Maximal Extension of Schwarzschild Metric*

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There is presented a particularly simple transformation of the Schwarzschild metric into new coordinates, whereby the "spherical singularity" is removed and the maximal singularity-free extension is clearly exhibited.

THE well-known Schwarzschild expression[1] for the metric around a center of mass $m(g)$ or $m^* = (Gm/c^2)$ (cm) is

$$ds^2 = -(1-2m^*/r)dT^2 + (1-2m^*/r)^{-1}dr^2 + r^2d\omega^2, \quad (1)$$

with

$$d\omega^2 \equiv d\theta^2 + \sin^2\theta \, d\varphi^2. \quad (2)$$

Kasner, Lemaître, Einstein and Rosen, Robertson, Synge, Ehlers, Finkelstein, and Fronsdal have shown[2] that the singularities at $r=0$ and $r=2m^*$ are very different in character (Table I). Their conclusion—that there is no real singularity at $r=2m^*$—can be demonstrated by a choice of coordinates seemingly simpler and more explicit than any introduced so far to this end. One way to find it is to seek a spherically symmetric coordinate system in which radial light rays everywhere have the slope $dx^1/dx^0 = \pm 1$:

$$ds^2 = f^2(-dv^2 + du^2) + r^2d\omega^2. \quad (3)$$

Identifying (3) with (1) and requiring f to depend on r alone and to remain finite and nonzero for $u=v=0$, one finds the following essentially unique equations of transformation between the exterior of the "spherical singularity," $r>2m^*$, and the quadrant $u>|v|$ in the plane of the new variables (Table II).

The new coordinates give an analytic extension, $\mathcal{E}$, of that limited region of space-time, $\mathcal{L}$, which is described without singularity by the Schwarzschild coordinates with $r>2m^*$. The metric in the extended region joins on smoothly and without singularity to the metric at the boundary of $\mathcal{L}$ at $r=2m^*$. That this extension is possible was already indicated by the fact that the curvature invariants of the Schwarzschild metric are perfectly finite and well behaved at $r=2m^*$.

The extended space, $\mathcal{E}$, moreover, is the *maximum* singularity-free extension of $\mathcal{L}$ that is at all possible, for the following reason: As may be seen by direct examination of the geodesics (perhaps most simply carried out mainly in the familiar r,T coordinates, with special attention to geodesics on which $r=2m^*$ either isolatedly or everywhere), every geodesic, followed in whichever direction, either runs into the "barrier" of intrinsic singularities at $r=0$ ($v^2-u^2=1$), or is continuable infinitely with respect to its "natural length." (This is measured in terms of the number of parallel transfers of an infinitesimal tangent vector, is determined only up to an arbitrary scale factor, and not only accords with the proper time or distance along time-like or space-like geodesics but is defined even for null geodesics.) But it is obvious that if there were a transformation (no matter how wild at the boundary) between $\mathcal{E}$ and a subregion $\mathcal{E}'$ of a still further singularity-free extension $\mathcal{F}$, there would have to be one geodesic (to say the least) running from $\mathcal{E}'$ into $\mathcal{F} - \mathcal{E}'$, in contradiction to the aforementioned property.

The maximal extension $\mathcal{E}$ has a *non-Euclidean topology* (Fig. 1) and therefore falls into the class of topologies considered by Einstein and Rosen, Wheeler, and Misner and Wheeler.[3] It is remarkable that it presents just such a "bridge" between two otherwise Euclidean spaces as Einstein and Rosen sought to obtain by modifying the field equations. It may also be interpreted as describing the "throat of a wormhole" in the sense of Wheeler, connecting two distant regions in *one* Euclidean space—in the limit when this separation of the wormhole mouths is very large compared to the circumference of the throat. The length of the wormhole connection may of course be exceedingly

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[1] K. Schwarzschild, Sitzber. Preuss. Akad. Wiss. Physik.-math. Kl. 189 (1916).

[2] E. Kasner, Am. J. Math. 43, 130 (1921); G. Lemaître, Ann. soc. sci. Bruxelles A53, 51 (1933); A. Einstein and N. Rosen, Phys. Rev. 48, 73 (1935); H. P. Robertson, lecture in Toronto, 1939 (unpublished), cited by J. L. Synge; J. L. Synge, Proc. Roy. Irish. Acad. A53, 83 (1950); J. Ehlers, thesis, Hamburg, 1957 (unpublished); D. Finkelstein, Phys. Rev. 110, 965 (1958); C. Fronsdal, Phys. Rev. 116, 778 (1959).

TABLE I. Singularities in Schwarzschild metric, real and apparent.

r	Invariant representation of curvature	Singularity in metric?	Singularity in Schwarzschild coordinate system?
0	Infinite as m^*/r^3	Yes	Yes
$2m^*$	Finite	No	Yes

[3] J. A. Wheeler, Phys. Rev. 97, 511 (1955); C. W. Misner and J. A. Wheeler, Ann. Physik 2, 525 (1957).

TABLE II. Relation of new coordinates to Schwarzschild coordinates.

New coordinates in terms of Schwarzschild coordinates	Schwarzschild coordinates in terms of new coordinates
$u=\left[\left(\dfrac{r}{2m^*}\right)-1\right]^{\frac{1}{2}}\exp\left(\dfrac{r}{4m^*}\right)\cosh\left(\dfrac{T}{4m^*}\right)$	$\left[\left(\dfrac{r}{2m^*}\right)-1\right]\exp\left(\dfrac{r}{2m^*}\right)=u^2-v^2$
$v=\left[\left(\dfrac{r}{2m^*}\right)-1\right]^{\frac{1}{2}}\exp\left(\dfrac{r}{4m^*}\right)\sinh\left(\dfrac{T}{4m^*}\right)$	$T/4m^*=\operatorname{arctanh}(v/u)$
	$\quad=\frac{1}{2}\operatorname{arctanh}[2uv/(u^2+v^2)]$

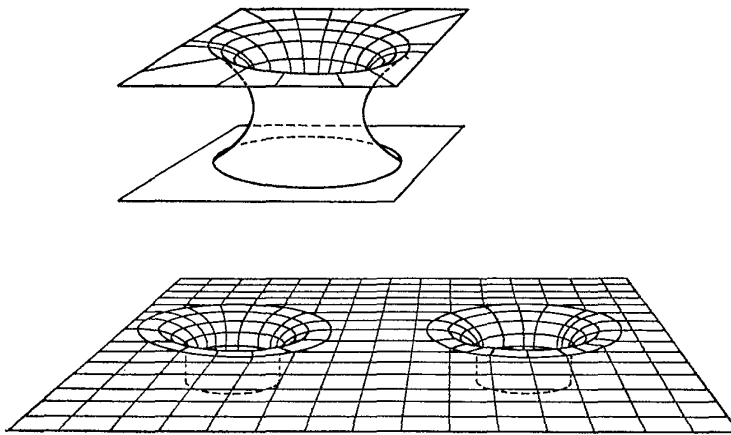$f^2=(32m^{*3}/r)\exp(-r/2m^*)=$ a transcendental function of (u^2-v^2)

FIG. 1. Two interpretations of the 3-dimensional "maximally extended Schwarzschild metric" at the time $T=0$. Above: A connection or bridge in the sense of Einstein and Rosen between *two* otherwise Euclidean spaces. Below: A wormhole in the sense of Wheeler connecting two regions in *one* Euclidean space, in the limiting case where these regions are extremely far apart compared to the dimensions of the throat of the wormhole.
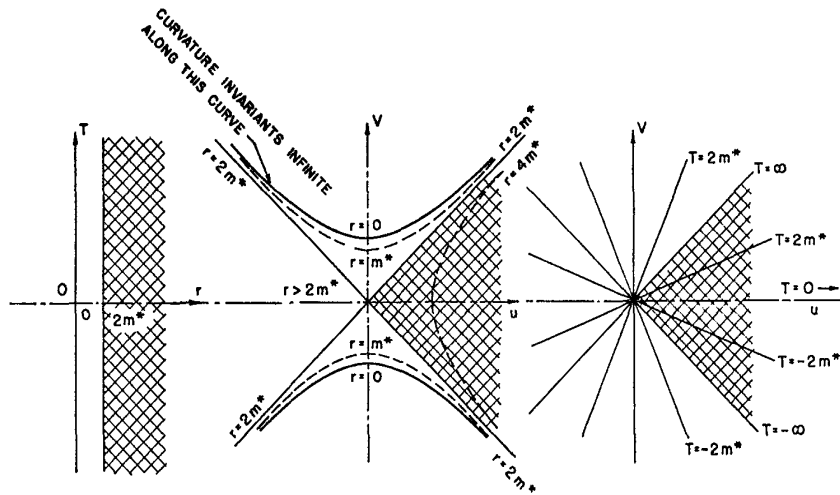

FIG. 2. Corresponding regions of the (r,T) and (u,v) planes. In the latter, curves of constant r are hyperbolas asymptotic to the lines $r=2m^*$ while T is constant on straight lines through the origin. The exterior of the singular sphere, $r>2m^*$, corresponds to the region $|v|<u$ (the hatched areas). The whole line $r=2m^*$ in the (r,T) plane corresponds to the origin $u=v=0$, while two one-dimensional families of ideal limit points with $r\to 2m^*$ and $T\to\pm\infty$ correspond to the remaining boundary points $u=|v|>0$.

In the (u,v) plane the metric is entirely regular not only in the hatched area but in the entire area between the two branches of the hyperbola $r=0$. This comprises two images of the exterior of the spherical singularity and two of its interior. (The expressions in Table II are valid in the right-hand quadrant $u>|v|$. To obtain formulas valid in the left-hand quadrant replace u and v by their negatives everywhere. To obtain formulas valid in the upper or lower quadrant replace u by $\pm v$, v by $\pm u$, and $r/2m^*-1$ by its negative everywhere. Note that the formula for r and the final formula for T remain invariant under these substitutions.) The purely radial $(d\theta=d\varphi=0)$ null geodesics are lines inclined at 45°. The points with $r=2m^*$ have no local topological distinction, but still a global one: if a test particle crosses $r=2m^*$ into the interior (where r is time-like and T space-like), it can never get back out but must inevitably hit the irremovable singularity $r=0$ (curvature invariants infinite). This circumstance guarantees that one cannot violate ordinary causality in the "main universe" by sending signals via the wormhole effectively faster than light.

short compared to the distance between the wormhole mouths in the approximating Euclidean space. However, as seen in Fig. 2, it is impossible to send a signal through the throat in such a way as to contradict the principle of causality; in effect the throat "pinches off" the light ray before it can get through. This pinch-off effect presents fundamental issues of principle which require further investigation.

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