

The Four-Frequency in Special Relativity

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Pasadena, California
November 2, 2007

Photons are particles of light that have many amazing properties. Being a composition of transverse electric and magnetic fields, they can exhibit various types of polarization and, being massless, they necessarily travel at the speed of light. Depending on how they're observed they can exhibit both particle and wave characteristics, and thus can behave like harmonic oscillators. Unlike sound, pressure and water-wave transmission, photons do not need a medium to propagate through. They have well-defined energies and momenta whose formulas were discovered at the very beginning of the quantum era. Indeed, the discovery that light can travel through a vacuum and exhibit discrete energy levels in atomic emission actually ushered in the quantum era in the latter part of the 19th century. Finally, the fact that the speed of a photon is a true invariant, unchanging regardless of the speed of the observer in any inertial frame of reference, is a lasting testament to the seeming craziness of relativistic physics.

Photons travel on what is called a *null geodesic*. This means that the world interval or line element ds for a photon vanishes identically:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = 0$$

Physically, this means that the photon has no concept of space or time. In a very real sense it is eternal, with no beginning or end, and it exists everywhere in the universe at the same time. In the ordinary world that we inhabit, photons spring into existence when we turn on a light, only to be annihilated a short time later when they impinge on our corneas. But the photon doesn't see it this way at all – according to the photon, it was never born and it will never die. This extreme difference in points of view between the world of observers and that of photons is a consequence of $ds = 0$. It is nothing more than the *twin paradox* of special relativity taken to its ultimate extent.

The New Testament is replete with references to the relationship between light and God. Light is often mentioned there in its more conventional sense as the agent of discovery and absolute truth. It clarifies, purifies, reveals what is hidden, and makes known what is good or abhorrent. Light in the Bible is the opposite of darkness, the enemy of lies. Is God himself light? Well, that's really a religious question, so we won't try to address it here.

1. Introduction

In special relativity, the mass-energy relationship is expressed as

$$E^2 = m^2 c^4 + c^2 p^2 \tag{1.1}$$

where, in Cartesian coordinates, the square of the three-momentum is $p^2 = p_x^2 + p_y^2 + p_z^2$. Equation (1.1) expresses the fact that the square of the length or magnitude of the momentum four-vector p^μ of a particle is related to its mass via

$$\eta_{\mu\nu} p^\mu p^\nu = m^2 c^2 \tag{1.2}$$

where

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z \right) \tag{1.3}$$

$$= mc \frac{dx^\mu}{ds} \tag{1.4}$$

and ds is the invariant line element. We will assume that the flat-space Lorentz metric $\eta_{\mu\nu}$ has signature -2 , or $\eta_{\mu\nu} = (1, -1, -1, -1)$. Since the mass m and the line element of a photon are both zero, the four-momentum as defined by (1.4) is indefinite, $0/0$. However, we can get around this difficulty by using the fact that the effective mass of a photon can be taken from $E = mc^2 = \hbar\omega$. We also have $E = cp$ and the de Broglie relationship $p = \hbar\omega/c = \hbar k$, where k is the magnitude of the wave number. Since \mathbf{k} is a

three-vector whose magnitude is just ω/c , its components can differ only in direction. Using the unit direction vector of spherical coordinates, we can write $\mathbf{k} = \omega/c \mathbf{r}_0$, where

$$\mathbf{r}_0 = \begin{bmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{bmatrix}$$

We can now express the four-momentum as the rather unusual but simple quantity

$$p^\mu = \frac{\hbar\omega}{c}(1, \mathbf{r}_0) \quad (1.5)$$

Because p^μ represents the four-momentum of a photon, it should not be surprising to note that its invariant length, given by $\eta_{\mu\nu}p^\mu p^\nu$, is identically zero. *The photon four-momentum is a null four-vector.*

Since p^μ is a four-vector, it transforms like any other four-vector under a Lorentz transformation:

$$\begin{aligned} p^{0'} &= \gamma(p^0 - \boldsymbol{\beta} \cdot \mathbf{p}) \\ \mathbf{p}' &= \gamma(\mathbf{p} - \boldsymbol{\beta}p^0) \\ \gamma &= \frac{1}{\sqrt{1 - \beta^2}} \end{aligned}$$

where $\mathbf{p} = \hbar\omega/c \mathbf{r}_0$ and where the primed system S' is moving at some velocity $\boldsymbol{\beta} = \mathbf{v}/c$ with respect to the unprimed system S . By exchanging systems, we get the equivalent expressions

$$p^0 = \gamma(p^{0'} + \boldsymbol{\beta} \cdot \mathbf{p}') \quad (1.6)$$

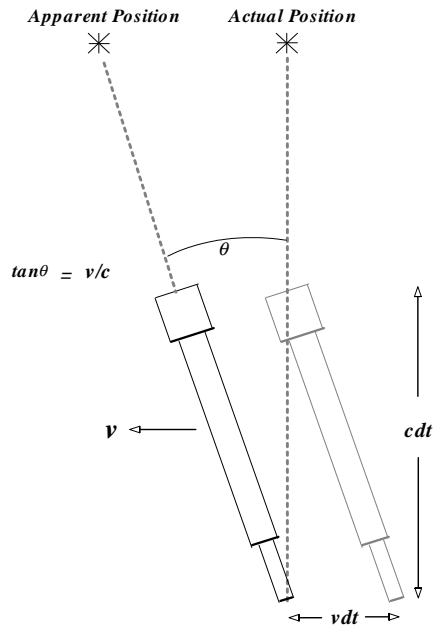
$$\mathbf{p} = \gamma(\mathbf{p}' + \boldsymbol{\beta}p^{0'}) \quad (1.7)$$

The photon four-momentum p^μ is also known as the *frequency four-vector* in view of its relationship with the frequency parameter ω .

2. The Aberration of Starlight

Many years ago, it was noticed that the positions of stars in the sky varied slightly over the course of a year. That is, if a telescope was pointed at a star directly overhead, six months later the star would seem to be pointed in a slightly different direction. This anomaly became known as the *aberration of starlight*. In pre-Einsteinian physics, it had a simple explanation.

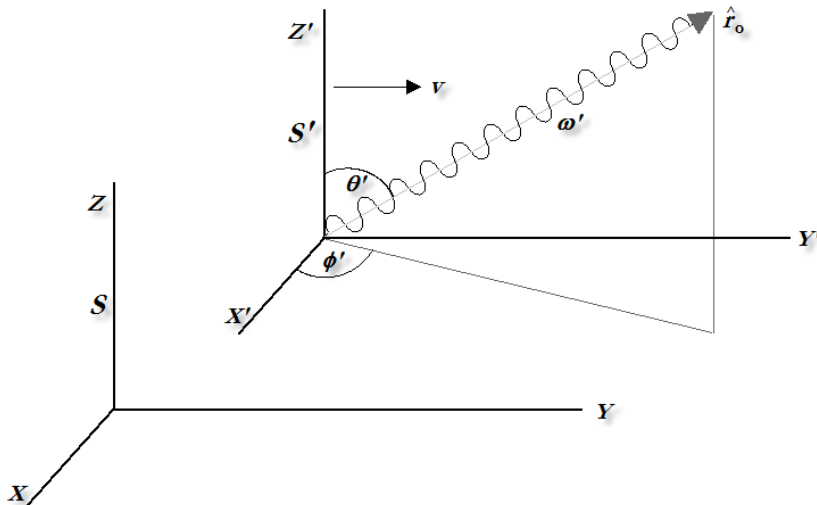
Picture a star fixed in the heavens and situated directly overhead of an Earth-based telescope moving through space at velocity v . If the light from the star is directed vertically downward at an angle of 90° , then the telescope would have to be tilted at an angle of roughly v/c . Because the Earth orbits the Sun with a velocity that is about $1 \times 10^{-4}c$, this angle works out to be approximately 20.6 seconds of arc (see diagram). Six months later, the Earth's position in its orbit around the Sun has shifted 180° so the Earth would appear to be moving in the opposite direction relative to the star. Consequently, the star would appear to have been shifted roughly v/c in the other direction, and the total yearly aberration comes to about 41 arc-seconds.



When Einstein published his theory of special relativity in 1905, it provided an exact explanation for the aberration of starlight. To first order, it agrees perfectly with the pre-Einsteinian prediction of 20.6 seconds of arc.

3. Aberration of Starlight in Special Relativity

Consider a photon radiating from a fixed star (which we will call system S') as observed by an observer on Earth (system S). Although the star is fixed, the motion of the Earth around the Sun gives it the appearance of movement with respect to the background. If we eliminate the effect of the Earth's rotational motion about its axis, the star will appear to shift slightly from one night to another. If we consider the star to be in an inertial frame moving with velocity v in the positive y -direction with respect to the observer, we can utilize the Lorentz transformation to analyze how the appearance of a photon changes as a result of this motion.



We will assume that the star emits a photon having a proper frequency ω' and an arbitrary direction specified by the proper angles θ' and ϕ' . Expanding the transformation equations in (1.6) and using the

identity in (1.5), we have

$$\begin{aligned} p^0 &= \gamma (p^{0'} + \boldsymbol{\beta} \cdot \mathbf{p}') & \text{or} \\ \omega &= \gamma \omega' (1 + \beta \sin \theta' \sin \phi') \end{aligned} \quad (3.1)$$

Similarly,

$$\begin{aligned} p_x &= p'_x & \text{or} \\ \omega \sin \theta \cos \phi &= \omega' \sin \theta' \cos \phi' \end{aligned} \quad (3.2)$$

along with

$$\begin{aligned} p_z &= p'_z & \text{or} \\ \omega \cos \theta &= \omega' \cos \theta' \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} p_y &= \gamma (p'_y + \beta p^{0'}) & \text{or} \\ \omega \sin \theta \sin \phi &= \gamma \omega' (\beta + \sin \theta' \sin \phi') \end{aligned} \quad (3.4)$$

where $\beta = v/c$. These expressions are valid for an arbitrary direction of the light ray. At this point we will simplify matters by demanding that the photon's direction lie in the yz plane, so that $\phi' = \pi/2$. From (3.2), we see that this requires $\phi = \pi/2$ as well. The other expressions then become

$$\omega = \gamma \omega' (1 + \beta \sin \theta') \quad (3.5)$$

$$\omega \cos \theta = \omega' \cos \theta' \quad (3.6)$$

$$\omega \sin \theta = \gamma \omega' (\sin \theta' + \beta) \quad (3.7)$$

Dividing (3.7) by (3.6), we get

$$\tan \theta = \gamma \frac{\sin \theta' + \beta}{\cos \theta'} \quad (3.8)$$

One of the central tenets of relativity is that there can be no preferred coordinate frame. This allows us to express (3.8) in terms of system S' by switching the primes and reversing the sign of β , getting

$$\tan \theta' = \gamma \frac{\sin \theta - \beta}{\cos \theta} \quad (3.9)$$

Let us now assume that system S measures the photon in the negative z direction. We then have $\theta = \pi$, so that

$$\tan \theta' = \beta \gamma \quad (3.10)$$

Equation (3.10) indicates that the photon would appear to have been emitted along the angle $\arctan(\beta \gamma)$ which, for $\beta \ll 1$, is approximately v/c , in accordance with the formula for the aberration of starlight.

4. The Doppler Effect

Let us now consider (3.5), which involves the relativistic transformation of the photon's frequency. It makes several interesting predictions, all of which have been famously verified. Let us first assume that system S' is approaching the observer in S from the negative- y direction (that is, from the left). Switching frames of reference as before, we have

$$\omega' = \gamma \omega (1 - \beta \sin \theta) \quad (4.1)$$

If S' is sufficiently far away, S will measure the photon moving almost head-on in the direction $\theta \simeq \pi/2$, so that

$$\begin{aligned} \omega &= \omega' \frac{\sqrt{1 - \beta^2}}{1 - \beta} & \text{or} \\ \omega &= \omega' \frac{\sqrt{1 + \beta}}{\sqrt{1 - \beta}} \end{aligned} \quad (4.2)$$

This shows that the frequency for an *approaching* photon appears to be increased or *blueshifted*, in agreement with observation. A similar analysis can be conducted for the case where S' is far to the right of the observer in S . In this case we find

$$\omega = \omega' \frac{\sqrt{1 - \beta}}{\sqrt{1 + \beta}} \quad (4.3)$$

so the frequency for light moving *away* from the observer is decreased or *redshifted*, again in agreement with experience. Equations (4.2) and (4.3) are the expressions for the *Doppler effect* for the frequency shift of light. Unlike the case for sound waves, it makes absolutely no difference whether the source of light or the observer is fixed or moving (or even if both are in motion); only the relative velocity is relevant.

5. Transverse Doppler Effect

The Doppler effect for light closely parallels that for sound waves, of which everyone is familiar. But special relativity makes another prediction for light that has no classical counterpart. Consider (3.11) once more, this time for the case $\theta = 0$. This corresponds to the observer in S measuring the frequency of the incoming photon in the positive z direction, which is transverse to the direction of motion. Equation (3.11) then gives us

$$\omega = \omega' \sqrt{1 - \beta^2} \quad (5.1)$$

That is, the observed frequency is *smaller* than the proper frequency. This phenomenon is known as the *transverse Doppler effect*. It is a purely relativistic effect and unknown in classical physics. It has been observed, and the experimental results agree with (5.1) to within a fraction of a percent.

Since a photon having a well-defined frequency can be viewed as a ticking clock, it should not be surprising that the transverse Doppler effect is just a consequence of relativistic time dilation, which is

$$t = \gamma t'$$

Moving clocks appear to run slow and, since frequency has the units of inverse time, the transverse Doppler effect makes perfect sense.