhat different approach.

7. The Second Cartan Structure Equation—the Riemann Curvature Tensor

In general relativity there are several ways to derive the Riemann curvature tensor, but the simplest involves taking the double covariant derivative of an arbitrary contravariant vector and subtracting its skew counterpart:

$$V^{\lambda}_{\ \ ||\mu||\nu} - V^{\lambda}_{\ \ ||\nu||\mu} = V^{\alpha} R^{\lambda}_{\ \alpha\mu\nu}$$

where $R^{\lambda}_{\ \alpha\mu\nu}$ is the Riemann curvature tensor

$$R^{\lambda}_{\ \alpha\mu\nu} = \Gamma^{\lambda}_{\alpha\nu|\mu} - \Gamma^{\lambda}_{\alpha\mu|\nu} + \Gamma^{\beta}_{\alpha\nu}\Gamma^{\lambda}_{\mu\beta} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\lambda}_{\nu\beta}$$

We can do exactly the same thing using a Lorentz vector, in which case we get the Lorentz form of the curvature tensor:

$$R^{a}_{\ b\mu\nu} = \omega^{\lambda}_{\ b\nu|\mu} - \omega^{\lambda}_{\ b\mu|\nu} + \omega^{a}_{\ c\mu}\,\omega^{c}_{\ b\nu} - \omega^{a}_{\ c\nu}\,\omega^{c}_{\ b\mu} \tag{7.1}$$

Now for a final leap of faith. We propose that the curvature tensor is in fact a differential 2-form which, following multiplying by $dx^{\mu}dx^{\nu}$ and consolidating, is

$$R^{a}_{\ b} = \frac{1}{2} R^{a}_{\ b\mu\nu}\,\omega^{a}_{\ b\nu|\mu}\,dx^{\mu}dx^{\nu} + \omega^{a}_{\ c\mu}\,\omega^{c}_{\ b\nu}\,dx^{\mu}dx^{\nu} \tag{7.2}$$

But $\omega^{a}_{\ b\nu|\mu}dx^{\mu}dx^{\nu} = d\omega^{a}_{\ b\nu}\,dx^{\nu}$, so we can write (7.2) as the curvature 2-form

$$R^{a}_{\ b} = d\omega^{a}_{\ b} + \omega^{a}_{\ c}\,\omega^{c}_{\ b} \tag{7.3}$$

or even more compactly as $R = d\omega + \omega^2 = (d+\omega)\,\omega$, with the covariant exterior derivative again making its presence known.

7.1. Several Properties of the Curvature Tensor

Let us take the second exterior derivative of the tetrad, which we know vanishes according to the $dd = 0$ rule:

$$d\,de^{a} = -d\left(\omega^{a}_{\ b\mu}e^{b}_{\ \nu}\right)dx^{\mu}dx^{\nu} = 0$$

Carrying out the differentiation in condensed notion, we get

$$\left(d\omega^{a}_{\ b}\right)e^{b} - \omega^{a}_{\ b}\left(de^{b}\right) = 0$$

where the minus sign results from the product law (4). Using $de^{a} = -\omega^{b}_{\ c}e^{c}$, this reduces to

$$\left(d\omega^{a}_{\ b} + \omega^{a}_{\ c}\omega^{c}_{\ b}\right)e^{b} = R^{a}_{\ b}e^{b} = 0 \tag{7.1.1}$$

Thus, the curvature 2-form is orthogonal to the tetrad 1-form. To see why this result is important, let us expand (7.1.1) in full differential form notation:

$$R^{a}_{\ b}e^{b} = \frac{1}{2}R^{a}_{\ b\mu\nu}\,e^{b}_{\ \lambda}\,dx^{\mu}dx^{\nu}dx^{\lambda} = \frac{1}{2}\,e^{a}_{\ \alpha}\,e^{\beta}_{\ b}\,e^{b}_{\ \lambda}\,R^{\alpha}_{\ \beta\mu\nu}\,dx^{\mu}dx^{\nu}dx^{\lambda} = 0$$

where we have introduced world indices into the curvature tensor using tetrads. Two of these tetrads contract, leaving

$$e^{a}_{\ \alpha}R^{\alpha}_{\ \lambda\mu\nu}dx^{\lambda}dx^{\mu}dx^{\nu} = 0$$

The remaining tetrad is now a mere spectator and can be divided out. By a cyclic permutation of the indices λ, μ, ν, the student should have no difficulty showing that this is equivalent to

$$R^{\alpha}_{\ \lambda\mu\nu} + R^{\alpha}_{\ \nu\lambda\mu} + R^{\alpha}_{\ \mu\nu\lambda} = 0 \tag{7.1.2}$$

which is a fundamental property of the Riemann curvature tensor.

The contracted curvature identity $R^{\lambda}{}_{\lambda\mu\nu} = 0$ is also easily derived from the above formalism. This corresponds to the contracted variant $R^a{}_a = R$ (not to be confused with the Ricci scalar). We therefore demand that

$$R^a{}_a = d\omega^a{}_a + \omega^a{}_b\omega^b{}_a = 0 \tag{7.1.3}$$

That (7.1.3) holds can be shown by the following argument. For one thing, the quantity $\omega^a{}_a = \eta^{ab}\omega_{ba}$ is identically zero in view of the symmetry of the Minkowski metric. The remaining term is a little more subtle. It is symmetric with respect to exchange of a, b but antisymmetric with respect to the form exchange $\omega^a{}_b\omega^b{}_a \to \omega^b{}_a\omega^a{}_b$, since it's a product of two 1-forms; see (4.1.3). This requires that $\omega^a{}_b\omega^b{}_a = 0$, establishing the Riemann identity (7.1.2).

7.2. Alternative Form of the Exterior Covariant Derivative

We would now like to derive the Bianchi identity from our formalism, which we assume will involve the use of the standard exterior covariant operator $d+\omega$ on the 2-form $R^a{}_b$. While we will indeed show that $(d+\omega)R = 0$, its connection with the Bianchi identity is not at all obvious, so we'll employ a somewhat different approach. We've in fact already used such an approach when we derived the tetrad postulate, when we showed that its covariant derivative vanishes.

Consider the possibility that there are actually two types of covariant derivatives, one that applies only to world vectors (involving the world connection Γ) and another that applies only to Lorentz vectors (involving the Lorentz connection ω). Call the first covariant derivative $D(\Gamma)$ and the other $D(\omega)$. When applied to vectors, we'll then have identities like

$$D(\omega)V^a = \left[V^a{}_{|\lambda} + V^b\omega^a{}_{b\lambda}\right]dx^\lambda \quad \text{and} \quad D(\Gamma)V^\mu = \left[V^\mu{}_{|\lambda} + V^\beta\,\Gamma^\mu_{\beta\lambda}\right]dx^\lambda \tag{7.2.1}$$

We'll also have the "null" identities

$$D(\omega)V^\mu = V^\mu{}_{|\lambda}dx^\lambda \quad \text{and} \quad D(\Gamma)V^a = V^a{}_{|\lambda}dx^\lambda \tag{7.2.2}$$

Let us now consider what we'll call the *total exterior covariant derivative operator* $D(\Gamma+\omega)$, which acts on mixed quantities. For example, the total exterior covariant derivative of the tetrad is

$$D(\Gamma+\omega)\,e^a{}_\mu = \left[e^a{}_{\mu|\lambda} + e^b{}_\mu\omega^a{}_{b\lambda} - e^a{}_\beta\Gamma^\beta_{\mu\lambda}\right]dx^\lambda = 0 \tag{7.2.3}$$

which confirms the earlier result in (6.2).

7.3. The Bianchi Identity in Differential Form Notation

We now proceed to derive the familiar Bianchi identity which, in world notation, is

$$R^\lambda{}_{\mu\nu\alpha||\beta} + R^\lambda{}_{\lambda\mu\nu||\alpha} + R^\lambda{}_{\alpha\lambda\nu||\mu} = 0 \tag{7.3.1}$$

which incidentally serves as the gateway to Einstein's gravitational field equations. So let us apply the total covariant derivative to the 2-form $R^a{}_b$:

$$D(\Gamma+\omega)R^a{}_b = D(\omega)R^a{}_b = R^a{}_{b|\lambda}dx^\lambda + R^c{}_b\,\omega^a{}_{c\lambda}dx^\lambda - R^a{}_c\,\omega^c{}_{b\lambda}dx^\lambda$$

Using the identities $R = d\omega + \omega^2$, $dd\omega = 0$ and the product rule for derivatives, we have, after restoring the appropriate indices,

$$D(\omega)R^a{}_b = \left[d\omega^a{}_c\,\omega^c{}_b - \omega^a{}_c\,d\omega^c{}_b + \left(d\omega^c{}_b + \omega^c{}_s\,\omega^s{}_b\right)\omega^a{}_c - \left(d\omega^a{}_c + \omega^a{}_s\,\omega^s{}_c\right)\omega^c{}_b\right]dx^\lambda$$

All of the terms on the right cancel, and we're left with

$$D(\omega)R^a{}_b = 0 \tag{7.3.2}$$

Let us now perform the same calculation using the total exterior covariant derivative on the full 2-form version of $R^a_{\ b} = e^a_{\ \alpha} e^\beta_{\ b} R^\alpha_{\ \beta}$, where we have converted the Lorentz indices in R to world indices with the tetrads. Since the tetrads act as constants under covariant differentiation, this is

$$D(\Gamma+\omega)R^a_{\ b} = \frac{1}{2}\, e^a_{\ \alpha} e^\beta_{\ b}\, D(\Gamma) R^\alpha_{\ \beta\mu\nu} dx^\mu dx^\nu = 0$$

But this is just

$$e^a_{\ \alpha} e^\beta_{\ b} R^\alpha_{\ \beta\mu\nu||\lambda}\, dx^\lambda dx^\mu dx^\nu = 0 \tag{7.3.3}$$

The two tetrads are spectators as far as the indices λ, μ, ν indices are concerned and can be dropped. By considering a cyclic permutation of these indices, along with the antisymmetry behavior of the $dx^\lambda dx^\mu dx^\nu$ term, the student should have no difficulty showing that (7.3.3) reduces to the Bianchi identity in (7.3.1).

8. The Electromagnetic Field as a Differential Form

As is noted in many texts, differential forms are almost tailor-made for the electromagnetic field, whose associated tensor is defined by the antisymmetric quantity

$$F_{\mu\nu} = A_{\nu|\mu} - A_{\mu|\nu}$$

where A_a is the electromagnetic four-potential, a 1-form:

$$A = A_\mu dx^\mu$$

This invites the identification of $F_{\mu\nu}$ as a 2-form,

$$F = \left(A_{\nu|\mu} - A_{\mu\nu}\right) dx^\mu dx^\nu = \frac{1}{2} A_{\nu||\mu}\, dx^\mu dx^\nu \tag{8.1}$$

which can be written as simply

$$F = dA \tag{8.2}$$

It is now quite easy to see that the quantity

$$dF = ddA = 0 \tag{8.3}$$

not only preserves the $dd = 0$ identity for differential forms but also expresses the homogeneous set of Maxwell's equations,

$$F_{\mu\nu|\lambda} + F_{\lambda\mu|\nu} + F_{\nu\lambda|\mu} = 0 \tag{8.4}$$

which correspond to

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0$$

of elementary electromagnetic theory.

For the non-homogeneous Maxwell equations, we need to introduce another kind of form, $*F$, where the so-called "Hodge star" formalism often makes its first appearance. Briefly (all too briefly), the Hodge star operator essentially *raises* the indices of a p-form, allowing subsequent differentiation of upper-indexed quantities. For the Maxwell source equations, the necessity of index-raising might have been expected, since

$$F^{\mu\nu}_{\ \ \ |\nu} = J^\mu$$

expresses the two non-homogeneous Maxwell equations. In differential form format, these equations appear simply as

$$d * F = J$$

where J is the electromagnetic source vector (note that charge conservation is implied by $dd*F = dJ = 0$). For the curious, the Hodge form for F is given by

$$*F = \frac{1}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\alpha\beta}\,F^{\mu\nu}dx^{\alpha}dx^{\beta} \tag{8.5}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$ and $\varepsilon_{\mu\nu\alpha\beta}$ is the completely antisymmetric Levi-Civita symbol. (Seeing this, the uninitiated student will probably be wondering where weird stuff like this comes from. She can thank the mathematicians.)

It was noted earlier that a general 2-form $CC = C^2$ like

$$C = \frac{1}{2}C_{\mu}C_{\nu}\,dx^{\mu}dx^{\nu}$$

will normally vanish, since the argument is symmetric with respect to interchange of C_{μ} and C_{ν}. But what if C_{μ} is not a number, but a matrix quantity? Then the argument need not commute, and we have the possibility of augmenting the electromagnetic form F with an additional term. In non-abelian quantum field theory, the four-potential A is indeed a matrix quantity that also acts as a connection term for the electromagnetic field. Since the non-abelian 1-form A does not vanish, we are free to add it to the 2-form F:

$$F = dA + A^2 = (d + A)\,A \tag{8.6}$$

It is tempting to consider $d + A$ as the exterior covariant derivative associated with electromagnetism, and indeed it is used to define the derivative of spinor quantities in the covariant treatment of quantum field theory in curved space. At our elementary level, however, it is more important that the student simply recognize the unexpectedly close similarity between the two form expressions

$$R = (d + \omega)\;\omega$$

$$F = (d + A)\,A$$

which serve as food for thought as to the possibility of an underlying theory unifying the gravitational and electromagnetic fields.

9. Final Comments

The calculus of differential forms is interesting, but it is hindered by a tendency to become overly abstract in the available literature, making it unappealing to the average physics student. The writer places the blame for this entirely on the part of mathematicians, whose typical quest for preciseness and love of mathematical abstraction often goes way beyond any relevance to physics. While this perhaps overly-harsh statement is not meant to diminish the countless indispensable contributions mathematicians have given to physicists, the writer simply does not feel that the calculus of differential forms is a truly necessary study topic for physics students.

The apparently deep connection between the gravitational and electromagnetic 2-forms revealed in this paper is fascinating, but the writer prefers to stick with the sentiment he expressed originally in the paper's abstract—the physics student need not be proficient in the theory of differential forms to acquire a working knowledge of gravity, electromagnetism and other basic fields in physics, up to and even beyond graduate school. It is hoped, however, that the student will acquire a deep and lasting appreciation for the concepts of *connections* and *covariant derivatives*, which profoundly link the physics of flat spaces with that of curved manifolds induced by electromagnetic and matter fields. Indeed, modern quantum *gauge theories*, which today are believed to underlie all mathematical theories of energy, matter and their interactions, would simply not be possible without these concepts.

References

1. R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity*, McGraw-Hill, 2nd Edition, 1975. Dated, but still one of the better texts of its kind for the undergraduate physics student who wants to learn the basic mathematics of general relativity.

2. M. Gasperini, *Theory of Gravitational Interactions*, Undergraduate Lecture Notes in Physics, Springer-Verlag Italia, 2013. Appendix A (The Language of Differential Forms) of this fine text on general relativity provides an ideal introduction to the subject for the advanced undergraduate/graduate student, with applications of the formalism to more advanced topics such as Einstein-Cartan torsion and supergravity. As of this writing, the appendix was available for download from

   `https://cds.cern.ch/record/1517921/files/978-88-470-2691-9_BookBackMatter.pdf`

3. T. Lancaster and S. Blundell, *Quantum Field Theory for the Gifted Amateur*, Oxford University Press, 2014. This text does not discuss differential forms, but provides a clear and understandable introduction to the notions of covariant derivatives, gauge transformations, and the Dirac, Weyl and Majorana spinor formalisms. One of the newer and better texts on a field that beginning graduate physics students often have trouble with.

4. A. Zee, *Einstein Gravity in a Nutshell*, Princeton University Press, 2013. Probably the best book available on gravitation for the student, Zee's section on differential forms is informative but too terse for a basic understanding of the subject, although it does provide some simple tetrad calculations. His presentation of the first Cartan structure equation in Chapter IX.7, however, avoids any mention of the tetrad postulate, making it incomprehensible.