Kaluza-Klein for Kids

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Abstract

A very elementary overview of the original Kaluza-Klein theory is presented, suitable for undergraduates who want to learn the basic mathematical formalism behind a revolutionary idea that was proposed one hundred years ago, yet today serves as the template for modern higher-dimensional particle and gravity theories.

1. Introduction

In the years immediately following Einstein's November 1915 announcement of the general theory of relativity (GTR), numerous attempts were made by Einstein and others to unify gravitation and electromagnetism, the only forces of Nature then known. These efforts generally involved theories that went outside the formalism of Riemannian geometry; both Einstein and Elie Cartan investigated the nonsymmetry of the GTR metric and connection term, while Hermann Weyl developed a non-Riemannian geometry that initially appeared to include all of electrodynamics as a purely geometrical construct. These efforts failed, as did all similar theories that were proposed over the following decades.

In 1919, the German mathematician Theodor Kaluza developed a theory that maintained all the formalism of Riemannian geometry but extended the geometry's reach by proposing the possibility that Nature in fact utilized a *five*-dimensional spacetime, with electromagnetism appearing as a natural consequence of the unseen fifth dimension (the same idea was actually proposed by the Finnish physicist Gunnar Nordström in 1914, but was ignored). Kaluza communicated his idea to Einstein in the form of a draft paper, who was initially very enthusiastic about the concept of electromagnetism springing from the fifth dimension. But despite promises to assist Kaluza in publishing, Einstein sat on the idea for another two years before he finally recommended Kaluza's work for publication.

When finally published in 1921, the immediate reaction to Kaluza's paper by the physics community was mixed, but not necessarily because of any strong resistance to a five-dimensional spacetime. After all, only a few years earlier Einstein had shown that the world had four dimensions, not three, so an added space dimension was not viewed as entirely absurd. In addition, the apparent invisibility of the extra space dimension was explainable by assuming that it was too small to observe directly (Kaluza assumed that all physical phenomena was independent of the fifth coordinate, thus effectively "shielding" the fifth dimension from view). But in order to work the theory had to assume several arbitrary parameters and conditions that the theory could not explain. Consequently, in 1926 the Swedish mathematician Oskar Klein reexamined Kaluza's theory and made several important improvements that also seemed to have application to the then-emerging quantum theory. Since that time, theories involving extra hidden (or *compactified*) dimensions have become known as *Kaluza-Klein theories*.

2. Notation

The notational history of higher spacetimes is annoyingly confusing (like that of early tensor calculus), mainly because today one normally denotes the time coordinate with $x^0 = ct$ and space with x^i $(i = 1, 2, 3)$. The use of x^4 to denote the fifth-dimensional space coordinate was obviously problematic, but no true consensus seems to have ever been reached by the scientific community. Here we will use x^5 to denote the fifth dimension, so that the indices for time and the four space dimensions will go like 0,1,2,3,5, even though this then invites the question of what happened to the fourth dimension. In addition, on occasion certain non-indexed quantities (like the metric determinant and the Ricci scalar) in five dimensions will be denoted using a squiggle above the quantity, so that $\sqrt{-g} \to \sqrt{-\tilde{g}}, R \to \tilde{R}$, etc. This notation still does not completely specify the dimensionality of the terms we'll be using, but hopefully the context of the formalism will resolve any confusion.

Furthermore, we will use Latin indices for all 5-dimensional subscripted and superscripted vector and tensor quantities ($A, B = 0, 1, 2, 3, 5$,etc.), while Greek indices will be used to denote all strictly 4-dimensional quantities ($\mu, \nu = 0, 1, 2, 3$, etc.). In many cases, quantities may exhibit a mixture of the two notations, such as $g_{A\mu}$, but utlimately these will all be resolved into their 4-dimensional counterparts.

We will denote ordinary partial differentiation with a single subscripted bar, as in

$$A_{\mu|\nu} = \partial_\nu A_\mu = \frac{\partial A_\mu}{\partial x^\nu}$$

while covariant differentiation will be denoted using a double bar, as in

$$F^\lambda_{\ \mu||\nu} = F^\lambda_{\ \mu|\nu} - F^\lambda_{\ \alpha} \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} + F^\alpha_{\ \mu} \begin{Bmatrix} \lambda \\ \alpha\nu \end{Bmatrix}$$

where the terms in braces are the usual Christoffel symbols of the second kind.

3. Assumptions and Conventions

The primary assumption of the original Kaluza-Klein theory (other than a fifth dimension actually exists) is the independence of all vector and tensor quantities with respect to the fifth coordinate. This assumption is due to Kaluza, who needed to make a more straightforward connection of his theory to the gauge transformation property of electromagnetism. Consequently, we will have identities such as $\tilde{g}_{AB|5} = 0, \tilde{g}_{\mu\nu|5} = 0, A_{\mu|5} = 0$, etc. This has come to be known as the "cylinder condition," since it implies that 4-dimensional spacetime underlies a cylindrical fifth dimension whose spacial extent is small enough to render it *invisible* to the underlying subspace. It was Klein who first postulated the idea that the fifth dimension is a cylindrical space having a radius roughly equal to the Planck length (10^{-35} meter), a concept that conveniently explains why the fifth dimension has never been directly observed. This same concept has been carried over to string theory, where it partially explains why that theory has been so difficult to verify experimentally. Indeed, a space having the dimensions of the Planck length would require energies equivalent to that of the Big Bang to resolve. If not overcome, this restriction may ultimately relegate string theory to a kind of unprovable religious faith.

The metric in five dimensions can be viewed notationally as $\tilde{g}_{AB} = (\tilde{g}_{\mu\nu}, \tilde{g}_{\mu 5}, \tilde{g}_{55})$. The metric $\tilde{g}_{\mu\nu}$ will of course represent the usual 4-dimensional metric tensor, while the $\tilde{g}_{\mu 5}$ is a four-vector that is assumed to be proportional to the electromagnetic four-potential field A_μ (the "bare" metric $g_{\mu\nu}$ will serve as the usual metric in the absence of the electromagnetic field). The remaining quantity $\tilde{g}_{55}$ appears as a superfluous quantity; it's usually normalized to unity, and that's what we'll do here.

4. A Brief (Very Brief!) Overview of Kaluza's Metric

Kaluza's basic idea was to add a fifth dimension to the symmetric metric tensor $g_{\mu\nu}$ by adding an additional row and column to the usual 4×4 metric matrix with the quantities shown as follows:

$$\tilde{g}_{AB} = \begin{Bmatrix} g_{00} & g_{01} & g_{02} & g_{03} & A_0 \\ g_{01} & g_{11} & g_{12} & g_{13} & A_1 \\ g_{02} & g_{12} & g_{22} & g_{23} & A_2 \\ g_{03} & g_{13} & g_{23} & g_{33} & A_3 \\ A_0 & A_1 & A_2 & A_3 & k \end{Bmatrix}$$

where A_μ is the electromagnetic 4-potential in some convenient set of units and k is a constant. Thus, in Kaluza's metric the components $g_{5\mu}$ represent the electromagnetic field. Using this metric, Kaluza noted that the 5D Ricci tensor $\tilde{R}_{AB}$ and the equations of the geodesics in five dimensions reduce to their usual forms along with terms that appeared to represent the electromagnetic tensor $F_{\mu\nu} = A_{\mu|\nu} - A_{\nu|\mu}$ in the context of the electromagnetic Lorentz force. In addition, Kaluza was able to show that the set of Maxwell's homogeneous equations

$$F_{\mu\nu|\lambda} + F_{\lambda\mu|\nu} + F_{\nu\lambda|\mu} = 0$$

is also satisfied in his theory. However, Kaluza's metric is seriously hindered by the fact that its upper-index form $\tilde{g}^{AB}$ (and the associated metric determinant $\tilde{g} = |\tilde{g}_{AB}|$) is enormously complicated, preventing even the most basic calculations involving its use. For this reason we will not consider this metric any further, but proceed to Klein's modification.

5. Klein's Revision

In 1926, Klein produced the first of two papers that considerably improved Kaluza's basic approach. Klein asserted that the metric actually takes the form

$$\tilde{g}_{AB} = \begin{Bmatrix} g_{00} + A_0 A_0 & g_{01} + A_0 A_1 & g_{02} + A_0 A_2 & g_{03} + A_0 A_3 & A_0 \\ g_{01} + A_0 A_1 & g_{11} + A_1 A_1 & g_{12} + A_1 A_2 & g_{13} + A_1 A_3 & A_1 \\ g_{02} + A_0 A_2 & g_{12} + A_1 A_2 & g_{22} + A_2 A_2 & g_{23} + A_2 A_3 & A_2 \\ g_{03} + A_0 A_3 & g_{13} + A_1 A_3 & g_{23} + A_2 A_3 & g_{33} + A_3 A_3 & A_3 \\ A_0 & A_1 & A_2 & A_3 & 1 \end{Bmatrix} \tag{5.1}$$

(Klein originally associated a constant k with each A_μ term, but since it's superfluous we're ignoring it here). We can write this revised Kaluza-Klein metric in more compact 2×2 notation as

$$\tilde{g}_{AB} = \begin{Bmatrix} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & 1 \end{Bmatrix}$$

A quick calculation shows that the 5D Kaluza-Klein metric determinant is identical to that of its 4-dimensional counterpart, or $\tilde{g} = g$ (this can be verified by direct expansion of the 5×5 determinant $|\tilde{g}_{AB}|$, if you have the energy). This in itself is something of a miracle: Klein's five-dimensional determinant $\tilde{g}$ is *independent* of the vector field A_μ. In addition, the inverse 5D metric also has a simple form:

$$\tilde{g}^{AB} = \begin{Bmatrix} g^{00} & g^{01} & g^{02} & g^{03} & -A^0 \\ g^{01} & g^{11} & g^{12} & g^{13} & -A^1 \\ g^{02} & g^{12} & g^{22} & g^{23} & -A^2 \\ g^{03} & g^{13} & g^{23} & g^{33} & -A^3 \\ -A^0 & -A^1 & -A^2 & -A^3 & 1 + A_\mu A^\mu \end{Bmatrix} \tag{5.2}$$

which, happily enough, gives us the familiar identity $\tilde{g}_{AB}\tilde{g}^{AD} = \delta^D_B$.

6. The Lorentz Force in Kaluza-Klein Theory

The geodesic equations associated with the Kaluza-Klein metric provide a tantalizing hint that the theory might indeed have something to do with electrodynamics. Let us first note that in ordinary four-dimensional spacetime the Lorentz force associated with a particle of charge q and mass m can be derived by an arbitrary coordinate variation δx^α of the invariant quantity

$$I = -mc \int ds - q \int A_\mu dx^\mu$$

Carrying out the variation and setting it to zero gives the familiar expression for the Lorentz force,

$$\frac{d^2 x^\alpha}{ds^2} + \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \frac{q}{mc} F_\mu^{\ \alpha} \frac{dx^\mu}{ds} \tag{6.1}$$

The Kaluza-Klein geodesics can be similarly derived from a variation of the single invariant

$$I = -mc \int d\tilde{s} = -mc \int \tilde{g}_{AB} \frac{dx^A}{d\tilde{s}}\frac{dx^B}{d\tilde{s}}\, d\tilde{s}$$

which leads to

$$\frac{d^2 x^A}{d\tilde{s}^2} + \widetilde{\begin{Bmatrix} A \\ BC \end{Bmatrix}} \frac{dx^B}{d\tilde{s}}\frac{dx^C}{d\tilde{s}} = 0 \tag{6.2}$$

Let us first expand this set of equations for the case $A=\alpha$. We get

$$\frac{d^2x^\alpha}{d\tilde{s}^2}+\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\frac{dx^\mu}{d\tilde{s}}\frac{dx^\nu}{d\tilde{s}}=F_\mu^{\ \alpha}\frac{dx^\mu}{d\tilde{s}}\frac{dx^5}{d\tilde{s}}+A_\nu F_\mu^{\ \alpha}\frac{dx^\mu}{d\tilde{s}}\frac{dx^\nu}{d\tilde{s}} \tag{6.3}$$

Here we see something like the classical Lorentz force for electrodynamics, although at first glance the $dx^5/d\tilde{s}$ term would seem to have no classical correspondence (this will shortly be fixed). Proceeding now for the case $A=5$, we have

$$\frac{d^2x^5}{d\tilde{s}^2}-A_\alpha\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\frac{dx^\mu}{d\tilde{s}}\frac{dx^\nu}{d\tilde{s}}=-A_{\mu|\nu}\frac{dx^\mu}{d\tilde{s}}\frac{dx^\nu}{d\tilde{s}}-A^\lambda F_{\alpha\lambda}\frac{dx^\alpha}{d\tilde{s}}\frac{dx^5}{d\tilde{s}}-A^\alpha A_\mu F_{\nu\alpha}\frac{dx^\mu}{d\tilde{s}}\frac{dx^\nu}{d\tilde{s}} \tag{6.4}$$

We can use this expression to eliminate the Christoffel term in (6.3) and show that the $dx^5/d\tilde{s}$ term has an interesting interpretation. If we multiply (6.3) by A_α and add the resulting expression to (6.4), we get

$$\frac{d^2x^5}{d\tilde{s}^2}+A_\mu\frac{d^2x^\mu}{d\tilde{s}^2}+A_{\mu|\nu}\frac{dx^\mu}{d\tilde{s}}\frac{dx^\nu}{d\tilde{s}}=0$$

But $A_{\mu|\nu}dx^\nu/d\tilde{s}=dA_\mu/d\tilde{s}$, so this goes over to

$$\frac{d}{d\tilde{s}}\left(\frac{dx^5}{d\tilde{s}}+A_\mu\frac{dx^\mu}{d\tilde{s}}\right)=0 \tag{6.5}$$

Note that the quantity in parentheses is a *global constant* with respect to the 5D invariant $\tilde{s}$. Let us call this constant ξ, so that

$$\frac{dx^5}{d\tilde{s}}=\xi-A_\mu\frac{dx^\mu}{d\tilde{s}}$$

We can then write (6.3) as

$$\frac{d^2x^\alpha}{d\tilde{s}^2}+\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\frac{dx^\mu}{d\tilde{s}}\frac{dx^\nu}{d\tilde{s}}=\xi F_\mu^{\ \alpha}\frac{dx^\mu}{d\tilde{s}}$$

which now strongly resembles the standard Lorentz force. To obtain a precise definition for the constant ξ, we note that direct expansion of $d\tilde{s}^2=\tilde{g}_{AB}dx^Adx^B$ gives the identity $d\tilde{s}\sqrt{1-\xi^2}=ds$. Therefore, since

$$\frac{dx^\alpha}{d\tilde{s}}=\frac{dx^\alpha}{ds}\frac{ds}{d\tilde{s}}$$

we see that the Kaluza-Klein set of geodesics is identical to that of the Lorentz force if we make the identification

$$\xi=\frac{q}{mc}\frac{1}{\sqrt{1+q^2/(mc)^2}} \tag{6.6}$$

Indeed, for a sufficiently massive particle $d\tilde{s}\approx ds$, and the Kaluza-Klein formalism reproduces the Lorentz force exactly without complication. However, there is still the possibility that the correspondence is merely a lucky coincidence. One way to address this issue is to see how the Kaluza-Klein vector A_μ behaves under a gauge transformation. As is well known in electrodynamics, Maxwell's equations are invariant under the arbitary gauge transformation

$$A_\mu\rightarrow A_\mu+\frac{\partial\lambda}{\partial x^\mu}$$

where λ is any function of the spacetime coordinates. Both Kaluza and Klein considered an infinitesimal change in the fifth coordinate,

$$\begin{aligned} x^5 &\rightarrow \overline{x}^5=x^5+\zeta(x^\mu), \quad \text{or} \\ \delta dx^5 &= d\overline{x}^5-dx^5=\zeta_{|\mu}dx^\mu \end{aligned}$$

where ζ is some arbitrary scalar field such that $|\zeta|\ll 1$. The five-dimensional line element $d\tilde{s}^2$, given by

$$d\tilde{s}^2=\tilde{g}_{AB}dx^Adx^B=g_{\mu\nu}dx^\mu dx^\nu+2A_\mu dx^\mu dx^5+A_\mu A_\nu dx^\mu dx^\nu+\left(dx^5\right)^2$$

must be invariant with respect to this variation. The subspace line element $g_{\mu\nu}dx^\mu dx^\nu$ is automatically invariant, so we are left with

$$\delta d\tilde{s}^2 = 2dx^\mu dx^5\,\delta A_\mu + 2A_\mu dx^\mu\,\delta dx^5 + 2A_\mu dx^\mu dx^\nu \delta A_\nu + 2dx^5\,\delta dx^5$$

It is a simple matter to show that $\delta d\tilde{s}^2$ vanishes if and only if the variation of the vector A_μ satisfies $\delta A_\mu = -\zeta_{|\mu}$; that is,

$$A_\mu \to A_\mu - \zeta_{|\mu}$$

This is the well-known *gauge transformation property* of the electromagnetic four-potential, and it strengthens the identification of the Kaluza-Klein vector A_μ with the electromagnetic field. This, together with the appearance of a Lorentz force-like term in the geodesic equations, provided both Kaluza and Klein a tempting, if still tentative, reason to believe that the fifth dimension has something to do with electrodynamics.

7. The Kaluza-Klein Action

It is well-known that Einstein's path to his theory of general relativity would have been significantly shortened if he had simply considered the action approach to gravitation, which lies in extremalizing the so-called Einstein-Hilbert integral

$$I = \int \sqrt{-g}\,R\,d^4x$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar. Variation of this integral with respect to the metric gives

$$\delta I = \int \sqrt{-g}\,\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu}\,d^4x$$

from which we get the celebrated Einstein field equation for free space,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

Kaluza and Klein naturally assumed that the action should generalize in five dimensions via

$$I = \int \sqrt{-\tilde{g}}\,\tilde{R}\,d^5x = \int \sqrt{-g}\,\tilde{R}\,d^5x \tag{7.1}$$

where the Kaluza-Klein Ricci tensor is given by

$$\tilde{R}_{AB} = \widetilde{\begin{Bmatrix} C \\ AC \end{Bmatrix}}_{|B} - \widetilde{\begin{Bmatrix} C \\ AB \end{Bmatrix}}_{|C} + \widetilde{\begin{Bmatrix} C \\ AD \end{Bmatrix}}\widetilde{\begin{Bmatrix} D \\ BC \end{Bmatrix}} - \widetilde{\begin{Bmatrix} C \\ CD \end{Bmatrix}}\widetilde{\begin{Bmatrix} D \\ AB \end{Bmatrix}}$$

Regrettably, expansion of this quantity into its 4D terms and those containing A_μ is tedious, and there is little recourse but to proceed with brute force. Thankfully, the first and last Christoffel terms in $\tilde{R}_{AB}$ collapse to their ordinary 4D counterparts, while the second is simplified by the fact that differentiation with respect to x^5 gives zero. A few of the terms we'll need are exhibited in the following:

$$\widetilde{\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}} = \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} + \frac{1}{2}\left(A_\mu F^\lambda_{\ \nu} + A_\nu F^\lambda_{\ \mu}\right), \quad \widetilde{\begin{Bmatrix} \lambda \\ \mu\lambda \end{Bmatrix}} = \begin{Bmatrix} \lambda \\ \mu\lambda \end{Bmatrix} + \frac{1}{2}A^\lambda F_{\lambda\mu}, \quad \widetilde{\begin{Bmatrix} 5 \\ \mu 5 \end{Bmatrix}} = \frac{1}{2}A^\lambda F_{\mu\lambda}$$

$$\widetilde{\begin{Bmatrix} A \\ 55 \end{Bmatrix}} = \widetilde{\begin{Bmatrix} \lambda \\ 55 \end{Bmatrix}} = 0, \quad \widetilde{\begin{Bmatrix} \lambda \\ \mu 5 \end{Bmatrix}} = \frac{1}{2}F^\lambda_{\ \mu}, \quad \widetilde{\begin{Bmatrix} \lambda \\ \lambda 5 \end{Bmatrix}} = 0, \quad \widetilde{\begin{Bmatrix} 5 \\ \mu\nu \end{Bmatrix}} = \frac{1}{2}\left(A_{\mu||\nu} + A_{\nu||\mu}\right) - \frac{1}{2}A_\lambda\left(A_\mu F^\lambda_{\ \nu} + A_\nu F^\lambda_{\ \mu}\right)$$

From these expressions (and similar ones that the student can work out for herself), it is straightforward to evaluate the following:

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\left(A_\mu F^\lambda_{\ \nu||\lambda} + A_\nu F^\lambda_{\ \mu||\lambda}\right) + \frac{1}{4}\left(F^\lambda_{\ \nu}F_{\lambda\mu} + F^\lambda_{\ \mu}F_{\lambda\nu}\right) - \frac{1}{4}A_\mu A_\nu F_{\alpha\beta}F^{\alpha\beta},$$

$$\tilde{R}_{\mu 5} = -\frac{1}{2}\, F^{\lambda}_{\ \mu||\lambda} - \frac{1}{4} A_{\mu}\, F_{\alpha\beta} F^{\alpha\beta}, \qquad \widetilde{R}_{55} = -\frac{1}{4}\, F_{\alpha\beta} F^{\alpha\beta}$$

The Ricci tensor, which is

$$\tilde{R} = \tilde{g}^{AB}\tilde{R}_{AB} = g^{\mu\nu}\tilde{R}_{\mu\nu} + 2\tilde{g}^{\mu 5}\tilde{R}_{\mu 5} + \tilde{g}^{55}\tilde{R}_{55}$$

can now be evaluated. From $\tilde{R} = \tilde{g}^{AB}\tilde{R}_{AB}$ we get the simple result

$$\tilde{R} = R + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \tag{7.2}$$

Amazingly, the 5D Kaluza-Klein action (7.1) thus reduces to the Einstein-Hilbert-Maxwell action, including the correct 1/4 factor:

$$I = \int \sqrt{-g}\left(R + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}\right) d^4x \int dx^5 \tag{7.3}$$

The integral over dx^5 would appear to be problematic, since it gives infinity. However, Klein and others recognized that if the fifth dimension were cylindrical, x^5 could be viewed as an angular coordinate having the period $2\pi r$, where r is the cylinder's radius, so that the integral becomes a trival constant. Upon further consideration (which we won't delve into here), Klein determined that this radius must be on the order of the Planck constant. Klein thus concluded that the fifth dimension would be strictly unobservable.

The automatic collapse of the Klauza-Klein five-dimensional Lagrangian to four dimensions is an example of *dimensional reduction*. This phenomenon has proved a useful tool in modern gauge theories, since a coordinate transformation in the higher space can appear as a gauge transformation in the subspace.

8. Final Remarks

The Kaluza-Klein model spurred considerable theoretical interest in the fifth dimension in the 1920s, and numerous physicists (including Einstein) tried to advance the theory, particularly with regard to the problem of matter and the possibility that gravity and the then-emerging field of quantum mechanics might somehow be connected in dimensions higher than (3+1). But in spite of its startling formal mathematical beauty, the theory made no new predictions with respect to gravity or electromagnetism, while the quantum connection seemed to lead nowhere. By the early 1930s, most researchers had lost interest, and the Kaluza-Klein model joined the ranks of other failed unified field theories.

In the early 1950s, Pauli tentatively proposed a six-dimensional Kaluza-Klein theory in an attempt to develop a non-abelian theory that would accommodate the weak and strong interactions. This too failed, and the concept of higher dimensions was pretty much scrapped until string theory began to make its appearance in the 1970s. The first string theories described only bosons, and to accomplish this theorists had to assume the existence of 26 spacetime dimensions. Subsequent developments in string theory brought that number down to ten, but there were still problems involving multiplicity. In 1995, Witten showed that a consistent if not entirely unique theory of strings involved an additional spacial dimension, bringing the total to eleven.

Nevertheless, the Kaluza-Klein approach shows that compactified extra dimensions lead naturally to locally gauge-invariant theories. If we view the fifth dimension x^5 as an angular coordinate then the smallness of the space renders this coordinate invisible. The associated four-dimensional space sees this as a local symmetry, and indeed it is a type of *local gauge symmetry*, as Kaluza and Klein demonstrated for their assumed four-potential A_μ. This gauge-invariant aspect of compactified dimensions persists whether the extra dimensions are real (as in the Kaluza-Klein theory) or related to internal degrees of freedom, like particle spin.

References

1. P. Bergmann, *Introduction to the Theory of Relativity*. Dover Publications, 1942.
2. M. Blagojevic, *Gravitation and Gauge Symmetries*. Institute of Physics Publishing, 2002.
3. T. Kaluza, *On the Unification Problem of Physics*. Sitz. Ber. Preuss. Akad. Wiss. (1921) 966. Reproduced in O'Raifertaigh's book.
4. O. Klein, *Quantum Theory and Five-Dimensional Relativity*. Zeit. f. Physik 37 (1926) 895. Reproduced in O'Raiferaigh's book.
5. L. O'Raifertaigh, *The Dawning of Gauge Theory*. Princeton Series in Physics, 1997.
6. W. Pauli, *Theory of Relativity*. Dover Publications, 1958. Pauli briefly discusses the KK theory in Supplementary Note 23.
7. W. Pauli, Correspondence to Abraham Pais, July and December 1953. In these informal letters to his friend and colleague Pais, Pauli explored the possibility of a six-dimensional Kaluza-Klein approach. The always-irrepressible Pauli begins the first letter with the note *Written down July 23-25 1953 in order to see how it looks*. The letters are also reproduced in O'Raifertaigh's book.
8. A. Zee, *Quantum Field Theory in a Nutshell.* Princeton University Press, 2003. See Chapter VIII.1 for a brief overview of the KK theory, along with some exercises.