Taming the Lambert W-Function

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Abstract

The Lambert W-function (named after the French-German polymath Johann Lambert for work he initiated in 1758) is used extensively for solving certain elementary but difficult algebraic problems, including combinatorics and various applications in physics. Incapable of being expressed in terms of elementary functions, until the advent of calculators and computers it was obtainable primarily from published tables.

Here we present a simple, semi-rational formula that provides nearly exact values for W(x) over a wide range of real and positive inputs x. While this approach is necessarily empirical and based on curve fitting, it provides a convenient method for the solution of various problems, or a least a starting point for the Newton-Raphson method.

Introduction

In 1758 Lambert considered problems of the form $x = a + bx^n$, where a, b, n are constants. At the time there was no rational approach to the solution other than successive approximation methods such as Newton-Raphson. A somewhat simpler but related problem was that of the self-power problem $x^x = n$, where n is a constant. Again, no method of solution other than approximation was available.

Lambert's work was expanded by Euler and others, eventually leading to the Lambert W-function, which is defined as

$$W(x)\, e^{W(x)} = x \tag{1}$$

where, if some function of x can be expressible as the function $W(x)$, the value of x itself is obtained. This discovery proved to be of enormous importance in solving a wide range of non-linear algebraic problems, while continued work showed it to be of use in many other applications, especially physics.

Approach

In this paper we will not be concerned with problems involving the W-function itself, such as solution existence or branches, as the goal is simply to derive a useful method of getting at the function itself for real and positive values of x. To start, consider taking the logarithm of $x^x = n$, in which we get

$$x \ln(x) = \ln(n) \text{ or } x = \frac{\ln(n)}{\ln(x)}$$

We immediately see that the solution x is proportional to $\ln(n)$, although the $\ln(x)$ term prevents a direct solution. So let us consider the semi-rational but otherwise purely empirical approach

$$x = a + b\,[\,\ln(n)\,]^c$$

where a, b, c are constants. By appealing to a table of known exact solutions to x, curve fitting can be used to derive these constants. While there is no guarantee that this approach will provide decent solutions, one finds the best-fit expression

$$x = 0.9356 + 0.8319\,[\,\ln(n)\,]^{0.7608} \tag{2}$$

over the range $1.2 < n < 1{,}000{,}000$. By a comparison of tables of exact solutions and those given by (2), one finds the coefficient of determination (COD) to be about 0.999998, lending some credibility to the approach. To test one example, for $n = 39.12$ we have

$$x = 0.9356 + 0.8319\,[\,\ln(39.12)\,]^{0.7608} = 3.1711$$

Comparing the actual solution of about $x = 3.1743$, one finds a difference of only 0.01 percent. Similar results are obtained for a wide range of n, so the empirical approach seems to have some validity.

Approximating the Lambert W-Function

The expression in (2) can be used to derive a useful approximation for the W-function. To see this, take the logarithm of $x^x = n$ so that $x\ln(x) = \ln(n)$. Now let $y = \ln(n)$ and $u = \ln(x)$. Exponentiating gives $x = e^u$, so that

$$u\,e^u = y$$

This is just the definition of the W-function for $W(x)$, or

$$W(\ln(x))\,e^{W(\ln(x))} = y$$

Now, since $y = \ln(x)$, we have the remarkable identity $W(x) = \ln(x)$. So, from (2), we have

$$W(x) = \ln\left(0.9356 + 0.8319\,x^{0.7608}\right) \tag{3}$$

Now, let's try a few problems.

Example1

To test (3), let us solve a typical problem using our empirical formula for the W-function:

$$x^{2.73} = 3.99\,(0.41)^{8.7x}$$

Taking the logarithm of both sides and dividing by 2.73, we get

$$\ln(x) = 0.5069 - 2.8413\,x$$

Exponentiating both sides, we have

$$x = e^{0.5069}\,e^{-2.8413x}$$

Dividing both sides by the exponential x-term, we get

$$x\,e^{2.8413x} = e^{0.5069} = 1.6601$$

Finally, multiplying both sides by 2.8413 gives

$$2.8413x\,e^{2.8413x} = 4.7168$$

But 4.7168 is the input for W(x) for the unknown x, so that, using (3), we have

$$W(x) = \ln\left(0.9356 + 0.8319\,(4.7168)^{0.7608}\right) = 1.2929$$

We have now found the solution, which is given by

$$2.8413x = 1.2929, \text{ or } x = 0.4550$$

By the Newton-Raphson method (or better, using Wolfram Alpha or Mathematica), the correct answer is $x = 0.4553$. The difference between the two values is negligible.

Example 2

This problem appeared on a Harvard University entrance exam. Solve for x, where

$$2^x + x = 5$$

Rewriting, we have

$$(5 - x)\,2^{-x} = 1$$

Multiplying both sides by 2^5, we have

$$(5-x)\,2^{5-x} = 32$$

or, exponentiating,

$$(5-x)\,e^{\ln(2)\,(5-x)} = 32$$

We multiply both sides by ln(2), getting

$$\ln(2)\,(5-x)\,e^{\ln(2)\,(5-x)} = 32\ln(2) = 22.1808$$

The W-function is then

$$W(22.1808) = \ln\left(0.9356 + 0.8319\,(22.1808)^{0.7608}\right) = 2.2750$$

We then have the solution:

$$\ln(2)\,(5-x) = 2.2750, \text{ or } x = 1.7179$$

The exact answer is $x = 1.7156$, and the difference is again negligible.

Comments

Because of the inherent mathematical properties of the actual Lambert W-function, not all problems of this sort can be solved. Issues such as non-existence and multiple solutions arise which cannot be handled by our empirical approach. In addition, whatever inaccuracies inherent in (2) may be compounded in (3), making solutions (when possible) less accurate than the example given here. For these reasons, caution should be exercised when using this approach.

Nevertheless, it is apparent that the empirical formulas presented here can be of real utility when solving real-life problems, and it is hoped they will be tried out by students searching for solutions to applicable problems.