On the Mannheim-Kazanas Spacetime

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Abstract

The Einstein-Hilbert action in general relativity has been remarkably successful, despite its lack of a fundamental mathematical symmetry known as *conformal invariance*, a symmetry that many consider natural and desirable in relativistic cosmology, if not in all of physics. Although a fully conformal action was proposed a century ago by Weyl, interest was largely ignored due to the complexity of the associated field equations and the fact that the action is necessarily of fourth order in the metric tensor and its derivatives—an undesirable aspect believed to result in unphysical quantities such as ghost fields. However, in 1989 Mannheim and Kazanas (MK) managed to solve the equations of motion using a Schwarzschild-like metric that involved two integration constants beyond that of the usual Schwarzschild point mass. By setting these constants to zero, the MK solution reduces to the Schwarzschild metric and thus fully reproduces the predictions of Einsteinian gravity for free space. This gave rise to the hope that a fully conformal approach to general relativity might address the problems of dark matter and dark energy. Indeed, the MK solution has had significant success in predicting the anomalous flat velocity curves of stars far from their galactic centers, a phenomenon that is currently associated with the presence of dark matter.

Recently it has been shown that the MK metric is equivalent to that of Schwarzschild-de Sitter spacetime, which predicts the future end point of the universe in which all ordinary matter and energy has devolved into a state of stray high-entropy radiation via particle decay and black hole evaporation, with dark energy dominating the continued evolution of the universe. We discuss the immediate consequences of this equivalency, along with a related aspect that comes from Weyl's failed 1918 theory of the unified gravitational-electromagnetic field.

1. Introduction

One hundred years ago the German mathematical physicist Hermann Weyl showed that the most general gravitational action that is invariant with respect to the conformal metric transformation $g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu}$ is

$$S = \int \sqrt{-g}\, C_{\mu\nu\alpha\beta}\, C^{\mu\nu\alpha\beta}\, d^4x \tag{1.1}$$

where, in four dimensions,

$$C_{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta} + \frac{1}{2}\left[g_{\mu\beta} R_{\nu\alpha} - g_{\mu\alpha} R_{\nu\beta} + g_{\nu\alpha} R_{\mu\beta} - g_{\nu\beta} R_{\mu\alpha}\right] + \frac{1}{6}\left[g_{\mu\alpha}\, g_{\beta\nu} - g_{\mu\beta}\, g_{\alpha\nu}\right] R$$

is the Weyl tensor. Unlike the Einstein-Hilbert Lagrangian $\sqrt{-g}\, R$, the equations of motion associated with (1.1) are complicated due mainly to the presence of the Riemann tensor $R_{\mu\nu\alpha\beta}$. (As an aside, we note that the matter action associated with (1.1) is likely going to be problematic as well, given that it is quadratic and of fourth order.)

However, with a little algebra (1.1) can be reduced to the much simpler quantity

$$S_G = \int \sqrt{-g}\left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2\right) d^4x \tag{1.2}$$

The reduction is based on a subtle appeal[1] to the Bianchi identities

$$\left(R^{\mu\nu} - \frac{1}{2}\, g^{\mu\nu} R\right)_{||\nu} = 0$$

[1]Textbooks invariably have the so-called Gauss-Bonnet quantity

$$\int \sqrt{-g}\left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2\right) d^4x$$

as a pure divergence. This is in fact not a divergence or surface term, but it can be eliminated from (1.1) as a direct consequence of the Bianchi identities. If we write the integrand as $R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + A R_{\mu\nu} R^{\mu\nu} + B R^2$ where A and B are constants, then the condition $A + 3B = -1$ is equivalent to the Gauss-Bonnet identity. Upon elimination of the $R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$ term, we recover (1.2). See Reference 2 for a detailed derivation.

where the double subscripted bar represents covariant differentiation. While the equations of motion associated with (1.2) for free space are still complicated, in 1989 Mannheim and Kazanas (see Reference 3) found that for the Schwarzschild line element

$$ds^2 = e^{\nu} c^2 dt^2 - e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

the general solution is

$$e^{\nu} = e^{-\lambda} = 1 - 3\beta\gamma - \frac{\beta(2-3\beta\gamma)}{r} + \gamma r - kr^2 \tag{1.3}$$

where β, γ, k are constants. Obviously, for $\gamma = k = 0$ the MK metric is identical to the Schwarzschild metric, with β representing the geometric source mass GM/c^2. The linear and quadratic terms in (1.3) represent expansion and acceleration terms that Mannheim and Kazanas thought might be linked to dark matter and dark energy. Of particular interest is the linear term in r, which we will see is absent in the Schwarzschild-de Sitter metric. One can show that stellar rotational velocities can be expressed as

$$v^2 = \frac{r^2}{e^{\nu}} \left(\frac{d\phi}{dt} \right)^2 = \frac{r}{2e^{\nu}} \frac{de^{\nu}}{dr}$$

which, given the presence of the linear term, describes flat stellar rotation curves for large r. However, while (1.3) accurately describes the effects of dark matter for many galaxies, it fails with respect to observations of many galactic clusters and cosmological lensing effects. Indeed, Hobson and Lasenby showed (see Reference 4) that the linear term in (1.3) can be eliminated by a suitable transformation of the radial parameter coupled with a conformal transformation of the metric.

2. Schwarzschild-de Sitter Spacetime

While (1.2) is considerably simpler than (1.1), the presence of the $R_{\mu\nu}R^{\mu\nu}$ term still greatly complicates the associated MK solution, and one wonders if a futher simplication might be possible. Let us consider the original de Sitter problem, which describes a universe completely devoid of mass-energy but with a cosmological constant Λ, a scenario that is believed to be valid with the continued expansion of the universe. It is described by the Einstein equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = 0$$

where the cosmological constant is assumed to be non-zero and positive for an accelerating universe. Contraction gives the equivalent traceless expression

$$R^{\mu\nu} - \frac{1}{4} g^{\mu\nu} R = 0 \tag{2.1}$$

where $R = 4\Lambda$. A Schwarzschild-like metric for (1.4) was discovered long ago, and is given by

$$e^{\nu} = -e^{-\lambda} = 1 - \frac{2GM}{c^2 r} - kr^2 \tag{2.2}$$

where k is a constant proportional to R. The similarity of (2.2) and (1.3) is evident, with the expressions differing only by a term linear in r and a trivial constant. However, Hobson and Lasenby have shown that (2.2) and (1.3) are equivalent, given the fact that a coordinate transformation in r, coupled with the conformal (Weyl) transformation $ds^2 \to \Omega^2 ds$, easily demonstrates their equivalence. To see this, we consider the MK line element in the Schwarzschild form

$$ds^2 = Ac^2 dt^2 - \frac{dr^2}{A} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \qquad A = 1 - 3\beta\gamma - \frac{\beta(2-3\beta\gamma)}{r} + \gamma r - Kr^2 \tag{2.3}$$

We wish to keep the coordinates t, θ, ϕ unchanged under the conformal transformation $ds^2 \to= \Omega^2 ds$. The only coordinate change will involve the radial parameter r, so we will have $dr \to dr'$. Nothing else changes, so we wish to consider the equivalent line element

$$ds^2 = \Omega^2 B\, c^2 dt^2 - \Omega^2 \frac{dr'^2}{B} - r'^2 d\theta^2 - r'^2 \sin^2\theta d\phi^2$$

Comparing the two line elements, we have $A=\Omega^2 B$ and $r=\Omega^2 r'$, the latter of which agrees with the transformation of the metric

$$g'_{11}=\left[\frac{\partial r}{\partial r'}\right]^2 g_{11}$$

We also have $\Omega^2\,dr'=dr$ which, assuming $\Omega=\Omega(r')$, integrates to

$$r=\frac{r'}{1-ar'}$$

where a is a constant of integration to be determined. We can now solve for the coefficient B in terms of the primed coordinates, which is

$$\begin{aligned}B=\ &1-3\beta\gamma+3a\beta\,(2-3\beta\gamma)-\frac{\beta(2-3\beta\gamma)}{r'}+\left[\gamma-2\alpha+6a\beta\gamma-3a^2\beta(2-3\beta\gamma)\right]r'\\&-\left[k+a^2-3a^2\beta\gamma-a\gamma+a^3\beta(2-3\beta\gamma)\right]r'^2\end{aligned}$$

The r' coefficient is a simple quadratic in the constant a, and it can be set to zero by choosing

$$a=\frac{\gamma}{2-3\beta\gamma}$$

so that the required coordinate change in r is

$$r=\frac{\gamma r'}{1-\gamma r'/(2-3\beta\gamma)}$$

with

$$\Omega=1+\frac{\gamma}{1-\gamma r'/(2-3\beta\gamma)}$$

This also wipes out the $3\beta\gamma$ term in (1.3), while the coefficient k' becomes

$$k'=k+\frac{\gamma^2(1-\beta\gamma)}{(2-3\beta\gamma)^2} \tag{2.4}$$

The transformed metric is thus

$$e^{\nu}=e^{-\lambda}=1-\frac{\beta(2-3\beta\gamma)}{r'}-k'r'^2 \tag{2.5}$$

Consequently, the MK metric and the Schwarzschild-de Sitter metric are effectively one and the same. This equivalence presents several problems, which are addressed in the following..

3. Concluding Remarks

In view of the equivalence of the MK and Schwarzschild-de Sitter metrics, there is no longer any need to consider the problematic $R_{\mu\nu}R^{\mu\nu}$ term in the MK metric. By simply omitting it from the associated Lagrangian we recover the much simpler Schwarzschild-de Sitter solution for free space. This appears to justify our stated desire to eliminate the term for the sake of simplicity.

However, this still leaves the problem of determining an appropriate mass-energy term for non-empty space. As noted earlier, an action that is quadratic in the Ricci scalar R presents problems with regard to the associated energy-momentum tensor $T^{\mu\nu}$, which traditionally is derived via the variational identity

$$T^{\mu\nu}=-2\frac{1}{\sqrt{-g}}\,\frac{\delta(\sqrt{-g}S_M)}{\delta g_{\mu\nu}}$$

where S_M is the mass-energy parameter. Clearly, $T^{\mu\nu}$ is of dimension zero, but it leaves open the possibility that it might also need to be quadratic like the gravitational equations of motion. Interestingly, the Schwarzschild-de Sitter action

$$S=\int\sqrt{-g}\,R^2\,d^4x$$

is dimensionless, eliminating the need for the $8\pi G/c^4$ term of traditional Einsteinian gravity. In addition, the Schwarzschild-de Sitter solution (2.1) is traceless, and a traceless matter term would seem to be required as well. The most familiar traceless energy-momentum term is the electromagnetic stress-energy tensor

$$T^{\mu\nu} = F^{\mu\alpha} F_{\alpha}^{\ \nu} - \frac{1}{4}\, g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

where $F^{\mu\nu}$ is the electromagnetic tensor. Lastly, we have to explain how the Schwarzschild-de Sitter action is conformally invariant despite the elimination of the $R_{\mu\nu}R^{\mu\nu}$ term from the MK action. A straightforward calculation shows that a conformal variation leads to

$$\frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{-g}\, R^2\, d^4x = -6 \int \left(\sqrt{-g}\, g^{\mu\nu} R_{|\mu} \right)_{|\nu} d^4x = 0$$

where the single subscripted bar represents ordinary partial differentiation (in Riemannian geometry, covariant derivatives and partial derivatives are the same for vector densities). In free space R is a constant proportional to the cosmological constant Λ, so conservation of the above vector density is trivially satisfied. But the divergence condition $\left(\sqrt{-g}\, g^{\mu\nu} R_{|\mu} \right)_{|\nu} = 0$ might still hold for a non-constant R in a space containing mass-energy. Proof of this assertion requires a suitable energy-momentum tensor (perhaps a quadratic), which at present we do not have.

In his failed 1918 theory of the unified gravitational-electromagnetic field, Weyl (see Reference 5) assumed the conformal invariance of his theeory's action. To accomplish this, he also had to assume the existence of a local four-vector field $\phi_\mu(x)$ which, when incorporated into the Ricci scalar R, achieved the desired invariance. But this also meant that his action had to be quadratic in the Ricci scalar R. Nevertheless, Weyl went on to show that in the absence of the ϕ_μ field, the vector density $\sqrt{-g}\, g^{\mu\nu} R_{|\mu}$ was a conserved quantity, and he boldly asserted that this scalar density was none other than the electromagnetic source density $\sqrt{-g}\, S^\mu$, where S^μ is the source four-vector.

It is interesting that the notion of conformal invariance, first proposed over a hundred years ago and later shown to be of fundamental importance in modern quantum theory, continues to be of significant theoretical and observational importance today.

References

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