A Brief Look at Gaussian Integrals

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Gaussian integrals appear frequently in mathematics and physics. Perhaps the oddest thing about these integrals is that they cannot be evaluated in closed form over finite limits but are exactly integrable over $[-\infty, \infty]$. A typical Gaussian expression has the familiar "bell curve" shape and is described by the equation

$$y = \exp(-\frac{1}{2}ax^2)$$

where a is a constant.

In the path integral approach to quantum field theory, these integrals typically involve exponential terms with quadratic and linear terms in the variables. They are always multi-dimensional with limits at $[-\infty, \infty]$, and are therefore difficult to solve. For example, the probability amplitude for a scalar field $\varphi(x)$ to transition from one spacetime point to another can be expressed by

$$Z = \int \mathcal{D}\varphi \, \exp[i/\hbar \int d^4x[-\frac{1}{2}\varphi(\partial^2 + m^2)\varphi + J\varphi]$$

where the first integral is of dimension n and $\mathcal{D}\varphi = d\varphi_1(x)d\varphi_2(x)...d\varphi_n(x)$, where n goes to infinity. Needless to say, it helps to have a few formulas handy to calculate such quantities.

In quantum field theory, Gaussian integrals come in two types. The first involves ordinary real or complex variables, and the other involves Grassmann variables. The second type is used in the path integral description of fermions, which are particles having half-integral spin. Grassmann variables are highly non-intuitive, but calculating Gaussian integrals with them is very easy. Here, we'll briefly consider both types.

Gaussian Integrals with Ordinary Variables

The simplest Gaussian integral is

$$I = \int_{-\infty}^{\infty} \exp(-x^2)\, dx \tag{1}$$

which can solved solved quite easily. Squaring the quantity I, we get

$$I^2 = \int_{-\infty}^{\infty} \exp(-x^2)\, dx \int_{-\infty}^{\infty} \exp(-y^2)\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-(x^2+y^2)]\, dx\, dy$$

Going over to polar coordinates, we have $r^2 = x^2 + y^2$, $dx\, dy = r\, dr\, d\theta$, and

$$I^2 = \int_0^{\infty}\int_0^{2\pi} \exp(-r^2)\, r\, dr\, d\theta$$

Integration of this double integral is now straightforward, and we have

$$I = \int_{-\infty}^{\infty} \exp(-x^2)\, dx = \sqrt{\pi}$$

Most problems of this sort will have the constant factor $a/2$ in the exponential term, but by substitution you should be able to easily show that

$$I = \int_{-\infty}^{\infty} \exp(-\frac{1}{2}ax^2)\, dx = \sqrt{\frac{2\pi}{a}} \tag{2}$$

The presence of the constant a comes in handy when encountering integrals of the kind

$$\int x^m \exp(-\frac{1}{2}ax^2)\,dx \tag{3}$$

where m is some positive integer. All we have to do is take the differential dI/da of both sides, getting

$$\frac{dI}{da} = -\frac{1}{2}\int x^2 \exp(-\frac{1}{2}ax^2)\,dx = -\frac{1}{2}\sqrt{2\pi}a^{-3/2}$$

from which we easily find that

$$\int x^2 \exp(-\frac{1}{2}ax^2)\,dx = \sqrt{2\pi}a^{-3/2}$$

By repeated differentiation, it is easy to show that

$$\begin{aligned}\int x^{2n} \exp(-\frac{1}{2}ax^2)\,dx &= (2n-1)!!\sqrt{2\pi}\,a^{-(2n+1)/2} \\ &= \frac{(2n-1)!!}{a^n}\sqrt{\frac{2\pi}{a}} \\ &= \frac{(2n)!}{a^n 2^n n!}\sqrt{\frac{2\pi}{a}} \end{aligned} \tag{4}$$

where $(2n-1)!! = 1 \cdot 3 \cdot 5 \ldots 2n-1$.

It is easy to see that (3) will always be zero when m is odd, because the integrand is then odd in x while the integration limits are even. The fact that the integral exists only for even powers of x is important, because in the path integral approach to quantum field theory the factor x^{2n} is related to the creation and annihilation of particles, which always occurs in *pairs*.

The next most complicated Gaussian integral involves a linear term in the exponential, as in

$$I = \int_{-\infty}^{\infty} \exp(-\frac{1}{2}ax^2 + Jx)\,dx \tag{5}$$

where J is a constant. To evaluate this integral, we use the old high school algebra trick of completing the square. Since

$$-\frac{1}{2}ax^2 + Jx = -\frac{1}{2}a(x - \frac{J}{a})^2 + \frac{J^2}{2a}$$

we can use the substitution $x \to x - J/a$ to see that this is essentially the same as (2) with a constant term, so that

$$\int_{-\infty}^{\infty} \exp(-\frac{1}{2}ax^2 + Jx)\,dx = \exp[\frac{J^2}{2a}]\sqrt{\frac{2\pi}{a}} \tag{6}$$

We can evaluate integrals like

$$I = \int_{-\infty}^{\infty} x^m \exp(-\frac{1}{2}ax^2 + Jx)\,dx$$

using the same differentiation trick we used earlier, but now we have the choice of differentiating with respect to either a or J. It really makes no difference, but differentiating with J is a tad easier. In that case we have

$$\frac{dI}{dJ} = \int_{-\infty}^{\infty} x \exp(-\frac{1}{2}ax^2 + Jx)\,dx = \frac{J}{a}\exp[\frac{J^2}{2a}]\sqrt{\frac{2\pi}{a}}$$

Note that when $J = 0$, we get the same odd/even situation as before, and the integral is zero. But now a second differentiation gives

$$\frac{d^2 I}{dJ^2} = \int_{-\infty}^{\infty} x^2 \,\exp(-\frac{1}{2}ax^2 + Jx)\, dx = \frac{1}{a}\,\exp[\frac{J^2}{2a}]\sqrt{\frac{2\pi}{a}}(1+\frac{J^2}{a})$$

which is the same as (4) if we set $J = 0$. Taking **successive** derivatives leads to more complicated expressions, but they all reduce to (4) when $J = 0$. The business of setting $J = 0$ after differentiation is a fundamental technique in quantum field theory, as it is used to generate particles from the source term J. And again, it always only applies to even powers in the variable x (which becomes the field $\varphi(x)$ in the path integral).

Up to this point we have dealt only with Gaussian integrals having the single variable x. But in quantum field theory there can be an infinite number of variables, and so we need to investigate how the Gaussian integrals behave when the variable x becomes the n-dimensional vector $\boldsymbol{x}$, where the dimension n may be infinite. So to begin, let's look at the generalization of (2) in n dimensions, which looks like

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \exp(-\frac{1}{2}\boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x})\, dx_1 dx_2 ... dx_n \tag{7}$$

where $\boldsymbol{x}^T$ is a row vector, $\boldsymbol{x}$ is a column vector, and $\boldsymbol{A}$ is a symmetric, non-singular $n \times n$ matrix. The situation at first glance appears hopeless, but we can simplify things by transforming the vector $\boldsymbol{x}$ to some other vector $\boldsymbol{y}$ with an orthogonal matrix $\boldsymbol{S}$ ($\boldsymbol{S}^T = \boldsymbol{S}^{-1}$) with a determinant of unity:

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{S}\boldsymbol{y} \\ d\boldsymbol{x} &= \boldsymbol{S} d\boldsymbol{y} \\ dx_1 dx_2 \ldots &= |\boldsymbol{S}|\, dy_1 dy_2 \ldots \end{aligned}$$

with $|\boldsymbol{S}| = 1$. The integral now looks like

$$\int \exp(-\frac{1}{2}\boldsymbol{y}^T \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}\boldsymbol{y})\, dy_1 dy_2 ... dy_n$$

where I am now using just one integral sign for compactness. We now demand that the matrix $\boldsymbol{S}$ be a diagonalizing matrix, which means that

$$\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \begin{bmatrix} d_1 & 0 & 0 & \cdots \\ 0 & d_2 & 0 & \cdots \\ 0 & 0 & d_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$\boldsymbol{S}^{-1}\boldsymbol{A}^{-1}\boldsymbol{S} = \begin{bmatrix} 1/d_1 & 0 & 0 & \cdots \\ 0 & 1/d_2 & 0 & \cdots \\ 0 & 0 & 1/d_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

This changes the integral to

$$\int_{-\infty}^{\infty} \exp[-\frac{1}{2}(d_1 y_1^2 + d_2 y_2^2 + \ldots d_n y_n^2)]\, dy_1 dy_2 ... dy_n$$

With the variables thus separated, we can use (2) to write the answer as

$$\int_{-\infty}^{\infty} \exp(-\frac{1}{2}\boldsymbol{x}^T \boldsymbol{A}\boldsymbol{x})\, dx_1 dx_2 ... dx_n = \sqrt{\frac{2\pi}{d_1}} \cdot \sqrt{\frac{2\pi}{d_2}} \cdots \sqrt{\frac{2\pi}{d_n}} = \frac{(2\pi)^{n/2}}{|\boldsymbol{A}|^{1/2}} \tag{8}$$

which reduces to (2) for the case $n = 1$ (remember that $|\boldsymbol{A}| = d_1 d_2 \cdots d_n$). In quantum field theory the numerator blows up as $n \to \infty$, but this term ends up in an overall normalization constant, so we don't worry about it too much.

Now we have to tackle the n-dimensional version of (5), which is a bit tricky. This integral looks like

$$\int_{-\infty}^{\infty} \exp(-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{J}^T\boldsymbol{x})\, dx_1 dx_2 ... dx_n \tag{9}$$

where $\boldsymbol{J}$ is a vector whose elements are all constants [in quantum field theory, the J's become source functions for the field $\varphi(\boldsymbol{x})$, and thus are very important]. We proceed as before, using the diagonalizing matrix $\boldsymbol{S}$, with the transformation

$$-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{J}^T\boldsymbol{x} \to -\frac{1}{2}\boldsymbol{y}^T\boldsymbol{D}\boldsymbol{y} + \boldsymbol{J}^T\boldsymbol{S}\boldsymbol{y}$$

Expanded, this looks like

$$-\frac{1}{2}(d_1 y_1^2 + d_2 y_2^2 + \ldots\, d_n y_n^2) + J_a S_{a1} y_1 + J_a S_{a2} y_2 + \ldots\, J_a S_{an} y_n$$

where summation over the index a is assumed. We now have to complete the square for every y_i term. For example, the first set of terms will be

$$-\frac{1}{2}d_1 y_1^2 + J_a S_{a1} y_1 = -\frac{1}{2}d_1(y_1 - \frac{J_a S_{a1}}{d_1})^2 + \frac{(J_a S_{a1})^2}{2d_1}$$

which can be expressed with matrix summation notation as

$$-\frac{1}{2}d_i(y_i - \frac{J_a S_{ai}}{d_i})^2 + \frac{(J_a S_{ai})^2}{2d_i}$$

Note that division by d_i [$= (S^{-1}AS)_{ii}$] looks a tad odd but, because d_i is a diagonal element, the summation notation is preserved. Now, by again making the substitution $y_i \to y_i - J_a S_{ai}/d_i$, we have (2) all over again, so the integral becomes simply

$$\int_{-\infty}^{\infty} \exp(-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{J}^T\boldsymbol{x})\, d^n x = \frac{(2\pi)^{n/2}}{|\boldsymbol{A}|^{1/2}} \exp[\frac{(J_a S_{ai})^2}{2d_i}]$$

While correct, the exponential term can be put into a more recognizable form. First, we have to ask ourselves: just what exactly is $(J_a S_{ai})^2/2d_i$? It has to be a scalar, and the only way to write this as such is

$$\frac{J_a S_{ai}(J_b S_{bi})^T}{2d_i} = \frac{J_a S_{ai} S_{ib}^{-1} J_b}{2d_i}$$

(remember that $\boldsymbol{S}^T = \boldsymbol{S}^{-1}$). It would appear now that $S_{ai}S_{ib}^{-1} = \delta_{ab}$, but this would give $\boldsymbol{J}^T\boldsymbol{J}/2d_i$, with an indexed denominator that has nowhere to go. So instead we put the $1/d_i$ *in the middle* and write

$$\frac{J_a S_{ai} S_{ib}^{-1} J_b}{2d_i} = \frac{1}{2}J_a S_{ai} \cdot \frac{1}{d_i} \cdot S_{ib}^{-1} J_b \tag{10}$$

Now recall that matrix theory guarantees that the diagonal matrix defined by $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{D}$ allows a similar expression for the inverse of $\boldsymbol{A}$, which is $\boldsymbol{S}^{-1}\boldsymbol{A}^{-1}\boldsymbol{S} = \boldsymbol{D}^{-1}$. In matrix summation notation, this is

$$S_{ik}^{-1} A_{kb}^{-1} S_{bi} = D_{ii}^{-1} = \frac{1}{d_i}$$

Plugging this into (10), the $\boldsymbol{S}^{-1}\boldsymbol{S}$ terms cancel at last, and we have

$$\begin{aligned}
\frac{1}{2}J_a S_{ai} \cdot \frac{1}{d_i} \cdot S_{ib}^{-1} J_b &= \frac{1}{2}J_a S_{ai} S_{ik}^{-1} A_{kb}^{-1} S_{bi} S_{ic}^{-1} J_c \\
&= \frac{1}{2}J_k A_{kc}^{-1} J_c \\
&= \frac{1}{2}\boldsymbol{J}^T\boldsymbol{A}^{-1}\boldsymbol{J}
\end{aligned}$$

The Gaussian integral is thus

$$\int_{-\infty}^{\infty} \exp(-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{J}^T\boldsymbol{x})\, dx_1 dx_2 ... dx_n = \frac{(2\pi)^{n/2}}{|\boldsymbol{A}|^{1/2}} \exp[\frac{1}{2}\boldsymbol{J}^T\boldsymbol{A}^{-1}\boldsymbol{J}] \tag{11}$$

In quantum field theory, the matrix $\boldsymbol{A}^{-1}$ is related to what is known as a *propagator*, and it's akin to the $\partial^2 + m^2$ term in the probability amplitude we presented earlier. It is essentially a Green's function that describes the physical propagation of a particle created at the source $\boldsymbol{J}$ and annihilated at the "antisource" or sink $\boldsymbol{J}^T$. The matrix identity $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$ gets transcribed into $(\partial^2 + m^2)D(\boldsymbol{x}-\boldsymbol{y}) = \delta^4(\boldsymbol{x}-\boldsymbol{y})$ for the path integral, where the quantity $D(\boldsymbol{x}-\boldsymbol{y})$ is the propagator.

There is one other Gaussian integral that I would like to look at, and it's the one with a function $f(x)$ sitting in the exponential term. In fact, we can let it be a function of many independent variables, $f(x_1, x_2 \ldots x_n)$. And, in order to bring up a salient point in quantum mechanics, let us write the integral as

$$I = \int_{-\infty}^{\infty} \exp[-i/\hbar\, f(x)]\, d^n x$$

where $\hbar$ is Planck's constant divided by 2π, which is roughly $1.05{\times}10^{-34}$ joule-seconds. (In quantum mechanics the function $f(x)$ might be the action Lagrangian, which expresses the many paths that a particle can take in going from one point to another.) Because $\hbar$ is a tiny number, the exponential term oscillates violently and the integral does not approach any finite value. However, amid all this oscillation there is a relatively stable region that occurs around some set of points (call it x_0) such that the function $f(x_0)$ is a *minimum.* A Taylor expansion around these points gives

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}\, f''(x_0)(x - x_0)^2 + \ldots$$

which, because $f'(x)$ vanishes at x_0, goes to

$$f(x) \approx f(x_0) + \frac{1}{2!}\, f''(x_0)(x - x_0)^2$$

where derivative terms higher than the second are ignored. Using (2), we can then straightforwardly write the integral as

$$\int_{-\infty}^{\infty} \exp[-i/\hbar\, f(x)]\, d^n x = \frac{(i\pi\hbar)^{n/2}}{|f''(x_0)|^{1/2}} \tag{12}$$

where $|f''(x_0)|$ is the determinant of the second derivative evaluated at the minimum. In the case of the action Lagrangian in quantum mechanics, the minimum occurs where the points describe the *classical path* of a particle. It is in this sense that classical physics is recovered in the limit $\hbar \to 0$.

Gaussian Integrals with Grassmann Variables

All of the Gaussian integrals we have looked at so far involve real variables, and the generalization to complex numbers presents no special problems. In quantum field theory, this situation corresponds to the path integral for a free scalar (bosonic) field. However, the path integral for the fermionic field involves field quantities $\psi_i(x,t)$ that anticommute ($\psi_i\psi_j = -\psi_j\psi_i$). Thus, the sequence of field terms in the integrand $d\psi_1 d\psi_2 \ldots d\psi_n$ needs careful attention. Classically, anticommuting variables are normally restricted to matrices, but the field quantities ψ_i are certainly not matrices. So what else is available?

Fortunately, there is a non-matrix, anticommuting mathematical quantity called a *Grassmann number* that obeys just the type of algebra we need. Discovered in 1855 by the German mathematician Hermann Grassmann, a Grassmann number (or G-number) anticommutes with other G-numbers under multiplication. Thus, given the G-quantities ψ and ξ, their product anticommutes, so that $\psi\xi = -\xi\psi$ and $\psi\psi = \xi\xi = 0$. This idea is perhaps disorienting, because it seems bizarre that a number (not an operator or matrix) can behave in such a manner. Grassmann algebra is just an abstract mathematical construct that, until the advent of quantum field theory, was of little practical use.

Abstract or not, G-numbers are easy to work with. For example, the Taylor series expansion of any function $f(\psi)$ of a G-variable is just $f(\psi) = a + b\psi$, since all higher powers of ψ are identically zero (the quantities a, b will be assumed here to be ordinary numbers, but they can be Grassmannian as well). Similarly, the exponential of of a G-variable is just $\exp(a\psi) = 1 + a\psi$. Functions of two or more G-variables can be also be expanded, as in $f(\psi, \Theta) = a + b\psi + c\Theta + e\psi\Theta$. The extension to functions of n G-variables is obvious, but only the last term in the expansion carries all n variables. In addition, ordinary numbers commute as usual with G-numbers ($a\psi = \psi a$). However, given a string of three G-numbers $\psi\xi\eta$, it is easy to see that $(\psi\xi)\eta = \eta(\psi\xi)$, so that the product of two Grassmann numbers can behave like an ordinary number.

Let us now consider the integral of some G-variable function, as in $\int d\Theta\, f(\Theta)$. Expanded, this is

$$\int_{-\infty}^{\infty} d\Theta\, f(\Theta) = \int d\Theta (a + b\Theta) \tag{13}$$

$$= a \int d\Theta + b \int d\Theta\, \Theta \tag{14}$$

It would appear that these integrals will blow up under direct integration, but Grassmann showed us a way around this. Consider the fact that ordinary integrals with infinite limits are invariant with respect to a translation of the independent variable:

$$\int_{-\infty}^{\infty} dx\, f(x) = \int_{-\infty}^{\infty} dx\, f(x + c)$$

Assuming that the corresponding Grassmann integrals behave similarly, we write

$$\int d\Theta\, f(\Theta + c) = \int d\Theta\, [a + b(\Theta + c)]$$
$$= (a + bc) \int d\Theta + b \int d\Theta\, \Theta$$

Equating this last equation with (14), we find that the integral of $d\Theta$ vanishes:

$$\int d\Theta = 0 \tag{15}$$

The remaining term, $\int d\Theta\, \Theta$, cannot be determined, so we are free to set it equal to anything we want; furthermore, it is the product of two Grassmann quantities, so it must behave as an ordinary number. A convenient value is unity:

$$\int d\Theta\, \Theta = 1 \tag{16}$$

Comparing the last two expressions, we see that the integral of $d\Theta$ is consistent with the remarkable operator identity

$$\int d\Theta = \frac{\partial}{\partial \Theta} \tag{17}$$

That is, integration and differentiation with respect to a G-variable is the same thing! (This certainly ranks as one of the strangest of all mathematical identities.) We can now see why $\int d\Theta$ vanishes, because trivially

$$\int d\Theta\, (1) = \frac{\partial (1)}{\partial \Theta} = 0$$

For consistency, all terms in the integrand must be located to the right of the differentials. However, differential operations by Grassmann integrals can go either way, left or right, but here we'll only be using them to the right.

It is a now simple matter to extend this formalism to multiple integrals (for compactness, we use only one integral sign). For example, in three dimensions we have the operator identity

$$\int d\Theta_1 d\Theta_2 d\Theta_3 = \frac{\partial}{\partial\Theta_1}\frac{\partial}{\partial\Theta_2}\frac{\partial}{\partial\Theta_3}$$
$$= \frac{\partial^3}{\partial\Theta_1\partial\Theta_2\partial\Theta_3}$$

Unlike ordinary multiple partial differentiation, the ordering of the differentials is important. If a quantity being operated on by the integral does not have a multiplicative term involving all three variables, the result is zero. For example, if we're given a function of only two G-variables, $f(\Theta_1, \Theta_2)$, then the integral $\int d\Theta_1 d\Theta_2 d\Theta_3\, f(\Theta_1, \Theta_2)$ vanishes. For $f(\Theta_1, \Theta_2, \Theta_3) = a + b\Theta_1 + c\Theta_2 + d\Theta_3 + e\Theta_1\Theta_2 + f\Theta_1\Theta_3 + g\Theta_2\Theta_3 + h\Theta_1\Theta_2\Theta_3$, we have

$$\int d\Theta_1 d\Theta_2 d\Theta_3\, f(\Theta_1, \Theta_2, \Theta_3) = h \int d\Theta_1 d\Theta_2 d\Theta_3\, \Theta_1\Theta_2\Theta_3$$
$$= -h \int d\Theta_1 d\Theta_2 d\Theta_3\, \Theta_3\Theta_2\Theta_1$$
$$= -h$$

where we have used the identity $\Theta_1\Theta_2\Theta_3 = -\Theta_3\Theta_2\Theta_1$. Only the last term survives; all of the other terms give zero because they have fewer than three variables. While the precise ordering of the $d\Theta_i$ terms in the integral is arbitrary, it is conventional to fix the sequence of the terms in the integral as either $d\Theta_1 d\Theta_2 \ldots d\Theta_n$ or $d\Theta_n \ldots d\Theta_2 d\Theta_1$; we then adjust the last term in $f(\Theta_i)$ to do the integration. (The phrase "doing the integral" seems out of place here, since integration in the usual sense never occurs!)

Obviously, doing Gaussian G-integrals is going to be very easy. The simplest Gaussian involves just two variables,

$$\int d\Theta_1 d\Theta_2\, \exp(-\frac{1}{2}a\Theta_1\Theta_2) = \int d\Theta_1 d\Theta_2\, (1 - \frac{1}{2}a\Theta_1\Theta_2)$$
$$= \frac{1}{2}a$$

but, for the sake of generality, integrals involving two or more variables should be expressed in a form similar to (7), and so we write

$$\int d^n\Theta\, \exp(-\frac{1}{2}\boldsymbol{\Theta}^T\boldsymbol{A}\boldsymbol{\Theta})$$

where $d^n\Theta = d\Theta_1 d\Theta_2 \ldots d\Theta_n$ and $\boldsymbol{\Theta}^T = [\Theta_1 \quad \Theta_2 \quad \ldots \Theta_n]$. But now we notice a peculiar thing–if, as before, we take the matrix $\boldsymbol{A}$ to be symmetric, we find that, because of the antisymmetry of the Grassmann variables, all of the diagonal terms of $\boldsymbol{A}$ will be missing from the expanded exponential, while all the off-diagonal terms in the integrand will appear like $(a_{ij} - a_{ji})\Theta_i\Theta_j$. Consequently, if $\boldsymbol{A}$ is symmetric the integral vanishes! Therefore, we take $\boldsymbol{A}$ to be an *antisymmetric* matrix. Furthermore, since the expansion

$$\int d^n\Theta\, \exp(-\frac{1}{2}\boldsymbol{\Theta}^T\boldsymbol{A}\boldsymbol{\Theta}) = \int d^n\Theta\, [1 - \frac{1}{2}\boldsymbol{\Theta}^T\boldsymbol{A}\boldsymbol{\Theta} + \frac{1}{2!}(\frac{1}{2}\boldsymbol{\Theta}^T\boldsymbol{A}\boldsymbol{\Theta})^2 - \ldots \frac{1}{m!}(\frac{1}{2}\boldsymbol{\Theta}^T\boldsymbol{A}\boldsymbol{\Theta})^m]$$

invariably involves $\Theta_1\Theta_2 \ldots \Theta_n$ powers of 2 in the exponential, we see that the dimension n must always be an *even* number, so $m = n/2$ (as you will soon see, this is closely related to the fact that the determinant of an odd-ranked antisymmetric matrix is zero). In fact, because the number of terms must be consistent, only the *last* term in the expansion of the exponential survives in the integration. For example,

$$\int d^6\Theta\, \exp(-\frac{1}{2}\boldsymbol{\Theta}^T\boldsymbol{A}\boldsymbol{\Theta}) = \int d^6\Theta\, [-\frac{1}{3!}(\frac{1}{2}\boldsymbol{\Theta}^T\boldsymbol{A}\boldsymbol{\Theta})^3]$$

For the sake of familiarity and practice, let's do the integral for the case $n = 4$, where $\boldsymbol{A}$ is a 4×4 antisymmetric matrix and $\boldsymbol{\Theta}^T = [\Theta_1 \quad \Theta_2 \quad \Theta_3 \quad \Theta_4]$. We thus have

$$\int d^4\Theta\, \exp(-\frac{1}{2}\boldsymbol{\Theta}^T\boldsymbol{A}\boldsymbol{\Theta}) = \int d\Theta_1 d\Theta_2 d\Theta_3 d\Theta_4\, [\frac{1}{2!}(-\frac{1}{2}\boldsymbol{\Theta}^T\boldsymbol{A}\boldsymbol{\Theta})^2]$$

First we must calculate $\boldsymbol{\Theta}^T \boldsymbol{A}\boldsymbol{\Theta}$, which is, after some simplification,

$$[\Theta_1 \quad \Theta_2 \quad \Theta_3 \quad \Theta_4] \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \Theta_4 \end{bmatrix} = 8\Theta_1\Theta_2\Theta_3\Theta_4(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})$$

But the quantity in parentheses is just the square root of the determinant $|\boldsymbol{A}|$, so

$$\int d^4\Theta \, \exp(-\frac{1}{2}\boldsymbol{\Theta}^T \boldsymbol{A}\boldsymbol{\Theta}) = |\boldsymbol{A}|^{1/2}$$

It can be shown that this result obtains for any n, so that

$$\int d^n\Theta \, \exp(-\frac{1}{2}\boldsymbol{\Theta}^T \boldsymbol{A}\boldsymbol{\Theta}) = |\boldsymbol{A}|^{1/2} \tag{18}$$

Note that, in contrast with (8), the determinant term is in the numerator.

The Grassmann version of (9) is complicated by several factors. First, "completing the square" must be handled differently, as all terms like Θ^2 and higher are zero. Second, $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$ is not diagonal for an antisymmetric matrix (although it can be *block diagonal*). Third, the exponential part involving the linear term $\boldsymbol{J}^T\boldsymbol{\Theta}$ must be expanded to a higher power than $\boldsymbol{\Theta}^T\boldsymbol{A}\boldsymbol{\Theta}$ to obtain the same sequence $\Theta_1\Theta_2\ldots$ required for the integration. And lastly, the J_i must also be Grassmann variables, so that $J_i\Theta_k = -\Theta_k J_i$. Because of these complications, we will only summarize the result, which is

$$\int d^n\Theta \, \exp(-\frac{1}{2}\boldsymbol{\Theta}^T A\boldsymbol{\Theta} + \boldsymbol{J}^T\boldsymbol{\Theta}) = |\boldsymbol{A}|^{1/2} \exp(\frac{1}{2}\boldsymbol{J}^T\boldsymbol{A}^{-1}\boldsymbol{J}) \tag{19}$$

which is structurally identical to (11).

Finally, let's consider a Gaussian integral in which there are two independent Grassmann fields $\boldsymbol{\psi}$ and $\bar{\boldsymbol{\psi}}$ along with their respective linear terms:

$$\int d^n\bar{\boldsymbol{\psi}} d^n\boldsymbol{\psi} \, \exp(-\frac{1}{2}\bar{\boldsymbol{\psi}}\boldsymbol{A}\boldsymbol{\psi} + \boldsymbol{J}_1^T\bar{\boldsymbol{\psi}} + \boldsymbol{J}_2^T\boldsymbol{\psi})$$

where $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are both n-dimensional Grassmann source fields. This is the standard form encountered for the fermion path integral, where ψ and $\bar{\psi}$ are spinor fields. We will not detail the solution to this integral, which happens to be

$$\int d^n\bar{\boldsymbol{\psi}} d^n\boldsymbol{\psi} \, \exp(-\frac{1}{2}\bar{\boldsymbol{\psi}}^T\boldsymbol{A}\boldsymbol{\psi} + \boldsymbol{J}_1^T\bar{\boldsymbol{\psi}} + \boldsymbol{J}_2^T\boldsymbol{\psi}) = |\boldsymbol{A}| \exp(\frac{1}{2}\boldsymbol{J}_2^T\boldsymbol{A}^{-1}\boldsymbol{J}_1) \tag{20}$$

Equations (11) and (20) are the simplified versions of the path integrals for bosonic and fermionic quantum field theory, respectively. The location of the matrix determinant $|\boldsymbol{A}|$ (numerator or denominator) depends on which type of particle is being described, but it's really not important, as it will get sucked into an overall normalization factor. The physics lies in the mathematics of the $\boldsymbol{J}_2^T\boldsymbol{A}^{-1}\boldsymbol{J}_1$ source term which, as we have noted, describes the creation, propagation and annihilation of particles. Unfortunately, the complete expressions cannot be integrated in closed form because the source term appears in another exponential integral in the integrand. However, the source term can be power-expanded, and it is this expansion that gives rise to multiple particles.

Ah, if we could only do the integral! But we can't. **– A. Zee, *Quantum Field Theory in a Nutshell***