Non-Metricity and the Riemann-Christoffel Tensor

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Abstract

The Riemann-Christoffel tensor lies at the heart of general relativity theory and much of differential geometry. Its very definition establishes the antisymmetry of its last two indices, but its remaining symmetry properties depend not only on the cyclicity condition but on the vanishing of the non-metricity tensor as well. In four dimensions the Riemann-Christoffel tensor traditionally exhibits 20 independent components, but when non-metricity is taken into account that number necessarily increases to 80. In this very brief paper, we show why this is true.

Introduction

For any vector or tensor quantity one can take the difference of its double covariant derivatives to obtain a derivation of the Riemann-Christoffel tensor $R^{\lambda}{}_{\mu\alpha\beta}$. For example, given an arbitrary covariant vector ξ_μ it is easy to show that

$$\xi_{\mu||\alpha||\beta} - \xi_{\mu||\beta||\alpha} = -\xi_\lambda R^{\lambda}{}_{\mu\alpha\beta} = -\xi^\lambda R_{\lambda\mu\alpha\beta} \tag{1}$$

(the double-bar convention of Adler-Bazin-Schiffer is assumed), where

$$R^{\lambda}{}_{\mu\alpha\beta} = \Gamma^{\lambda}_{\mu\alpha|\beta} - \Gamma^{\lambda}_{\mu\beta|\alpha} + \Gamma^{\lambda}_{\beta\nu}\Gamma^{\nu}_{\mu\alpha} - \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\nu}_{\mu\beta}$$

where the $\Gamma^{\lambda}_{\mu\alpha}$ quantities are connection coefficients (Christoffel symbols in Riemannian geometry) and the single subscripted bar represents ordinary partial differentiation. Lowering the upper index with the metric tensor $g_{\mu\lambda}$ gives us $R_{\mu\nu\alpha\beta}$ which, considering the antisymmetry in the last two indices, gives the Riemann-Christoffel tensor a total of $n^3(n-1)/2$ independent components, or 96 in 4-dimensional space. However, a condition called *cyclicity* reduces this number substantially. It is a simple matter to show that

$$R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0, \tag{2}$$

a result that holds also in a non-Riemannian manifold.

Riemannian geometry is characterized by the vanishing of the *non-metricity tensor* $g_{\mu\nu||\lambda}$, which is the covariant derivative of the metric tensor. Assuming $g_{\mu\nu||\lambda} = 0$, it is a standard exercise to show that the RC tensor displays the additional symmetry conditions

$$R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}, \qquad R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$$

the latter property being the familiar *index-pair exchange* property. Using these various symmetry properties, it can be shown that the lower-index RC tensor has $n^2(n^2-1)/12$ independent components, or a total of 20 components in 4-dimensional space.

Note also that there are only two possible contractions of $R^{\lambda}{}_{\mu\alpha\beta}$ in Riemannian space,

$$R^{\lambda}{}_{\lambda\alpha\beta} = R_{\alpha\beta} - R_{\beta\alpha} = 0, \qquad R^{\lambda}{}_{\mu\lambda\beta} = R_{\mu\beta}$$

the latter quantity being the (symmetric) Ricci tensor.

The RC Tensor in Non-Riemannian Space

In non-Riemannian space the non-metricity tensor $g_{\mu\nu||\lambda}$ no longer vanishes. However, as in (1) the difference of its double covariant derivatives can be written as usual as

$$g_{\mu\nu||\alpha||\beta} - g_{\mu\nu||\beta||\alpha} = -g_{\mu\lambda}R^{\lambda}{}_{\nu\alpha\beta} - g_{\lambda\nu}R^{\lambda}{}_{\mu\alpha\beta}$$

or

$$g_{\mu\nu||\alpha||\beta} - g_{\mu\nu||\beta||\alpha} = -(R_{\mu\nu\alpha\beta} + R_{\nu\mu\alpha\beta}) \tag{3}$$

In non-Riemannian space, therefore, the antisymmetry of the first two indices of $R_{\mu\nu\alpha\beta}$ is lost and, lacking a specific definition for the non-metricity tensor, this antisymmetry property cannot be used to reduce the number of independent components of the RC tensor. We will now use the cyclicity condition in (2) to show that the index-pair exchange property is invalid as well. Using the two equivalent expressions

$$R_{\mu\nu\alpha\beta} + R_{\mu\beta\nu\alpha} + R_{\mu\alpha\beta\nu} = 0$$

and

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$$

we can easily show, by repeated exchange of the antisymmetric index pairs in each, that

$$R_{\mu\alpha\beta\nu} + R_{\alpha\mu\beta\nu} = \left(R_{\alpha\beta\mu\nu} - R_{\mu\nu\alpha\beta}\right) + \left(R_{\mu\beta\alpha\nu} - R_{\alpha\nu\mu\beta}\right) \tag{4}$$

Both sides of this expression vanish identically in Riemannian space, but that is no longer the case when $g_{\mu\nu||\lambda} \neq 0$. Consequently, the number of independent components in the Riemann-Christoffel tensor depends only on the antisymmetry property of the last two indices in $R_{\mu\nu\alpha\beta}$ and the cyclicity condition. Careful consideration of (2) shows that cyclicity eliminates $n^2(n-1)(n-2)/6$ terms; subtracting this from the inital $n^3(n-1)/2$ terms leaves us with exactly $n^2(n^2-1)/3$ independent components in the Riemann-Christoffel tensor. In 4-dimensional space this comes out to 80 components, far more than the 20 terms one has for the Riemannian case. Precisely what this large number of components means geometrically or what additional degrees of freedom it gives to the Riemann-Christoffel tensor is not known.

Comments

In his 1918 theory of the unified gravitational/electromagnetic field, the German mathematical physicist Hermann Weyl developed a non-Riemannian geometry in which the non-metricity tensor was given by

$$g_{\mu\nu||\lambda} = 2g_{\mu\nu}\varphi_\lambda$$

where φ_λ was assumed to be proportional to the four-potential of the electromagnetic field. Despite its stunning mathematical beauty and the apparent initial success of Weyl's unification plan, Einstein showed the theory to be unphysical, and it was eventually discarded. But it led directly to the notion of *gauge invariance* in physics, where it had enormous impact on quantum theory. Subsequent investigations by Einstein, Eddington and Schrödinger resulted in similar identifications for the non-metricity tensor, usually in terms of the electromagnetic four-potential. These efforts also failed, and today it is fair to say that no one really knows what $g_{\mu\nu||\lambda}$ represents geometrically nor what significance it might have in physics.

References

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