How to Graph the Lorentz Transformation
and Why You Don't Want To

William O. Straub
Pasadena, California 91104
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Abstract

In special relativity, calculations involving the Lorentz transformation are straightforward, but the transformation can be represented graphically as well. Most undergraduate students intuitively recognize that the Lorentz transformation equations involve rotations, but it is not obvious that the rotations differ for space and time coordinates. For one spacial dimension, this difference is a natural consequence of the invariant line element $\Delta s^2 = \Delta x^2 - c^2 \Delta t^2$. Similarly, the scales of the space and time coordinates differ as well.

The graphical representation of the Lorentz transformation is usually ignored at the undergraduate level for reasons of brevity, but it offers a pictorial version of the transformation that provides a means of visualizing what is taking place. Nevertheless, graphing the transformation is not only tedious but impractical and imprecise. Here we present a simple summary of the graphical method for the sake of completing the student's basic understanding of special relativity.

1. Review of Special Relativity

Einstein's special theory of relativity relates how space and time measurements differ between two inertial observers moving with respect to one another at some constant velocity v. For simplicity, we will consider the standard case of an observer in the reference system S who makes observations of another observer S' who is traveling in the positive x-direction of S at velocity v.

Measurements of space differences and time intervals are made using the familiar Lorentz transformation equations given by

$$\Delta x' = \gamma \left(\Delta x - \beta \Delta x^0 \right), \qquad \Delta x^{0\prime} = \gamma \left(\Delta x^0 - \beta \Delta x \right)$$

where $x^0 = ct$, $\beta = v/c$ and γ is the coefficient

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Conversely, we have the associated equations

$$\Delta x = \gamma \left(\Delta x' + \beta \Delta x^{0\prime} \right), \qquad \Delta x^0 = \gamma \left(\Delta x^{0\prime} + \beta \Delta x' \right)$$

The familiar notions of simultaneity, length contraction and time dilation all spring from a careful consideration of what each observer sees using these expressions, as do somewhat more involved notions of the Doppler shift of light frequency. These issues are all described in elementary form in standard texts, and will not be discussed here.

The Lorentz transformation for (1) can be expressed in matrix form via

$$\begin{bmatrix} x^{0\prime} \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x \\ y \\ z \end{bmatrix}$$

This simple form becomes vastly more complicated when problems involving all four components x^μ are encountered.

The student will note the similarity of the matrix form with an ordinary rotation about some fixed axis. However, in view of the fact that the Lorentz transformation leaves intact the invariant line element $\Delta s^2 = \Delta x^2 - (\Delta x^0)^2$,

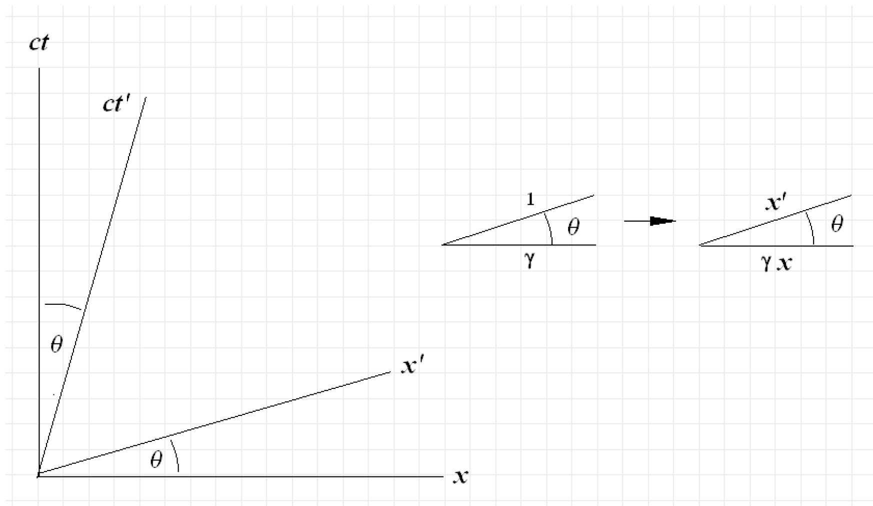

Figure 1:

we are not dealing with ordinary rotations. We shall see that the x axis is indeed rotated as expected, but the time axis simultaneously undergoes a *reverse* rotation by the same angle. A graph of the Lorentz transformation therefore has a scissors-like appearance with respect to the S' axes, as will be shown shortly.

2. Graph Details

For small β, the x' axis will of course lie close to the x axis, rotated about the origin as shown in the figure. To find the rotation angle, we'll start at the origin at time $x^{o\prime} = 0$. Then

$$\Delta x^{0\prime} = \gamma \left(\Delta x^0 - \beta x \right) \to x^0 = \beta x$$

Some finite distance x along the x-axis and time x^0 along the x^0 axis, we have

$$\tan\theta = \frac{x^0}{x} = \beta$$

Therefore, the rotation angle is simply β in the positive (counterclockwise) sense. The asymmetry of the line element $\Delta s^2 = \Delta x^2 - (\Delta x^0)^2$ immediately implies that the same rotation will occur for the $x^{0\prime}$ axis in the clockwise sense, but to see this we note that in a time x^0 the S' system moves a distance $x = \beta x^0$ to the right, so that

$$\tan\theta = \frac{\beta x^0}{x^0} = \beta$$

The resulting graph resembles a scissors, as shown in Figure 1.

In the system S, the world lines of events occurring in the system move perpendicularly away from the x axis and are parallel to the x^0 axis. Less intuitively, this notion of world-line propagation is carried over into the axes of the S' system: On a graph, these lines will both be diagonal, requiring some care in their construction. When done by hand, this invariably introduces errors, which will be apparent when we get to the exercises.

Lastly, we need to point out that the relative scales of the S and S' axes differ with respect to one another. From the above graph we have the identity $\cos\theta = x\gamma/x'$, which means that for each *unit* of distance in S the *unit* scale of x' in the S' system will always be larger. It is easy to show that from the *measured* lengths of x' and $x^{0\prime}$ from the graph we have

$$\Delta x'_{\text{actual}} = \sqrt{\frac{1-\beta^2}{1+\beta^2}}\, \Delta x'_{\text{measured}} \ , \qquad \Delta x^{0\prime}_{\text{actual}} = \sqrt{\frac{1-\beta^2}{1+\beta^2}}\, \Delta x^{0\prime}_{\text{measured}}$$

With these details in mind, the student can graphically reproduce any calculation involving the Lorentz transformation. While the graph pictorially demonstrates what is going on, it goes without saying that graphs take more time to construct, with hand measurements of angles and lengths being inherently imprecise.

3. Exercises

For a first example, consider two *simultaneous* events ($\Delta x^0 = 0$) occurring in S, as shown above in Figure 2. System S will see both events taking place at time ct_0 and separated by a distance of 5 units of length. What will an observer S' moving to the right of S at a velocity $v = 0.36c$ see? With $\beta = 0.36$, we have $\theta = \arctan(0.36) = 20°$ and $\gamma = 1.07$. The resulting graph appears as shown above. This demonstrates the fact that what might be seen as simultaneous events in one system look different to other inertial systems moving at high velocities. Note also that in the S' system, Event 2 occurs prior to Event 1.

The scale of the graph is set by $\Delta x = 5$ units , so we have the exact quantities

$$\Delta x'_{\text{actual}} = \gamma\left(\Delta x - \beta\Delta x^0\right) = 1.07\,(5-0) = 5.35 \text{ units}$$

and

$$\Delta x^{0\prime}_{\text{actual}} = \gamma\left(\Delta x^0 - \beta\Delta x\right) = 1.07\,(0-.36(5)) = -1.93 \text{ units}$$

Measurement of the associated S' distances in the graph gives $\Delta x' \approx 6.2$ units, $\Delta x^{0\prime} \approx -2.25$ units. But these have to be adjusted by the scale factor $\sqrt{(1-\beta^2)/(1+\beta^2)} = 0.877$, so

$$\Delta x'_{\text{measured}} = 0.877(6.2) = 5.43 \text{ units}$$

and

$$\Delta x^{0\prime}_{\text{measured}} = 0.877(-2.25) = -1.97 \text{ units}$$

Comparing these figures, we see that graph measurements are off by one or two percent.

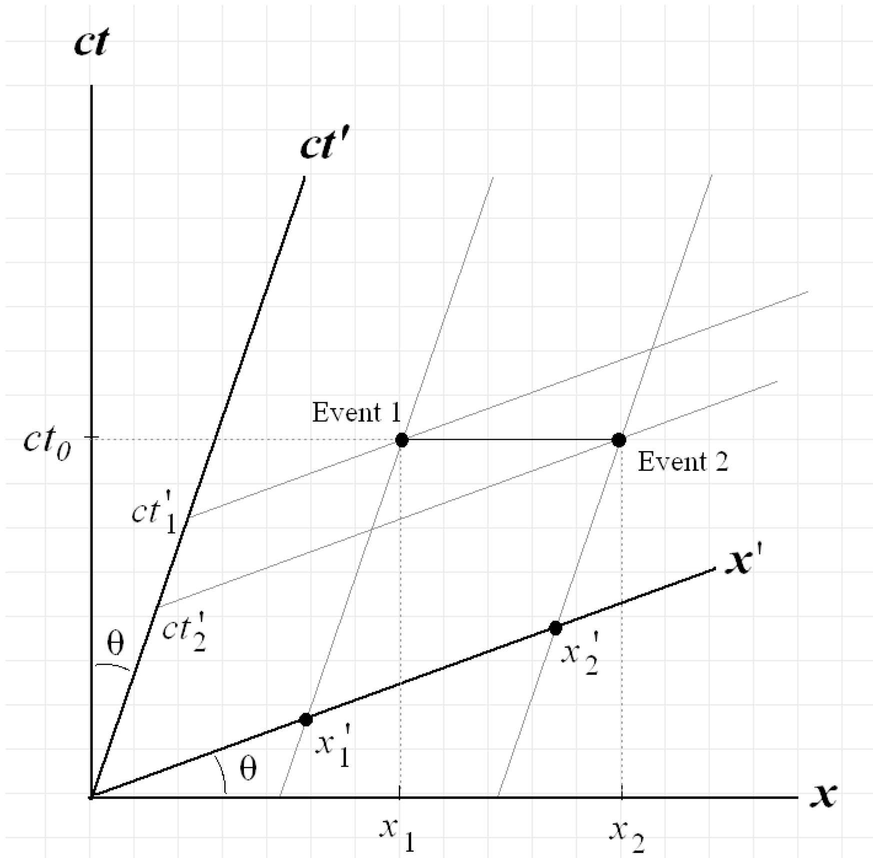

Figure 2:

Next, consider a more typical case as shown in Figure 3. We now have $\beta = 0.29$, with $\theta = \arctan(0.29) = 16°$, $\gamma = 1.04$ and $\sqrt{(1-\beta^2)/(1+\beta^2)} = 0.92$, and the scale of the graph is again set at $\Delta x = 5$ units, while x^0 is about 6.80 units. The student should be able to show that

$$\Delta x'_{\text{actual}} = 1.04\,(5 - .29(6.80)) = 1.07\,(5 - 0) = 3.15 \text{ units}$$

and

$$\Delta x^{0\prime}_{\text{actual}} = 1.04\,(6.80 - .29(5)) = 5.56 \text{ units}$$

Measurement of the associated S' distances in the graph gives $\Delta x' \approx 3,44$ units, $\Delta x^{0\prime} \approx 6.10$ units. But these have to be adjusted by the scale factor $\sqrt{(1-\beta^2)/(1+\beta^2)} = 0.92$, so

$$\Delta x'_{\text{measured}} = 0.92(3.44) = 3.16 \text{ units}$$

and

$$\Delta x^{0\prime}_{\text{measured}} = 0.92(6.10) = 5.61 \text{ units}$$

Again, the measurement approach is accurate only to a few percent.

4. Comments

As noted, the Lorentz transformation can be graphed, although it is much easier and accurate to use the Lorentz transformation equations themselves. Perhaps the only benefit of the graph approach is to show how simultaneity is relative and not absolute.

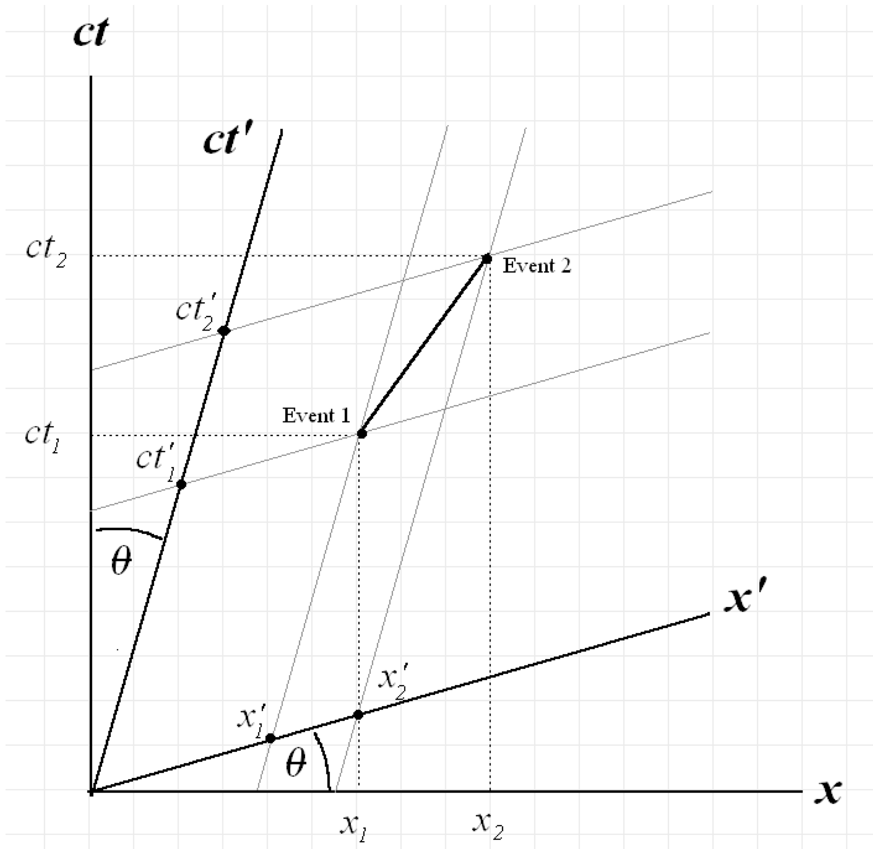

Figure 3:

The student might want to consider the situation where the S' axes both lie on the line given by $\theta = 45°$. This is the *light cone*, which corresponds to $\beta = 1, \gamma = \infty$, and distances and times lose their meaning entirely. The light cone is the realm of light itself, which experiences no time and no space.

It should now be obvious that the use of graphs to calculate Lorentz transformations is largely (in the immortal words of actor William Shatner) “a colossal waste of time”.