Can Weyl geometry explain acceleration?

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**Abstract.** We study homogeneous and isotropic cosmologies in a Weyl spacetime. We show that for homogeneous and isotropic spacetimes, the field equations can be reduced to the Einstein equations with a two-fluid source. We write the equations as a two-dimensional dynamical system and analyze the qualitative, asymptotic behavior of the models. We examine the possibility that in certain theories the Weyl 1-form may give rise to a late accelerated expansion of the Universe and conclude that such behaviour is not met as a generic feature of the simplest cosmologies.

## 1. Introduction

Recent observations regarding the evolution of the Universe have urged the theorists on cosmology to establish the model of the accelerated expansion and to pursue the origin of the acceleration. To this end several authors propose a variety of departures from the standard cosmology. These proposals can be roughly grouped into two categories. First, there exists a dark energy of unknown nature which is responsible for the accelerating expansion of the Universe, (cf. [1, 2] for comprehensive reviews and references). Alternatively general relativity requires a modification at cosmological distance scales [3, 4] (for a pedagogical review see [5]). Less explored is the idea that the geometry of spacetime is not the so far assumed Lorentz geometry (see for example [6]). In this circumstance we need not only to look for a new idea, but also to revisit every type of the old models from a new viewpoint. Weyl geometry is certainly one candidate to be reviewed and reanalysed. Due to its simplicity Weyl geometry is considered as the most natural candidate for extending the Riemannian structure. In a Weyl space the covariant derivative of the metric tensor is not zero and the geometry is determined not only from the metric $g_{\alpha\beta}$, but also from the Weyl vector field $Q_\mu$.

In this paper we study homogeneous and isotropic cosmologies in a Weyl framework. More precisely, we explore the field equations derived with the constrained variational principle from the Lagrangian $L = R + L_{matter}$, under the condition that the geometry be Weylian [7]. If the matter Lagrangian is chosen so that ordinary matter is described by a perfect fluid with equation of state $p_2 = (\gamma_2 - 1)\rho_2$, we show that for homogeneous and isotropic spacetimes, the field equations reduce to the Einstein equations with a two-fluid source. The first fluid with equation of state $p_1 = \rho_1$ (i.e. stiff matter), stems from the Weyl vector field and the second fluid describes ordinary matter as mentioned above.

The plan of the paper is the following. In Section 2 we review the constrained variational principle, obtain the field equations and express them in terms of the Levi-Civita connection. In Section 3 we show that homogeneous and isotropic models can be interpreted as two-fluid

           

models. In Section 4, following the method of Coley and Wainwright in [8], we write the equations as a two-dimensional dynamical system and analyze the asymptotic behavior of the models. In Section 5 we consider the two-fluid model resulting from a modification of the Einstein-Hilbert Lagrangian, cf. (22), and study the possibility of providing a mechanism of accelerating expansion. Readers unfamiliar with Weyl geometry may find in the Appendix an exposition of the techniques involved. In this Appendix we also derive the Bianchi identities and write the Einstein tensor and the Bianchi identities in terms of the Levi-Civita connection.

## 2. Constrained variations and field equations

We recall that a Weyl space is a manifold endowed with a metric **g** and a linear symmetric connection $\nabla$ which are interrelated via

$$\nabla_\mu g_{\alpha\beta} = -Q_\mu g_{\alpha\beta}, \tag{1}$$

where the 1-form $Q_\mu$ is customarily called Weyl covariant vector field (see the Appendix for a full explanation of the notation involved below). We denote by $D$ the Levi-Civita connection of the metric $g_{\alpha\beta}$.

Weyl geometry can be incorporated a priori in a theory or a posteriori, for example as a consequence of the variational principle involved. This is the case of the application of the Palatini method to gravitational Lagrangians of the form $L = f(R)$ [9, 10], where it turns out that $Q_\mu = \partial_\mu \ln f'(R)$. The Palatini variational principle leads to the field equations

$$f' R_{(\mu\nu)} - \frac{1}{2} f g_{\mu\nu} = 0, \tag{2}$$

and taking the trace of (2) we find

$$f'(R) R = 2f(R). \tag{3}$$

This equation is identically satisfied by the function

$$f(R) = R^2, \tag{4}$$

apart from a constant rescaling factor. We see that the Palatini method does not accept general Lagrangians of the form $L = f(R)$, but forces the theory to be purely quadratic. Accordingly the field equation (2) is

$$R_{\alpha\beta} - \frac{1}{4} R g_{\alpha\beta} = 0, \tag{5}$$

provided that $R \neq 0$. Observe that the scalar curvature is undetermined because (5) is traceless, which is another peculiarity of the Palatini method. However, the quadratic solution (4) to the trace equation (3) is not the only possibility. Given an arbitrary differentiable function $f$, equation (3) can be regarded as an algebraic equation to be solved for $R$. The situation is even worse if matter fields are included in the Lagrangian $L = f(R)$. In that case the stress-energy tensor no longer satisfies the conservation equation. The remedy is to define a new stress-energy tensor which is conserved (see [11] for the curvature squared Lagrangian, [9] for an $1/R$ correction and a generalization in [12]). However, the physical interpretation of this generalized conservation law is put in doubt. In the weak field limit the equation of motion of test particles derived from the 'conservation equation' exhibits undesirable terms which disagree with Newton's law [12]. We argue that the Palatini device suffers from serious problems and leads to inconsistencies when applied to general Lagrangians for the construction of a gravity theory (see for example [13, 14] for a thorough critique of the Palatini device).

In [7] it was shown that a consistent way to incorporate an arbitrary connection into the dynamics of a gravity theory is the so-called constrained variational principle. According to this method one adds to the original Lagrangian a constraint with suitable Lagrange multipliers (see also [15]). To outline the method, consider the most general higher order Lagrangian with general matter couplings. For an arbitrary symmetric connection $\Gamma^{\gamma}_{\alpha\beta}$ we introduce the difference tensor between $\Gamma^{\gamma}_{\alpha\beta}$ and the Levi-Civita connection $\left\{ {\gamma \atop \alpha\beta} \right\}$

$$C^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} - \left\{ {\gamma \atop \alpha\beta} \right\}.$$

The constrained Palatini variation is defined by adding to the original Lagrangian the following term as a constraint (with Lagrange multipliers $\Lambda$)

$$L_c\left(g, \Gamma, \Lambda\right) = \Lambda_{\rho}{}^{\mu\nu} \left[ \Gamma^{\rho}_{\mu\nu} - \left\{ {\rho \atop \mu\nu} \right\} - C^{\rho}_{\mu\nu} \right]. \tag{6}$$

For instance, in Riemannian geometry (6) takes the form

$$L_c\left(g, \Gamma, \Lambda\right) = \Lambda_{\rho}{}^{\mu\nu} \left[ \Gamma^{\rho}_{\mu\nu} - \left\{ {\rho \atop \mu\nu} \right\} \right],$$

while, in Weyl geometry

$$L_c\left(g, \Gamma, \Lambda\right) = \Lambda_{\rho}{}^{\mu\nu} \left[ \Gamma^{\rho}_{\mu\nu} - \left\{ {\rho \atop \mu\nu} \right\} - \frac{1}{2} g^{\rho\sigma} \left( Q_{\nu} g_{\mu\sigma} + Q_{\mu} g_{\sigma\nu} - Q_{\sigma} g_{\mu\nu} \right) \right].$$

We then express all the covariant derivatives appearing in the Lagrangian in terms of the connection $\Gamma$, and vary the resulting action with respect to the independent fields $g$, $\Gamma$, $\Lambda$ and matter fields collectively denoted by $\psi$.

In [7] it was shown that application of the constrained variational principle to general Lagrangians of the form $L = f\left(R\right)$, in the context of Weyl geometry, yields the field equations obtained via the metric variation in Riemannian spaces with a source tensor depending on the Weyl vector field. The simplest theory that can be constructed with the constrained variational principle is obtained from the Lagrangian $L = R$.[1] The field equations are (see equations (30) and (31) in [7])

$$G_{(\mu\nu)} = -\nabla_{(\mu} Q_{\nu)} + Q_{\mu} Q_{\nu} + g_{\mu\nu} \left( \nabla^{\alpha} Q_{\alpha} - Q^{\alpha} Q_{\alpha} \right) =: M_{\mu\nu}. \tag{7}$$

If we express the tensor $M_{\mu\nu}$ in terms of the quantities formed with the Levi-Civita connection $D$ and take into account of (A.8), the field equations become

$$\overset{\circ}{G}_{\mu\nu} = \frac{3}{2} \left( Q_{\mu} Q_{\nu} - \frac{1}{2} Q^2 g_{\mu\nu} \right). \tag{8}$$

In the case of integrable Weyl geometry, i.e., when $Q_{\mu} = \partial_{\mu} \phi$, the source term is that of a massless scalar field. Taking the divergence of (8) and using the Bianchi identities (A.10) we conclude that

$$D^{\mu} Q_{\mu} = 0. \tag{9}$$

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[1] A theory with a first order dependence of the Lagrangian on the scalar curvature is not gauge invariant (cf. (A.1)). In his original theory, Weyl postulated that the *natural gauge* is determined by the condition $R = 4\lambda$, where $\lambda$ is a constant. Later, in the paper in 1921 [16], he considered a first order Lagrangian as an alternate and extension of his original theory.

### 3. A two-fluid model

In this paper we will be concerned with spatially homogeneous and isotropic spacetimes. Therefore we have to make the assumption that $Q^\mu$ is hypersurface orthogonal. That means that $Q^\mu$ is proportional to the unit timelike vector field $n^\mu$ which is orthogonal to the homogeneous hypersurfaces,

$$Q^\mu =: q n^\mu, \quad Q^2 = Q_\mu Q^\mu = -q^2.$$

Formally the field equations (8) can be rewritten as

$$\overset{\circ}{G}_{\mu\nu} = (\rho_1 + p_1)\, n_\mu n_\nu + p_1 g_{\mu\nu}, \tag{10}$$

with

$$\rho_1 = p_1 = \frac{3}{4} q^2, \tag{11}$$

and we see that the equation of state of the $q-$fluid corresponds to stiff matter. For spatially homogeneous and isotropic models, the field equations (10) become the system of the equations: the Friedmann equation[2]

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{1}{4} q^2,$$

and the Raychaudhuri equation

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -\frac{3}{4} q^2.$$

Equation (9) becomes

$$\dot{q} + 3\frac{\dot{a}}{a} q = 0,$$

which implies that

$$q = \frac{C}{a^3},$$

where $C$ is a constant. Therefore the energy density and pressure of the $q-$fluid evolve as $a^{-6}$.

In the following we assume that ordinary matter is described by a perfect fluid with energy-momentum tensor,

$$T_{\mu\nu} = (\rho_2 + p_2)\, u_\mu u_\nu + p_2 g_{\mu\nu}, \tag{12}$$

where $u^\mu$ denotes the fluid velocity. Although conceptually different, $u^\mu$ and $\eta^\mu$ coincide in the case of spatially homogeneous and isotropic spacetimes, i.e.,

$$Q^\mu = q u^\mu, \quad Q^2 = Q_\mu Q^\mu = -q^2.$$

Therefore we are dealing with a two-fluid model with total energy density and pressure given by

$$\rho = \rho_1 + \rho_2, \quad p = p_1 + p_2, \tag{13}$$

respectively, where

$$p_1 = \rho_1, \quad p_2 = (\gamma_2 - 1)\, \rho_2, \tag{14}$$

i.e., $\gamma_1 = 2$ and $\gamma_2 < \gamma_1$.

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[2] We adopt the metric and curvature conventions of [17]. Here, $a(t)$ is the scale factor, an overdot denotes differentiation with respect to time $t$, and units have been chosen so that $c = 1 = 8\pi G$.

The Bianchi identities applied to the equations derived from the whole Lagrangian $L = R + L_{matter}$ imply that the total energy-momentum tensor is conserved, and therefore a non-trivial interaction between the two fluids is induced. However, the energy-momentum of ordinary matter is expected to be conserved, although this fact does not follow from the field equations. If the condition $D_\mu T^\mu{}_\nu = 0$ is satisfied, then the Weyl fluid is also conserved due to (9). It is necessary to make an assumption about the interaction between the two fluids (cf [8]), otherwise the field equations are an underdetermined system of differential equations. The simplest assumption is that the energy-momentum of each fluid is separately conserved, so that the two fluids do not interact. We restrict our analysis to this case and refer to [18] for more general situations.

The field equations take the final form

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{1}{3}\left(\rho_1 + \rho_2\right) \tag{15}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\left[4\rho_1 + (3\gamma_2 - 2)\rho_2\right] \tag{16}$$

$$\dot{\rho}_1 = -6\rho_1\frac{\dot{a}}{a} \tag{17}$$

$$\dot{\rho}_2 = -3\gamma_2\rho_2\frac{\dot{a}}{a}. \tag{18}$$

It is well known that the field equations for one fluid can be written as a two-dimensional dynamical system for the Hubble variable $H := \dot{a}/a$ and the density parameter $\Omega := \rho/3H^2$. As mentioned in [17], a drawback of this analysis is that it does not give a complete description of the evolution for closed models. In fact, at the time of maximum expansion the Hubble parameter becomes zero and therefore, the time coordinate defined in [17] p. 58, cannot be used past the instant of maximum expansion. Instead, we deal with closed models by defining the compactified density parameter $\omega$, (see [19])

$$\Omega = \frac{1}{\tan^2\omega}, \tag{19}$$

or

$$\omega = \arctan\left(\frac{\sqrt{3}H}{\sqrt{\rho}}\right), \quad \text{with} \quad -\pi/2 \leq \omega \leq \pi/2.$$

We see that $\omega$ is bounded at the instant of maximum expansion ($H = 0$) and also as $\rho \to 0$, in ever-expanding models.

## 4. Phase plane analysis

We now adopt the Coley and Wainwright formalism for a general model with two fluids with variable equations of state (cf. [8]). Assuming that $\gamma_1 > \gamma_2$ and $\gamma_1 > 2/3$, we define the transition variable $\chi \in [-1, 1]$

$$\chi = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \tag{20}$$

which describes which fluid is dominant dynamically. The total density parameter $\Omega = \Omega_1 + \Omega_2$ can be compactified as in (19) and the evolution equation of the variable $\chi$ is obtained by applying the conservation equation to $\rho_1$ and $\rho_2$. Defining a new time variable $\tau$ by

$$\frac{d}{dt} = \frac{3(\gamma_1 - \gamma_2)}{2}\sqrt{\frac{\rho}{3}}\frac{1}{\cos\omega}\frac{d}{d\tau},$$
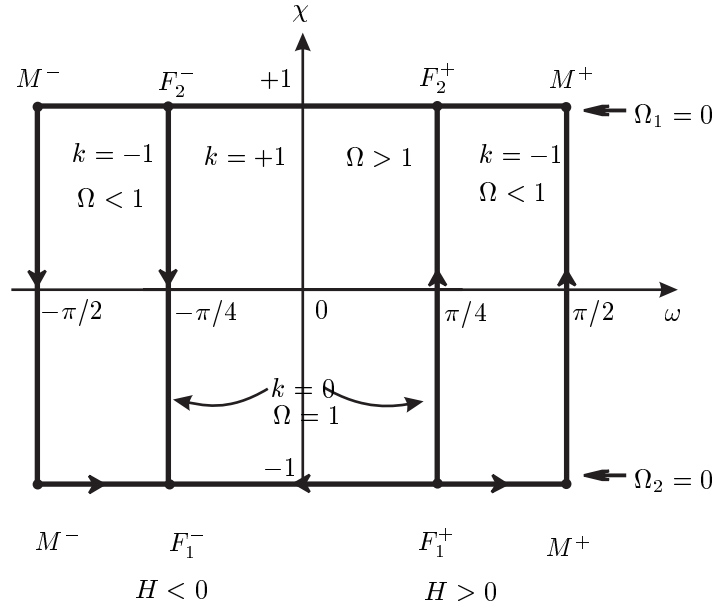
**Figure 1.** The invariant sets and equilibrium points of (21)

and with the same kind of manipulations as in [8], one obtains the following dynamical system

$$
\begin{aligned}
\frac{d\omega}{d\tau} &= -\frac{1}{2}(b-\chi)\cos 2\omega \cos \omega \\
\frac{d\chi}{d\tau} &= (1-\chi^2)\sin\omega,
\end{aligned}
\tag{21}
$$

where the constant $b$ is

$$
b = \frac{3(\gamma_1 + \gamma_2) - 4}{3(\gamma_1 - \gamma_2)} > -1.
$$

In our case, we always have $\gamma_1 = 2$.

The phase space of the two-dimensional system (21) is the closed rectangle

$$
D = [-\pi/2, \pi/2] \times [-1, 1]
$$

in the $\omega - \chi$ plane (see Figure 1).

By inspection we can see that the line segment $\{(\omega, \chi) \in D : \omega = \pi/4\}$ is an invariant set of (21). It consists of three trajectories, the line segment

$$
\{(\omega, \chi) \in D : \omega = \pi/4, \; -1 < \chi < 1\}
$$

and the equilibrium points $(\pi/4, -1)$ and $(\pi/4, +1)$. Similarly we can specify the following invariant sets.

| | | |
|---|---|---|
| $\omega = -\pi/2$ | contracting empty models | $\Omega = 0, H < 0$ |
| $\chi = -1$ | one-fluid models | $\Omega_2 = 0$ |
| $\chi = +1$ | one-fluid models | $\Omega_1 = 0$ |
| $\omega = \pi/4$ | expanding flat models | $\Omega = 1, H > 0$ |
| $\omega = -\pi/4$ | contracting flat models | $\Omega = 1, H < 0$ |
| $\omega = \pi/2$ | expanding empty models | $\Omega = 0, H > 0$ |

It is easy to verify that the equilibrium points lie at the intersection of these sets. The equilibrium points customarily denoted as $F_{1,2}^+$, $F_{1,2}^-$, $M^+$, $M^-$ are

$$
\begin{aligned}
F_{1,2}^+ : & \quad \omega = \pi/4 & \text{expanding flat model} & \quad \Omega = 1, \ k = 0, \ H > 0 \\
F_{1,2}^- : & \quad \omega = -\pi/4 & \text{contracting flat model} & \quad \Omega = 1, \ k = 0, \ H < 0 \\
M^+ : & \quad \omega = \pi/2 & \text{expanding Milne model} & \quad \Omega = 0, \ k = -1, \ H > 0 \\
M^- : & \quad \omega = -\pi/2 & \text{contracting Milne model} & \quad \Omega = 0, \ k = -1, \ H < 0
\end{aligned}
$$

and the subscripts indicate which fluid survives. Each of these eight equilibrium points corresponds to an exact solution of the Einstein equations.

We shall carry out in some detail the stability analysis of the equilibrium points of (21) in the case $\gamma_2 = 1$, corresponding to dust. It turns out that linearization of (21) is sufficient to determine the global phase portrait of the system. In fact, the derivative matrix $J(\omega, \chi)$ of the vector field of (21) is non-singular and, $J$ computed at each of the eight equilibrium points, has two real eigenvalues. Therefore, the Hartman-Grobman theorem applies in the case of (21). It is easy to verify that $J$ computed at all equilibrium points is diagonal. Therefore, we conclude in a straightforward manner that $(-\pi/2, -1)$, $(-\pi/4, +1)$, $(\pi/4, +1)$, $(\pi/2, -1)$ are saddle points, $(-\pi/2, +1)$, $(\pi/4, -1)$ are unstable nodes and $(-\pi/4, -1)$, $(\pi/2, +1)$, are stable nodes. The phase portrait is shown in Figure 2.
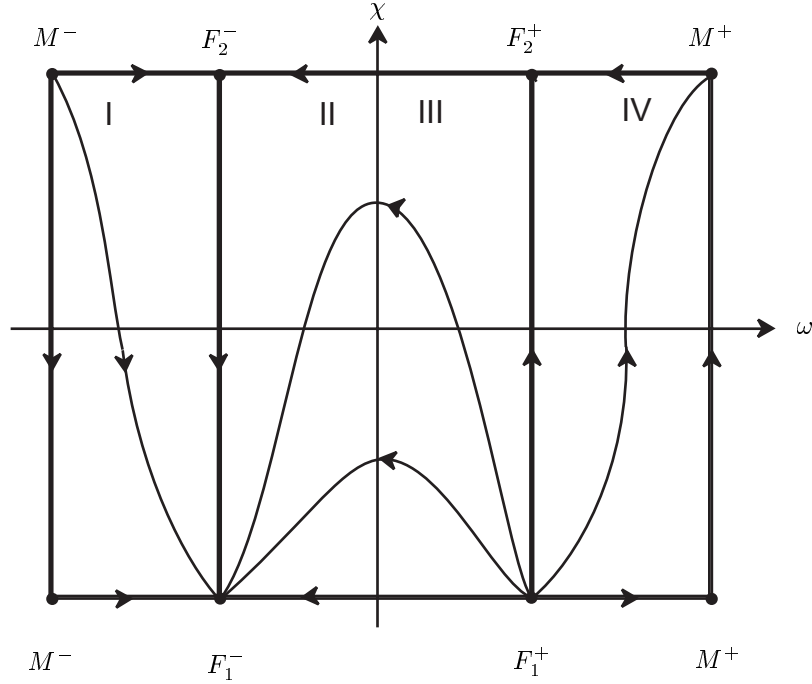


**Figure 2.** The phase portrait of (21) with $\gamma_2 = 1$.

Regions III and IV correspond to expanding models. The $F_1^+$ is a past attractor of all models with $\Omega > 0$, i.e., the evolution near the big bang is approximated by the flat FRW model where the Weyl fluid dominates. Open models expand indefinitely and the evolution is approximated by the Milne universe at late time. Flat models expand indefinitely and the evolution is approximated by the flat FRW universe at late time. In both cases the "real" second fluid dominates at late times while the $q$−fluid becomes insignificant. On the other hand, any

initially expanding closed model in region III, however close to $F_2^+$, eventually recollapses and the evolution is approximated by the flat FRW model where the Weyl fluid dominates.

In the case $\gamma_2 = 0$ corresponding to a positive cosmological constant, a new equilibrium point $(0, 2/3)$ appears, denoted by $E$ (see Figure 3). It corresponds to the Einstein static model.
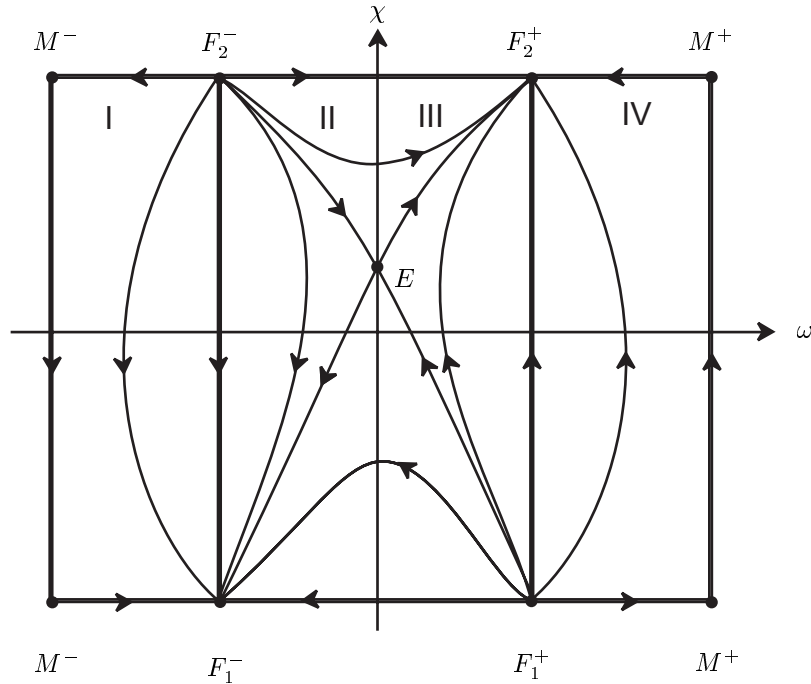


**Figure 3.** The phase portrait of (21) with $\gamma_2 = 0$.

Consider for example a closed model in region III starting close to the $F_1^+$ model. Its trajectory passes close to the Einstein model, indicating a phase of halted expansion and asymptotically approaches the de Sitter model $F_2^+$. Similarly, an open model in region IV starting close to $F_1^+$ asymptotically approaches the de Sitter model. This attracting property of the de Sitter solution for all expanding models is not restricted only to isotropic cosmology. In fact, the cosmic no-hair conjecture states that all expanding universe models with a positive cosmological constant, asymptotically approach the de Sitter solution (see for example [21]).

We therefore conclude that the Weyl fluid has significant contribution only near the cosmological singularities. In expanding models the "real" fluid always dominates at late times and therefore the contribution of the Weyl fluid to the total energy-momentum tensor is important only at early times.

## 5. Extensions

The field equations (8) constitute the generalization of the Einstein equations in a Weyl spacetime in the sense that they come from the Lagrangian $L = R$. There is however an alternative view, namely that the pair $(Q, \mathbf{g})$ which defines the Weyl spacetime also enters into the gravitational theory and therefore, the field $Q$ must be contained in the Lagrangian independently from $\mathbf{g}$. In the case of integrable Weyl geometry, i.e. when $Q_\mu = \partial_\mu \phi$ where $\phi$ is a scalar field, the pair $(\phi, g_{\mu\nu})$ constitute the set of fundamental geometrical variables. A simple Lagrangian involving this set is given by

$$L = R + \xi \nabla^\mu Q_\mu, \tag{22}$$

where $\xi$ is a constant. Motivations for considering theory (22) can be found in [20, 18] (see also [22] for a multidimensional approach and [23] for an extension of (22) to include an exponential potential function of $\phi$). By varying the action corresponding to (22) with respect to both $g_{\mu\nu}$ and $\phi$ one obtains

$$\overset{\circ}{G}_{\mu\nu} = \frac{3 - 4\xi}{2} \left( Q_\mu Q_\nu - \frac{1}{2} Q^2 g_{\mu\nu} \right), \quad \text{and} \quad \overset{\circ}{\Box} \phi = 0. \tag{23}$$

We emphasize that the field equations (8) and (23) are derived via different variational principles. Note that the second of (23) comes from the variational procedure while (9) is a consequence of the Bianchi identities.

For isotropic cosmologies one can again interpret the source term in (23) as a perfect fluid with density and pressure given by

$$\rho_1 = p_1 = \lambda q^2, \qquad \lambda = \frac{3 - 4\xi}{2} \tag{24}$$

respectively. However, if we allow for $\xi$ to be a free parameter, equation (24) implies that the energy density and pressure of the $q$-fluid may take negative values. An important consequence is that open models ($k = -1$) in vacuum with $\lambda < 0$ avoid the initial singularity [20].

In the following we assume that $\lambda < 0$ and apply again the two-fluid analysis of Section 4. A sufficient negative contribution of the $q$-fluid to the total energy density and pressure, cf. (13), may provoke an accelerating expansion of the universe. In fact, the $\rho + 3p$ term in the Raychaudhuri equation

$$\frac{\ddot{a}}{a} = -\frac{1}{6} \left( \rho + 3p \right),$$

becomes negative provided that $4\rho_1 < (2 - 3\gamma_2) \rho_2$. Unfortunately, this cannot explain the observed acceleration of the Universe, as we shall see in a moment.

The Friedmann constraint (15) implies that the total density parameter $\Omega = \Omega_1 + \Omega_2$ is still non-negative for flat and closed models, but $\Omega$ may be negative for open models. We conclude that $\Omega$ cannot be compactified as in (19) and the transition variable $\chi$ defined by (20) is unbounded. Therefore we cannot apply the Coley and Wainwright formalism developed in Section 4.

Nevertheless, we can infer about the asymptotic behavior of the system by looking at the original field equations, (15)-(18). From these equations we see that the state $(a, \dot{a}, \rho_1, \rho_2) \in \mathbb{R}^4$ of the system lies on the hypersurface defined by the constraint (15) and the remaining evolution equations can be written as a constrained four-dimensional dynamical system. By standard arguments one can show that for flat and open models the sign of $H$ is invariant (see for example [24]), therefore an initially expanding universe remains ever expanding. Suppose that $\ddot{a}(t_0) > 0$ at some time $t_0$. Then no solution of (16)-(18) exists such that $\ddot{a}(t) > 0$ for all $t > t_0$. In fact, (17) and (18) can be solved to give

$$\rho_1 = \frac{C_1}{a^6}, \quad \rho_2 = \frac{C_2}{a^{3\gamma_2}}, \tag{25}$$

where $C_1$ and $C_2$ are constants. Since $\gamma_2 < 2$, the "real" second fluid in an expanding universe eventually dominates. Therefore, the term $4\rho_1 + (3\gamma_2 - 2) \rho_2$ in (16) becomes positive at some time $t_1 > t_0$ and evidently $\ddot{a}(t) < 0$ for all $t > t_1$. We conclude that even if the universe expands initially with acceleration, it eventually evolves according to the Friedmann cosmology.

Note that for flat and open models there exist solutions without an initial singularity. To see this, consider for example $\gamma_2 = 1$ and substitute (25) into the Friedmann equation (15) with

$k = -1$. Then (15) can be interpreted as describing the motion of a particle with total energy $E = 1/2$ in the effective potential

$$V(a) = -\frac{A}{a} + \frac{B}{a^4}, \quad A, B > 0. \tag{26}$$
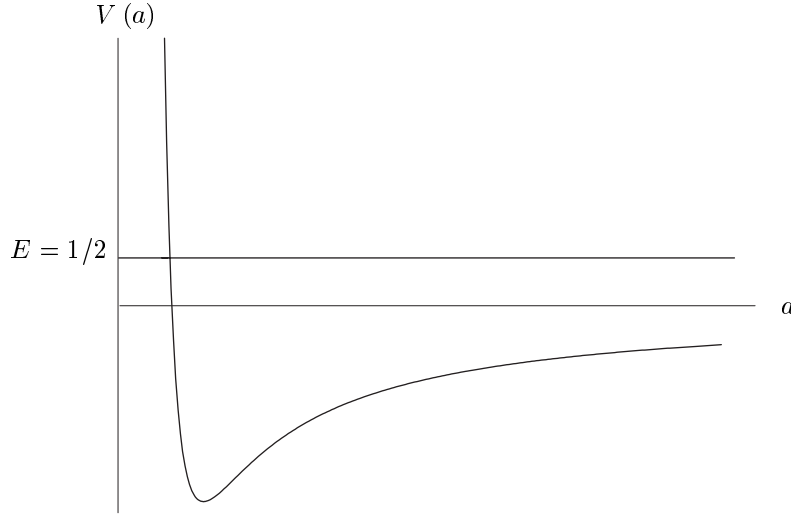


**Figure 4.** The effective potential (26).

In Figure 4, we see that motion is impossible for values of the scale factor smaller than some minimum $a_{\min}$ and therefore, $a(t) \geq a_{\min}$ for all $t$. A similar argument shows that flat models, also avoid the initial singularity, a conclusion already obtained using phase portrait analysis by Oliveira *et al* [23].

To summarize, the extension to Weyl geometry in theory (22) cannot explain the observed acceleration of the Universe. Since the real fluid dominates at late times, the accelerated expansion due to the Weyl fluid is important only at early times. Nevertheless, it is possible that theory (22) could provide a geometric explanation of an inflationary phase present in the early universe. Considering more general Lagrangians, or weakening the requirement of separate conservation of the two fluids should be the subject of further research in the framework of Weyl geometry.

**Acknowledgments**
I am grateful to S Cotsakis for reading the manuscript and commenting on this work.

**Appendix: Weyl geometry**
The connection $\nabla$ of the Weyl space is determined by the symmetric functions

$$\Gamma^\alpha_{\beta\gamma} = \{^\alpha_{\beta\gamma}\} + C^\alpha_{\beta\gamma},$$

where $\{^\alpha_{\beta\gamma}\}$ are the Levi-Civita connection coefficients and the tensor field $C^\alpha_{\beta\gamma}$ is given by

$$C^\alpha_{\beta\gamma} = \frac{1}{2}\left(\delta^\alpha_\beta Q_\gamma + \delta^\alpha_\gamma Q_\beta - g_{\beta\gamma} Q^\alpha\right).$$

Given a metric and a connection satisfying (1), *viz.* $\nabla_\mu g_{\alpha\beta} = -Q_\mu g_{\alpha\beta}$, then for every differentiable function $\sigma$, the metric and the 1-form defined by

$$g'_{\alpha\beta} = \sigma g_{\alpha\beta}, \qquad Q'_\mu = Q_\mu - \partial_\mu \ln \sigma \tag{A.1}$$

respectively, also satisfy (1). Thus, $g_{\alpha\beta}$ and $Q_\mu$ are far from unique; rather **g** belongs to an equivalence class $[g]$ of metrics and for each $\mathbf{g} \in [g]$, there exists a unique 1-form $Q$ such that (1) is satisfied. A particular choice of a pair $(Q, \mathbf{g})$ is called a gauge and (A.1) is a gauge transformation.

The Riemann tensor is defined by

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\beta\delta} - \partial_\delta \Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\sigma\gamma} \Gamma^\sigma_{\beta\delta} - \Gamma^\alpha_{\sigma\delta} \Gamma^\sigma_{\beta\gamma}. \tag{A.2}$$

In contrast to the familiar property in Riemannian geometry, in Weyl geometry $R_{\mu\nu\alpha\beta}$ is not antisymmetric in its first two indices; it satisfies

$$2R_{(\mu\nu)\alpha\beta} = g_{\mu\nu} H_{\alpha\beta}, \tag{A.3}$$

where

$$H_{\alpha\beta} := \partial_\alpha Q_\beta - \partial_\beta Q_\alpha. \tag{A.4}$$

Note that $H_{\alpha\beta}$ is the same for all derivative operators, for example $H_{\alpha\beta} = \nabla_\alpha Q_\beta - \nabla_\beta Q_\alpha$. When first introduced by Weyl, $H_{\alpha\beta}$ was supposed to represent the Faraday 2-form. The relation $2R_{(\mu\nu)\alpha\beta} = g_{\mu\nu} H_{\alpha\beta}$ can be obtained either directly from (A.2), or more quickly by applying on the metric the anticommutativity property of $\nabla_\alpha$:

$$\left( \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha \right) g_{\mu\nu} = R^\rho{}_{\mu\beta\alpha} g_{\rho\nu} + R^\rho{}_{\nu\beta\alpha} g_{\mu\rho}.$$

The Ricci tensor is defined by

$$R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}$$

and is not a symmetric tensor as we are accustomed to in Riemannian geometry; its antisymmetric part is given by

$$R_{[\alpha\beta]} = H_{\alpha\beta}.$$

The geometric meaning of (A.3) is that the measuring units change from point to point (see [25] for a discussion). In fact, the length $l^2 = g_{\mu\nu} V^\mu V^\nu$ of a vector field $V$ is not invariant when $V$ is parallely translated in a small circuit ; instead, using the standard argument one may show that for a closed circuit enclosing an elementary area $\delta S^{\alpha\beta}$, the variation of the length is

$$\delta \left( l^2 \right) = R_{(\mu\nu)\alpha\beta} V^\mu V^\nu \delta S^{\alpha\beta} = l^2 H_{\alpha\beta} \delta S^{\alpha\beta}.$$

As a consequence, there is an additional loss of synchronization for two initially synchronized clocks following different paths from one point to another, due to the distinct variation of the units of measure along the two paths. This is the so-called second clock effect [26].

When the Weyl vector field is the gradient of a scalar function, then the curl $H_{\alpha\beta}$ vanishes identically and we have the so-called integrable Weyl geometry. In that case the spacetime is not a genuine Weyl space, but a conformally equivalent Riemann space. In fact, if $Q_\mu = \partial_\mu \phi$, where $\phi$ is a scalar field, it is easy to see that it can be gauged away by the conformal transformation $\widetilde{g}_{\alpha\beta} = (\exp \phi) \, g_{\alpha\beta}$ and therefore the original space is not a general Weyl space, but a Riemann space with an undetermined gauge [27]. However, most studies of gravity theories were developed in the framework of integrable Weyl geometry in which the second clock effect is eliminated [20, 22, 18, 23].

**Bianchi identities**

For the convenience of the reader we present a few identities. We denote by $D$ the Levi-Civita connection.

1. $\nabla^\alpha Q_\alpha + Q^2 = \nabla_\alpha Q^\alpha$
2. $\Box := g^{\alpha\beta}\nabla_\alpha\nabla_\beta = \nabla^\alpha\nabla_\alpha = \nabla_\alpha\nabla^\alpha - Q_\alpha\nabla^\alpha$
3. $Q_\alpha\nabla^\alpha = Q^\alpha\nabla_\alpha$
4. $\nabla_\mu A^\nu = D_\mu A^\nu + C^\nu_{\mu\alpha}A^\alpha, \ \ \nabla_\mu B_\nu = D_\mu B_\nu - C^\alpha_{\mu\nu}B_\alpha$
5. $C^\alpha_{\alpha\mu} = 2Q_\mu, \ \ \ g^{\mu\nu}C^\alpha_{\mu\nu} = -Q^\alpha$
6. $\nabla_\mu Q_\nu = D_\mu Q_\nu - Q_\mu Q_\nu + \frac{1}{2}Q^2 g_{\mu\nu}$
7. $\nabla^\alpha Q_\alpha = D^\alpha Q_\alpha + Q^2, \ \ \ \nabla_\alpha Q^\alpha = D_\alpha Q^\alpha + 2Q^2$
8. $\nabla_\alpha Q^2 = Q_\alpha Q^2 + 2Q^\mu\nabla_\alpha Q_\mu = Q^\mu\nabla_\alpha Q_\mu + Q_\mu\nabla_\alpha Q^\mu$
9. $\left(\nabla^\mu\nabla_\alpha - \nabla_\alpha\nabla^\mu\right)Q_\mu = Q_\mu R^\mu{}_\alpha + Q^\mu H_{\alpha\mu} = Q_\mu R^\mu{}_{\underline{\alpha}}$

We now define the auxiliary operator $\overline{\nabla}_\alpha := \nabla_\alpha - Q_\alpha$; thus $\overline{\nabla}_\alpha$ commutes with $g^{\mu\nu}$. Note that $\overline{\nabla}_\alpha$ is not a derivative operator. For every torsion-free connection, the Bianchi identities take the form

$$\nabla_\rho R^\alpha{}_{\mu\sigma\nu} + \nabla_\nu R^\alpha{}_{\mu\rho\sigma} + \nabla_\sigma R^\alpha{}_{\mu\nu\rho} = 0.$$

We contract on $\alpha - \sigma$ to obtain

$$\nabla_\rho R_{\mu\nu} - \nabla_\nu R_{\mu\rho} + \nabla_\alpha R^\alpha{}_{\mu\nu\rho} = 0.$$

Transvection with $g^{\rho\mu}$ yields

$$\overline{\nabla}_\mu R^\mu{}_\nu - \overline{\nabla}_\nu R + \overline{\nabla}_\alpha R^{\alpha\mu}{}_{\nu\mu} = 0 \Rightarrow \overline{\nabla}_\mu R^\mu{}_\nu - \overline{\nabla}_\nu R - \overline{\nabla}_\alpha\left(g^{\alpha\mu}H_{\mu\nu}\right) + \overline{\nabla}_\alpha R^{\mu\alpha}{}_{\mu\nu} = 0$$

$$\Rightarrow 2\overline{\nabla}_\mu R^\mu{}_\nu - \overline{\nabla}_\nu R - \overline{\nabla}_\alpha\left(g^{\alpha\mu}H_{\mu\nu}\right) = 0. \tag{A.5}$$

On the other hand,

$$R^\mu{}_\nu = g^{\mu\alpha}R_{\alpha\nu} = g^{\mu\alpha}\left(R_{(\alpha\nu)} + H_{\alpha\nu}\right) =: R^\mu{}_{\underline{\nu}} + g^{\mu\alpha}H_{\alpha\nu},$$

hence equation (A.5) yields

$$2\overline{\nabla}_\mu R^\mu{}_{\underline{\nu}} - \overline{\nabla}_\nu R + \overline{\nabla}_\alpha\left(g^{\alpha\mu}H_{\mu\nu}\right) = 0$$

and since $\overline{\nabla}_\alpha$ commutes with $g^{\alpha\mu}$ we obtain

$$\overline{\nabla}_\mu\left(R^\mu{}_{\underline{\nu}} - \frac{1}{2}\delta^\mu_\nu R\right) = -\frac{1}{2}\overline{\nabla}^\mu H_{\mu\nu}. \tag{A.6}$$

Denoting by $G^\mu{}_{\underline{\nu}}$ the mixed tensor corresponding to the symmetric part of the Einstein tensor, i.e.,

$$G^\mu{}_{\underline{\nu}} = g^{\mu\alpha}G_{(\alpha\nu)},$$

the Bianchi identities (A.6) can be finally written as

$$\left(\nabla_\mu - Q_\mu\right)G^\mu{}_{\underline{\nu}} = -\frac{1}{2}\left(\nabla^\mu - Q^\mu\right)H_{\mu\nu}. \tag{A.7}$$

Note that in the case of integrable Weyl geometry, equation (A.7) reduces to $\nabla_\mu G^\mu{}_{\underline{\nu}} = Q_\mu G^\mu{}_{\underline{\nu}}$ which is usually referred in the literature as Bianchi identity (cf. [12]).

**Relation with the Levi-Civita connection**

It is useful to express the Einstein tensor and the Bianchi identities in terms of quantities formed with the Levi-Civita connection, $D$. Starting with the definition of the Riemann tensor we arrive at

$$
\begin{aligned}
R^{\alpha}{}_{\beta\gamma\delta} &= \mathring{R}^{\alpha}{}_{\beta\gamma\delta} + \partial_{\gamma}C^{\alpha}_{\beta\delta} - \partial_{\delta}C^{\alpha}_{\beta\gamma} + C^{\alpha}_{\mu\gamma}C^{\mu}_{\beta\delta} - C^{\alpha}_{\mu\delta}C^{\mu}_{\beta\gamma} \\
&+ \left\{{}^{\alpha}_{\mu\gamma}\right\}C^{\mu}_{\beta\delta} + \left\{{}^{\mu}_{\beta\delta}\right\}C^{\alpha}_{\mu\gamma} - \left\{{}^{\alpha}_{\mu\delta}\right\}C^{\mu}_{\beta\gamma} - \left\{{}^{\mu}_{\beta\gamma}\right\}C^{\alpha}_{\mu\delta},
\end{aligned}
$$

where the accent $\circ$ denotes a quantity formed with the Levi-Civita connection. We contract on $\alpha, \gamma$ and take account of $C^{\alpha}_{\alpha\mu} = 2Q_{\mu}$, $g^{\mu\nu}C^{\alpha}_{\mu\nu} = -Q^{\alpha}$, $\left\{{}^{\alpha}_{\alpha\mu}\right\} = \partial_{\mu}\ln\sqrt{-g}$ to obtain for the symmetric part of the Ricci tensor,

$$
R_{(\beta\delta)} = \mathring{R}_{\beta\delta} - D_{(\beta}Q_{\delta)} + \frac{1}{2}Q_{\beta}Q_{\delta} - \frac{1}{2}g_{\beta\delta}\left(D^{\mu}Q_{\mu} + Q^2\right)
$$

and upon a new contraction,

$$
R = \mathring{R} - 3D_{\alpha}Q^{\alpha} - \frac{3}{2}Q^2.
$$

Therefore the Einstein tensor takes the form

$$
G_{(\alpha\beta)} = \mathring{G}_{\alpha\beta} - D_{(\alpha}Q_{\beta)} + \frac{1}{2}Q_{\alpha}Q_{\beta} + g_{\alpha\beta}\left(D^{\mu}Q_{\mu} + \frac{1}{4}Q^2\right). \tag{A.8}
$$

We now turn to the Bianchi identities. We calculate the right-hand side (RHS) of (A.7)

$$
\left(\nabla_{\alpha} - Q_{\alpha}\right)G^{\alpha}{}_{\underline{\nu}} = D_{\alpha}G^{\alpha}{}_{\underline{\nu}} + C^{\alpha}_{\alpha\beta}G^{\beta}{}_{\underline{\nu}} - C^{\beta}_{\alpha\nu}G^{\alpha}{}_{\underline{\beta}} - Q_{\alpha}G^{\alpha}{}_{\underline{\nu}}. \tag{A.9}
$$

Taking into account of (A.8) the first term in the RHS of (A.9) can be written as

$$
\begin{aligned}
D_{\alpha}G^{\alpha}{}_{\underline{\nu}} &= D_{\alpha}\mathring{G}^{\alpha}{}_{\nu} - \frac{1}{2}\mathring{\Box}Q_{\nu} - \frac{1}{2}D_{\alpha}D_{\nu}Q^{\alpha} + D_{\nu}D_{\alpha}Q^{\alpha} \\
&+ \frac{1}{2}Q_{\nu}D_{\alpha}Q^{\alpha} + \frac{1}{2}\left(Q^{\alpha}D_{\alpha}\right)Q_{\nu} + \frac{1}{4}D_{\nu}Q^2.
\end{aligned}
$$

We use the anticommutativity of the derivative operator $D_{\alpha}$ to write the terms $-\frac{1}{2}D_{\alpha}D_{\nu}Q^{\alpha} + D_{\nu}D_{\alpha}Q^{\alpha}$ as

$$
-Q_{\alpha}\mathring{R}^{\alpha}{}_{\nu} + \frac{1}{2}D_{\alpha}D_{\nu}Q^{\alpha}.
$$

The second and the last terms in the RHS of (A.9) can be written as

$$
\begin{aligned}
C^{\alpha}_{\alpha\beta}G^{\beta}{}_{\underline{\nu}} - Q_{\alpha}G^{\alpha}{}_{\underline{\nu}} &= Q_{\alpha}\mathring{G}^{\alpha}{}_{\nu} - \frac{1}{2}\left(Q^{\alpha}D_{\alpha}\right)Q_{\nu} \\
&- \frac{1}{4}D_{\nu}Q^2 + \frac{1}{2}Q_{\nu}Q^2 + Q_{\nu}D_{\alpha}Q^{\alpha} + \frac{1}{4}Q_{\nu}Q^2.
\end{aligned}
$$

The third term in the RHS of (A.9) can be written as

$$
\begin{aligned}
-C^{\beta}_{\alpha\nu}G^{\alpha}{}_{\underline{\beta}} &= -\frac{1}{2}\left(\delta^{\beta}_{\alpha}Q_{\nu} + \delta^{\beta}_{\nu}Q_{\alpha} - g_{\alpha\nu}Q^{\beta}\right)G^{\alpha}{}_{\underline{\beta}} = \frac{1}{2}Q_{\nu}R \\
&= \frac{1}{2}Q_{\nu}\mathring{R} - \frac{3}{2}Q_{\nu}D_{\alpha}Q^{\alpha} - \frac{3}{4}Q_{\nu}Q^2.
\end{aligned}
$$

Putting all these together we obtain from equation (A.9)

$$\left(\nabla_\alpha - Q_\alpha\right) G^\alpha_{\underline{\ \nu}} = -\frac{1}{2} \overset{\circ}{\Box} Q_\nu + \frac{1}{2} D_\alpha D_\nu Q^\alpha.$$

On the other hand,

$$\begin{aligned}
\left(\nabla^\alpha - Q^\alpha\right) H_{\alpha\nu} &= g^{\alpha\mu}\left(D_\mu H_{\alpha\nu} - C^\rho_{\mu\alpha} H_{\rho\nu} - C^\rho_{\mu\nu} H_{\alpha\rho}\right) - Q^\alpha H_{\alpha\nu} \\
&= \overset{\circ}{\Box} Q_\nu - D_\alpha D_\nu Q^\alpha - Q^\alpha H_{\alpha\nu}.
\end{aligned}$$

Therefore the Bianchi identities (A.7) reduce to

$$Q^\alpha H_{\alpha\nu} = 0. \tag{A.10}$$

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