

The Four-Frequency of Light

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November 2, 2007
(Revised June 8, 2014)

Photons are particles of light that have many amazing properties. Being a composition of transverse electric and magnetic fields, they can exhibit various types of polarization and, being massless, they necessarily travel at the speed of light. Depending on how they're observed they can exhibit both particle and wave characteristics, and thus can behave like harmonic oscillators. Unlike sound, pressure and water-wave transmission, photons do not need a medium to propagate through. They have well-defined energies and momenta whose formulas were discovered at the very beginning of the quantum era. Indeed, the discovery that light can travel through a vacuum and exhibit discrete energy levels in atomic emission actually ushered in the quantum era in the latter part of the 19th century. Finally, the fact that the speed of a photon is a true invariant, unchanging regardless of the speed of the observer in any inertial frame of reference, is a lasting testament to the seeming craziness of relativistic physics.

Photons travel on what is called a *null geodesic*. This means that the world interval or line element ds for a photon vanishes identically:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = 0$$

Physically, this means that the photon has no concept of space or time. In a very real sense it is eternal, with no beginning or end, and it exists everywhere in the universe at the same time. In the ordinary world that we inhabit, photons spring into existence when we turn on a light, only to be annihilated a short time later when they impinge on our corneas. But the photon doesn't see it this way at all — according to the photon, it was never born and it will never die. This extreme difference in points of view between the world of observers and that of photons is a consequence of $ds = 0$. It is nothing more than the *twin paradox* of special relativity taken to its ultimate extent.

Introduction

In special relativity, the mass-energy relationship is expressed as

$$E^2 = m^2 c^4 + c^2 p^2 \quad (1)$$

where, in Cartesian coordinates, the square of the three-momentum is $p^2 = p_x^2 + p_y^2 + p_z^2$. Equation (1) expresses the fact that the square of the length or magnitude of the momentum four-vector p^μ of a particle is related to its mass via

$$\eta_{\mu\nu} p^\mu p^\nu = m^2 c^2$$

where

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z \right)$$

or

$$p^\mu = mc \frac{dx^\mu}{ds} \quad (2)$$

and ds is the invariant line element. Since the mass m and the line element ds of a photon are both zero, the four-momentum as defined by (2) is indefinite, $0/0$. However, we can get around this difficulty by using the fact that the effective mass of a photon can be taken from $E = mc^2 = \hbar\omega$. We also have $E = cp$ and the de Broglie relationship $p = \hbar\omega/c = \hbar k$, where k is the magnitude of the wave number. Since \mathbf{k} is a three-vector whose magnitude is just ω/c , its components can differ only in direction. Using the unit direction vector of spherical coordinates, we can write $\mathbf{k} = \omega/c \mathbf{r}_0$, where \mathbf{r}_0 is the unit vector

$$\mathbf{r}_0 = \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix}$$

We can now express the four-momentum by the simple formula

$$p^\mu = \frac{\hbar\omega}{c} (1, \mathbf{r}_0) \quad (3)$$

Because p^μ represents the four-momentum of a photon, it should not be surprising to note that its invariant length, given by $\eta_{\mu\nu}p^\mu p^\nu$, is identically zero. *The photon four-momentum is a null four-vector.*

Since p^μ is a four-vector, it transforms like any other four-vector under a Lorentz transformation:

$$\begin{aligned} p^{0'} &= \gamma (p^0 - \beta \cdot \mathbf{p}) \\ \mathbf{p}' &= \gamma (\mathbf{p} - \beta p^0) \\ \gamma &= \frac{1}{\sqrt{1 - \beta^2}} \end{aligned}$$

where $\mathbf{p} = (\hbar\omega/c)\mathbf{r}_0$ and where the primed system S' is moving at some velocity $\beta = \mathbf{v}/c$ with respect to the unprimed system S . By exchanging systems, we get the equivalent expressions

$$p^0 = \gamma (p^{0'} + \beta \cdot \mathbf{p}') \quad (4)$$

and

$$\mathbf{p} = \gamma (\mathbf{p}' + \beta p^{0'}) \quad (5)$$

The photon four-momentum p^μ is also known as the *frequency four-vector* in view of its relationship with the frequency parameter ω .

The Aberration of Starlight, Pre-Einstein

Many years ago it was noticed that the positions of stars in the sky varied slightly over the course of a year. That is, if a telescope was pointed at a star at some angle in the heavens, six months later the star would seem to be pointed in a slightly different direction. This anomaly became known as the *aberration of starlight*. In pre-Einsteinian physics, it had a simple explanation.

Picture a star fixed in the heavens and situated directly overhead of an Earth-based telescope moving through space at velocity v (Figure 1). If the light from the star is directed vertically downward at an angle of 90° , then the telescope would have to be tilted at an angle of roughly v/c . Because the Earth orbits the Sun with a velocity of about $1 \times 10^{-4}c$, this angle works out to be approximately 20.6 seconds of arc (see diagram). Six months later, the Earth's position in its orbit around the Sun has shifted 180° so the Earth would appear to be moving in the opposite direction relative to the star. Consequently, the star would appear to have been shifted roughly v/c in the other direction, and the total yearly aberration comes to about 41 arc-seconds. When Einstein published his theory of special relativity in 1905, it provided an exact explanation for the aberration of starlight. To first order, it agrees perfectly with the pre-Einsteinian prediction of 20.6 seconds of arc.

The Aberration of Starlight in Special Relativity

The *relativistic* explanation of the aberration of starlight and the Doppler effect offer good opportunities to brush up on one's skills in special relativity. Many students learning SR for the first time get confused by the notions of reference frame, who's doing what, and how the Lorentz transformation relates to everything. Students often decide that they will be the observers in the non-moving reference frame S , while the moving frame S' flies by. Any frame is fine, but it should be remembered that if you set any *a priori* condition about what action is being taken, you must fix yourself in the frame in which it's taking place and stay there.

Keeping these basic notions in mind, let us consider a photon radiating from a star (which we will call system S') as seen by observer S on Earth (Figure 2). Although the star appears fixed in the heavens, the motion of the Earth around the Sun gives it the appearance of movement (taking into account the rotation of the Earth about its axis). If we consider the star to be in an inertial frame moving with constant velocity v in the positive y -direction with

respect to the observer, we can utilize the Lorentz transformation to analyze how the appearance of a photon changes as a result of this motion.

We will assume that the star emits a photon having a proper frequency ω' in an arbitrary direction specified by the proper angles θ' and ϕ' . Expanding the transformation equations in (4) and (5) and using the identity in (3), we have

$$p^0 = \gamma (p^{0'} + \beta \cdot \mathbf{p}')$$

or

$$\omega = \gamma \omega' (1 + \beta \sin \theta' \sin \phi') \quad (6)$$

Similarly,

$$p_x = p'_x \quad \text{or} \quad \omega \sin \theta \cos \phi = \omega' \sin \theta' \cos \phi' \quad (7)$$

along with

$$p_z = p'_z \quad \text{or} \quad \omega \cos \theta = \omega' \cos \theta' \quad (8)$$

and, lastly,

$$p_y = \gamma (p'_y + \beta p^{0'}) \quad \text{or} \quad \omega \sin \theta \sin \phi = \gamma \omega' (\beta + \sin \theta' \sin \phi') \quad (9)$$

where $\beta = v/c$, taken in the positive y -direction.

Let us now assume that a photon is observed from Earth along the angle $\theta = 0$; that is, directly above (appropriate for an Earthbound astronomer looking *up* into the night sky). This condition effectively places us in the S reference frame. We then have $\sin \theta = 0$, and the above equations reduce to

$$\omega = \gamma \omega' (1 + \beta \sin \theta' \sin \phi') \quad (10)$$

$$\sin \theta' \cos \phi' = 0 \quad (11)$$

$$\omega = \omega' \cos \theta' \quad (12)$$

and

$$\sin \theta' \sin \phi' = -\beta \quad (13)$$

Equations (10) and (13) immediately give

$$\omega = \gamma \omega' (1 - \beta^2)$$

or

$$\omega = \sqrt{1 - \beta^2} \omega' \quad (14)$$

The observer in the S frame thus sees the photon's frequency *redshifted*. (This seemingly innocuous identity has a very interesting property that we will return to shortly.) Using this result, it is now easy to show that

$$\cos \theta' = \sqrt{1 - \beta^2}$$

$$\sin \theta' = \pm \beta$$

or

$$|\tan \theta'| = \beta \gamma \quad (15)$$

which, for $\beta \ll 1$, gives approximately $\theta' = v/c$. Equation (15) indicates that for S to receive the photon from angle $\theta = 0$, it would have to be emitted from S' at an angle of v/c , in accordance with the pre-relativistic formula for the aberration of starlight.

To see this same effect from a different point of view, assume now that the photon is emitted instead from S' *downward* along the angle $\theta' = \pi$, a condition that effectively places us in the S' reference frame. We then have $\sin \theta' = 0$, and the above equations reduce to

$$\omega = \gamma \omega' \quad \text{or} \quad \omega' = \sqrt{1 - \beta^2} \omega \quad (16)$$

$$\sin \theta \cos \phi = 0 \quad (17)$$

$$\omega \cos \theta = -\omega' \quad (18)$$

and

$$\omega \sin \theta \sin \phi = \beta \gamma \omega' \quad (19)$$

Equation (16) now seems to be saying the exact opposite of (14)! But remember that we are now in the S' frame so, whatever S measures for the photon frequency, S' will see it *redshifted* as well (note that ω is now effectively the proper frequency of the photon). Equations (16) and (18) now give $\cos \theta = -\sqrt{1-\beta^2}$, $\sin \theta = \pm\beta$, or $|\tan \theta| = \beta\gamma$, which means that S must be viewing the photon along the angle v/c , consistent with our previous result.

The Relativistic Doppler Effect

Let us now assume that the system S' is *approaching* the observer in S *head-on* from the negative y -direction. Looking in that direction from the S frame means we will set $\theta = \phi = -\pi/2$. Equations (6) and (9) then give us $\sin \theta' \sin \phi' = 1$, so that

$$\omega = \gamma \omega' (1 + \beta)$$

or

$$\omega = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} \omega' \quad (20)$$

Therefore, the frequency of a light source approaching the observer head-on will appear to be increased — it is *blueshifted*. For a light source that is *receding* from the observer, we merely have to make the change $\beta \rightarrow -\beta$ in Equation (20), giving us

$$\omega = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} \omega' \quad (21)$$

In this case, the light frequency is *reduced*, so the light is *redshifted*. The amount of blue- or redshift in the elemental spectra from a star can be precisely determined using a spectrometer. In 1929, Edwin Hubble noticed that the stars in almost every galaxy he studied are redshifted, from which he deduced that the universe is expanding — a truly historic discovery.

Equations (20) and (21) are the expressions for the *Doppler effect* for the frequency shift of light. Unlike the case for sound waves, it makes no difference whether the source of light or the observer is fixed or moving (or even if both are in motion); only the relative velocity is relevant.

Transverse Doppler Effect

The Doppler effect for light closely parallels that for sound waves, of which everyone is familiar. But special relativity makes another prediction for light that has no classical counterpart. Consider (14) or (16) once again, which each correspond to the redshift in light frequency that appears in *both* reference frames when they pass perpendicularly or *transverse* to each frame's line of sight. Intuitively, one might expect no shift whatsoever, like the sound waves from a police siren passing perpendicularly to your line of sight. But the observed frequency is always *smaller* than the proper frequency of the source, a phenomenon known as the *transverse Doppler effect*. It is a purely relativistic effect, unknown in classical physics. It has been observed, and the experimental results agree with these results to within a fraction of a percent.

Since a photon having a well-defined frequency can be viewed as a ticking clock, it should not be surprising that the transverse Doppler effect is just a consequence of relativistic time dilation, which is given by

$$t = \gamma t'$$

Moving clocks appear to run slow and, since frequency has the units of inverse time, the transverse Doppler effect makes perfect sense.

References

1. French, A.P. (1968). Introduction to special relativity. W.W. Norton and Co., New York.
2. Jackson, J.D. (1975). Classical electrodynamics, 2nd. Ed. Wiley and Sons, New York.

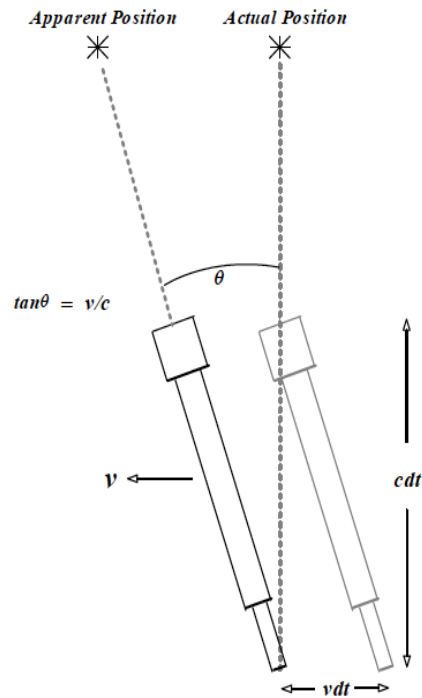


Figure 1: The aberration of starlight. Because of the Earth's orbital velocity with respect to a star (assumed fixed in the sky), a telescope must be tilted slightly to avoid the star's light from hitting the side of the telescope. The angle of tilt is approximately equal to v/c .

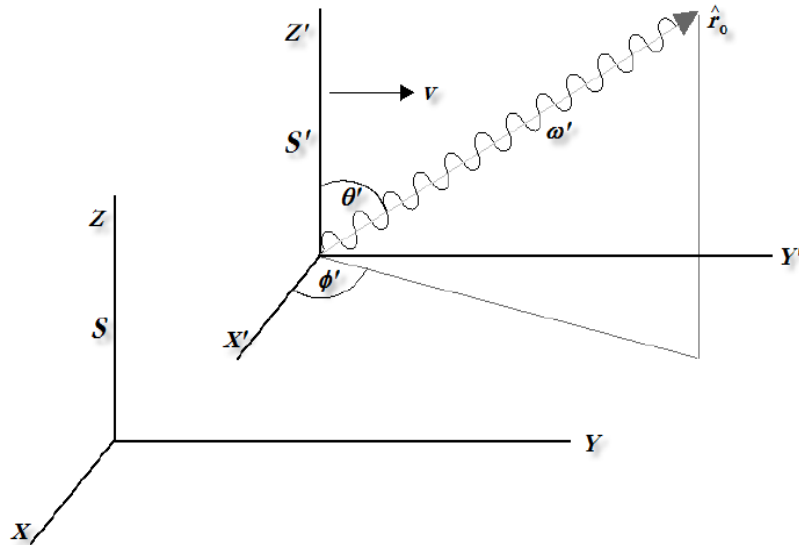


Figure 2: Definition of terms used in the text for a source of light in the system S' moving in the y -direction with respect to system S .