FEYNMAN'S PATH INTEGRAL APPROACH TO QUANTUM FIELD THEORY

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"*Consider the lilies of the goddamn field.*"
– Ulysses Everett McGill, *O Brother, Where Art Thou?*

Here's an elementary explanation of the mathematics behind Feynman's path integral, along with a *very* simplified overview of its application to self-interacting quantum field theory (QFT), also known as ϕ^4 scalar field theory. Although it's elementary, there's enough information to provide a basic understanding of what the path integral is and how it leads to a many-particle interpretation in QFT. The discussion is very detailed in some of the finer points, and some of the material is just plain overkill, so it's rather longer than I wanted. But if you can stand my frankly insufferable didactic style, it may help you fill in the blanks.

Notation

Most of the integrals in QFT are four dimensional, but for brevity I have used dx in lieu of d^4x whenever possible. Similarly, dk and k are four dimensional (although occasionally k is the 3-momentum), while kx is shorthand for $k_\mu x^\mu$. The Dirac delta function is denoted as $\delta^4(x)$, which may occasionally look like the fourth-power functional derivative operator $\delta^4/\delta J^4$, so don't get them confused. Unless specified otherwise, integration limits are $-\infty$ to $+\infty$. Einstein's summation convention is assumed, as is Dirac notation.

Prerequisites

For basic background material on quantum mechanics, I very strongly recommend that you read J.J. Sakurai's *Modern Quantum Mechanics*, which is probably the best book on the subject at the advanced undergraduate or beginning graduate level. The first few chapters provide an especially clear overview of the basic principles, with emphasis on the mathematical notation I use here (the so-called Dirac notation), along with Sakurai's extremely lucid exposition on quantum dynamics. As a student, I didn't like Sakurai's book much at first because I didn't want to work with Dirac notation and because Sakurai seemed to leave out too many steps in his derivations. Now I cannot imagine anyone learning quantum theory without his book. At the same time, I cannot imagine how anyone could learn anything from Dirac's *Principles of Quantum Mechanics* – it seems ancient, dry and boring to me, even though it was written (in my opinion) by the greatest theoretical physicist who ever lived. But if Sakurai isn't your cup of tea, try the second edition of R. Shankar's *Principles of Quantum Mechanics*, which is probably just as good and devotes two chapters to path integrals to boot.

As for QFT itself, there are many books available, all of them somewhat difficult and obtuse, in my opinion (this is most likely because I'm an engineer, and there are many aspects of QFT that strain my logical abilities). The best advice I can give you is to find a text that speaks to you at your level, then try more advanced subjects as your confidence increases. For me, L. Ryder's *Quantum Field Theory* is about as comprehensible as they come, and I would recommend it as a starting point. It has a very readable introduction to the Lagrangian formulation and canonical quantization, the latter of which should be read so that the reader will fully appreciate how much simpler the path integral approach is. Another relatively understandable text is M. Kaku's *Quantum Field Theory*, although the notation is occasionally a trifle bizarre (for example, he expresses the closure relation as $|p\rangle \int dp \, \langle p| = 1$). If you're desperate to learn QFT, and the above books are still over your head, then try A. Zee's excellent book *Quantum Field Theory in a Nutshell*. The author practically takes you by the hand for the first 60 pages or so, and if that doesn't do the trick for you then you're probably a hopeless case.

Understanding the path integral is a snap, but picking up quantum field theory is a difficult task. It's somewhat like learning a new language; it takes a while, but soon it starts to make sense, and then things get much easier. I hope you have a great deal of intellectual curiosity, because in the end that's the main thing that will motivate you to learn it. Hopefully, you'll also learn to appreciate what a truly strange and wonderful world God created for us undeserving and witless humans. Good luck!

1. Derivation of the Propagator In Quantum Mechanics

In order to derive Feynman's path integral, we first need to develop the concept of the *propagator* in quantum dynamics using the time translation operator $\hat{U}(t)$. To do this we shall need to review the distinction between operators in the Schrödinger and Heisenberg "pictures."

In the so-called Schrödinger operator picture, state vectors are assumed to be time-dependent, whereas operators are taken as time-independent quantities. In this picture, all time dependence is assumed to come from the "moving" state ket $|\alpha, t\rangle$, while operators stay fixed in time. Even the Hamiltonian operator $\hat{H}$ normally is not itself dependent on time, but instead operates on state vectors whose time-dependent parts "feel" the effect of $\hat{H}$.

In elementary quantum dynamics, we define the time translation operator $\hat{U}(t'', t')$, which takes the value that some state ket $|\alpha, t'\rangle$ has at time t' and returns the value that the ket would have at time t''; thus,

$$\begin{aligned} |\alpha, t''\rangle &= \hat{U}(t'', t')\,|\alpha, t'\rangle\,, \quad \text{where} \\ U(t'', t') &= \exp[-i\hat{H}(t'' - t')/\hbar] \end{aligned}$$

(Please note that I'm going to drop the hat notation on the U and H operators from here on out.) Similarly, a state bra changes according to

$$\langle \alpha, t''| = \langle \alpha, t''|U^\dagger(t'', t')$$

where $U^\dagger(t'', t')$ is the hermitian adjoint of U (I suppose that in this case, the quantity $U(t'', t')$ voids our definition of a Schrödinger operator as being strictly time-independent, but we'll have to overlook this for now).

Now consider $\langle \alpha, t'|\alpha, t'\rangle$. If the state vector is normalized, this quantity is unity. Obviously, if we evaluate this quantity at some other time t'' nothing changes; that is,

$$\begin{aligned} \langle \alpha, t''|\alpha, t''\rangle &= \langle \alpha, t'|U^\dagger(t'', t')\,U(t'', t')\,|\alpha, t'\rangle \\ &= \langle \alpha, t'|\alpha, t'\rangle \end{aligned}$$

(Actually, this holds trivially for any unitary operator, since the product of the operator and its adjoint is unity.) But now let's see how the situation changes when we look at the expectation value of some operator $\hat{A}$, which we denote as $\langle \alpha, t'|\hat{A}|\alpha, t'\rangle$. At time t'', this goes over to

$$\begin{aligned} \langle \alpha, t''|\hat{A}|\alpha, t''\rangle &= \langle \alpha, t'|U^\dagger(t'', t')\ \hat{A}\,U(t'', t')\,|\alpha, t'\rangle \\ &= [\,\langle \alpha, t'|U^\dagger]\ \hat{A}\ [U\,|\alpha, t'\rangle\,] \qquad (1.1) \\ &= \langle \alpha, t'|\ [U^\dagger \hat{A} U]\ |\alpha, t'\rangle \qquad (1.2) \end{aligned}$$

Notice that we have used the associativity law of operator multiplication to write this in two ways. In (1.1), the operator $\hat{A}$ is sandwiched between time-translated state vectors, while in (1.2) the product operator $U^\dagger \hat{A} U$ appears wedged between *unchanged* state kets. The first situation (where the operator $\hat{A}$ is time-independent) corresponds to the Schrödinger picture, whereas the second involves the time-dependent *Heisenberg operator* $U^\dagger \hat{A} U$ operating on the state vector $|\alpha, t'\rangle$, which keeps whatever value it has at time t' for all time. We therefore have two ways of looking at dynamical systems, either of which is completely valid:

$$\begin{aligned} \text{Schrödinger Picture} &: \quad \hat{A} \text{ is static, } |\alpha, t'\rangle \longrightarrow |\alpha, t''\rangle \\ \text{Heisenberg Picture} &: \quad |\alpha, t'\rangle \text{ is static, } \hat{A} \longrightarrow U^\dagger \hat{A} U \end{aligned}$$

We now need to look at how eigenkets change in the Heisenberg picture. First, let's rename the time-independent operator $\hat{A}$ as $\hat{A}_S$, where S stands for Schrödinger. Similarly, the quantity $U^\dagger \hat{A} U$ is renamed $\hat{A}_H$ for the obvious reason. We then have

$$\hat{A}_H = U^\dagger \hat{A}_S U$$

Now let's right-multiply both sides of this by the quantity $U^\dagger|a_i\rangle$, where $|a_i\rangle$ is a base ket of the operator $\hat{A}_S$. We get

$$\begin{aligned} \hat{A}_H\, U^\dagger|a_i\rangle &= U^\dagger \hat{A}_S U U^\dagger|a_i\rangle \\ &= U^\dagger \hat{A}_S\,|a_i\rangle \\ &= a_i U^\dagger|a_i\rangle \end{aligned}$$

Evidently, the quantity $U^\dagger|a_i\rangle$ is the base ket in the Heisenberg picture, and the eigenvalue a_i is the same in both pictures. Consequently, even though state vectors are considered time-independent in this picture, the base kets are not. To summarize, we have, in the Heisenberg picture,

$$\begin{aligned} |a_i, t''\rangle &= U^\dagger(t'', t')|a_i, t'\rangle \quad \text{and} \\ \langle a_i, t''| &= \langle a_i, t'|U(t'', t') \end{aligned}$$

where $U(t'', t') = \exp[-iH((t'' - t')/\hbar]$. Thus, the situation is reversed from what you'd normally expect – we don't operate with $U(t'', t')$ on a base ket, we use $U^\dagger(t'', t')$ instead to get the time-translated version. That's all there is to it.

If you're confused, please try to think of it this way: the operator $U^\dagger(t'', t')$ replaces the t' it finds in an eigenket to its right with t'', while the operator $U(t'', t')$ does exactly the same thing to an eigenbra to its left. If a base ket or bra doesn't specify any t (for example, $|a_i\rangle$), then it's assumed that $t' = 0$. In that case, $U(t'', 0) = U(t'')$, which replaces $t = 0$ with $t = t''$, etc.

Now let's do something interesting with all this. For any state ket $|\alpha, t'\rangle$, we have as usual

$$|\,\alpha,\, t''\rangle = \exp[-i/\hbar\, H(t'' - t')]|\,\alpha,\, t'\rangle$$

The operator thus moves the state ket from its value at t' to the value it would have at time t'' (we assume $t'' > t'$). We now move over to the Heisenberg picture. Let's multiply the above ket by the position eigenbra $\langle x''|$ (note that $t' = 0$ is assumed here). We then get

$$\begin{aligned} \langle x''\,|\,\alpha,\, t''\rangle &= \langle x''|\exp[-i/\hbar\, H(t'' - t')]\,|\,\alpha,\, t'\rangle \\ &= \langle x''|\exp[-i/\hbar\, H\, t'']\exp[i/\hbar\, H\, t']\,|\,\alpha,\, t'\rangle \\ &= \langle x''|U(t'')U^\dagger(t')|\,\alpha,\, t'\rangle \\ &= \langle x'', t''|U^\dagger(t')\,|\,\alpha,\, t'\rangle \end{aligned}$$

Notice how $U(t'')$ found a bra to its left with $t = 0$ and stuck t'' into it. We now insert the closure relation

$$\int dx'\,|x'\rangle\langle x'| = 1$$

immediately to the right of $U^\dagger(t')$ and write this as

$$\langle x''|\,\alpha,\, t''\rangle = \int dx'\,\langle x'',\, t''|\,U^\dagger(t')|\,x'\rangle\langle x'|\,\alpha,\, t'\rangle$$

Again, the Heisenberg picture changes this to

$$\langle x''|\,\alpha,\, t''\rangle = \int dx'\,\langle x'',\, t''|\,x',\, t'\rangle\langle x'|\,\alpha,\ t'\rangle$$

Now, you might recall that the wave function is just $\Psi_\alpha(x,\, t) = \langle x|\alpha, t\rangle$, so this can be expressed simply as

$$\Psi_\alpha(x'', t'') = \int dx'\,\langle x'', t''|\,x', t'\rangle\,\Psi_\alpha(x', t') \tag{1.3}$$

How do we interpret this? Whatever the quantity $\langle x'', t''|\,x', t'\rangle$ is, it acts kind of like the double Dirac delta function $\delta^3(x'' - x')\,\delta(t'' - t')$. In fact, for $t'' = t'$ it is *precisely* the delta function $\delta^3(x'' - x')$. However, when $t'' \neq t'$ we give it the name *propagator*. The propagator $\langle x'', t''|x', t'\rangle$ is actually a Green's function that determines how the wave function develops in time and space (it's often written as $K(x'', t''; x', t')$, which to me detracts from the Dirac notation we've been using). It can (and should) be viewed as the probability amplitude for a particle located originally at x', t' to be found at x'', t'' (remember that we always read probability amplitudes from right to left, like Hebrew or Arabic). In anticipation of where this is all going, I'll mention here that the propagator can easily be extended to the case of a quantum field propagating from one state to another. For example, in the path integral approach to quantum field theory,

the propagator $\langle 0,\infty|0,-\infty\rangle$ expresses the amplitude of the vacuum state "$|0\rangle$" to transition from minus infinity back to itself in the distant future via an infinite number of non-vacuum states involving particle creation and annihilation.

Note also that the propagator can be broken into multiple steps; that is,

$$\langle x_n, t_n|\, x_0, t_0\rangle \sim \langle x_n, t_n|\, x, t\rangle\, \langle x, t|\, x_0, t_0\rangle$$

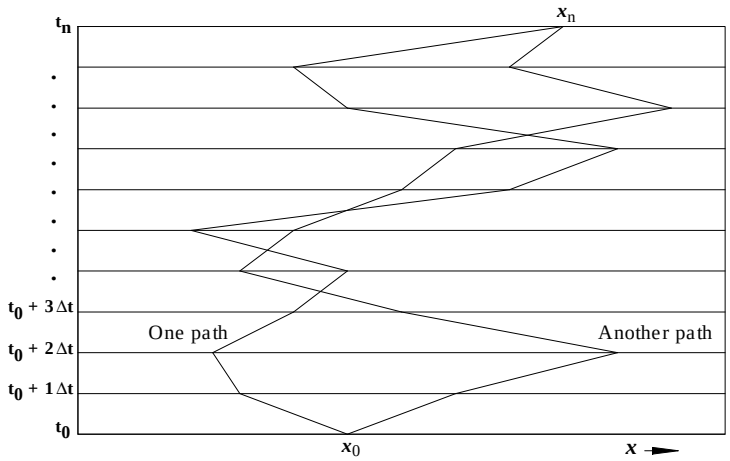
where the spacetime point x, t is intermediate between x_0, t_0 and x_n, t_n. Obviously, we can promote this to the equality

$$\langle x_n, t_n|x_0, t_0\rangle = \int dx\, \langle x_n, t_n|\, x, t\rangle\, \langle x, t|\, x_0, t_0\rangle$$

where we have used a closure relation to link the two propagators. We can do this again and yet again:

$$\langle x_n, t_n|\, x_0, t_0\rangle = \iiint dx_1\, dx_2\, dx_3\, \langle x_n, t_n|\, x_3, t\rangle\langle x_3, t|\, x_2, t\rangle\langle x_2, t|\, x_1, t\rangle\, \langle x_1, t|\, x_0, t_0\rangle$$

In fact, we can do this n times, breaking the total time into short pieces of duration $\Delta t = (t_n - t_0)/n$ (see figure). As n approaches infinity, Δt will go to zero, but the domain of integration is $\pm\infty$ for *any* path, so in effect a particle's "velocity," $(x_n - x_0)/(t_n - t_0)$, can be infinite because $x_n - x_0$ is finite. Consequently, particle velocity is essentially meaningless in the path integral formalism.



In spite of this, there doesn't appear to be much physics in the propagator at first glance. It has a certain superficial charm in the fact that its square gives you the probability that a particle will go from here to there (possibly via an infinite number of intermediate points), but that's about it. After all, we are far more interested in particles that interact with fields or other particles, *anything* but just the probability that something goes from point A to point B. But the simplicity of the propagator is illusory. In many ways, it's the key to everything that quantum mechanics represents, and it can even be looked upon as an allegory of life itself. Take the quantity $\langle 0,\infty|0,-\infty\rangle$, for example: in the beginning you arise out of nothingness, you have a life of some kind as you go from one point to another, and then you're dust again. Almost biblical, when you think about it. But is it any more interesting than this?

On my bookshelf is a copy of *The Principles of Quantum Mechanics*, the first edition of which was written in 1930 by Paul Adrien Maurice Dirac, one of the founders of quantum mechanics (that's putting it too mildly – he is arguably the greatest theoretical physicist who ever lived). Anyway, on page 128 of this book Dirac says (in slightly modernized notation)

" ... we then have $\langle x_n, t_n|x_0, t_0\rangle$ as the analogue of $e^{i/\hbar\, S}$."

Here, Dirac's S is the familiar action quantity $\int L\,dt$, where $L(x,\dot{x},t)$ is the Lagrangian of classical mechanics. Now, how does this come about? The Lagrangian has everything built into it – kinetic and potential energy, including interaction terms – so if Dirac's remark is true, then the propagator is truly a wonderful discovery. Could it be that a principle of least action holds in quantum mechanics (as it does in classical mechanics), such that by minimizing S the quantity $\langle x_n, t_n|x_{0,}t_0\rangle$ will describe the true path of a particle? As a young graduate student at Princeton University, Richard Feynman is said to have been fascinated by this rather brusque, throw-away remark by Dirac. What did Dirac mean by "analogue," and how does the Lagrangian enter into it, anyway? (Dirac made a similar remark in a seminal paper he published in 1933 [apparently, this is the one Feynman was intrigued by]; see the references). To make a long story short, Feynman took the idea and made it the basis of his 1942 PhD dissertation (*The Principle of Least Action in Quantum Mechanics*), and in doing so discovered a completely new approach to quantum theory – the *path integral*. You can do it, too – just recognize that $\langle x_n, t_n|x_{0,}t_0\rangle = \langle x_n|U(t_n - t_0)|x\rangle$, and remember that U contains the Hamiltonian operator H, which provides all the physics. Getting the $e^{i/\hbar\, S}$ term out of this is just one step in what all physicists consider to be one of the most profound discoveries of quantum physics – the path integral approach to quantum field theory.

2. Derivation of the Path Integral

Let us write the Heisenberg-picture propagator as $\langle x_n\, t_n|x_0\, t_0\rangle$, which represents the transition amplitude for a particle to move from the point x_0 at time t_0 to some other point x_n at another time t_n(again, assume $t_n > t_0$). In this picture, we'll use the shorthand notation

$$\langle x_n\, t_n|x_0\, t_0\rangle \equiv \langle n|0\rangle = \langle x_n\, t|\exp[-i/\hbar H(t_n - t_0)]\,|x_0\, t\rangle$$

Remember that the two t's on the right hand side are completely arbitrary, because the time translation operator is going to replace them with t_n and t_0. In most texts, they're not even shown.

Let's now split the time up into n equal pieces by setting $\Delta t = (t_n - t_0)/n$. We can then write the time translation operator as the n-term product

$$\exp[-i/\hbar\hat{H}(t_n - t_{n-1})]\,\exp[-i/\hbar H(t_{n-1} - t_{n-2})]\;\ldots\,\exp[-i/\hbar H(t_2 - t_1)]\,\exp[-i/\hbar H(t_1 - t_0)]$$

where $t_{j+1} - t_j = \Delta t$, $j = 0, 1, 2 ...n - 1$. We now insert a position eigenket closure relation just before the last exponential term:

$$\langle n|0\rangle = \int dx_1\,\langle x_n t|\underbrace{\exp[-i/\hbar H\Delta t]}_{n-1\,\text{times}}|x_1\, t\rangle\langle x_1\, t|\,\exp[-i/\hbar H(t_1 - t_0)]\,|x_0 t\rangle$$

(where the time t is arbitrary and the limits of integration are from $-\infty$ to ∞). The last term is

$$\langle x_1 t|\exp[-i/\hbar H(t_1 - t_0)]\,|x_0 t\rangle = \langle x_1 t_1|x_0 t_0\rangle$$

The integral over dx_1 means summation over every possible spacetime point along the $t = 1$ time step, so in effect we are integrating over every possible path between t_0 and t_1.

Let's repeat this procedure by inserting another closure relation $\int dx_2|x_2 t\rangle\langle x_2 t|$ just to the left of the next exponential operator. Using

$$\langle x_2 t|\exp[-i/\hbar H(t_2 - t_1)]\,|x_1 t\rangle = \langle x_2 t_2|x_1 t_1\rangle$$

we now have

$$\langle n|0\rangle = \iint dx_1 dx_2\,\langle x_n t|\underbrace{\exp[-i/\hbar H\Delta t]}_{n-2\,\text{times}}|x_2 t\rangle\langle x_2 t_2|x_1 t_1\rangle\langle x_1 t_1|x_0 t_0\rangle$$

The integrals over x_1 and x_2 mean that every path between t_0 and t_2 has been accounted for. Continuing this process of closure insertion a total of $n - 1$ times, we have

$$\langle n|0\rangle = \int dx_1 dx_2 \ldots dx_{n-1}\,\langle x_n|x_{n-1}t_{n-1}\rangle\langle x_{n-2}t_{n-2}|x_{n-3}t_{n-3}\rangle\ldots\langle x_1 t_1|x_0 t_0\rangle \tag{2.1}$$

where the single integral sign is now shorthand for an $n-1$-dimensional integral (please note that we are not integrating over the initial and final points). This simply shows that the propagator $\langle x_n t_n | x_0 t_0 \rangle$ can be expressed as the product of smaller propagator terms, as I indicated earlier. No big surprise here, and you may think that it is all a big waste of time and effort to write this as an $n-1$-dimensional integral, which at first glance appears impossible to evaluate anyway. What we have then in the above integral is an expression for taking $n-1$ paths to get from the starting point to the end, where each path is itself integrated over every time step Δt.

Of course, you can see what's going to happen – we're going to let n go to infinity, which means that we will consider *all* possible paths in the complete propagator, which includes *every possible point in spacetime* from the starting point 0 to the end point n. We will do this not to make life more difficult, but to see what happens when we retain the exponential terms $\exp[-i/\hbar H \Delta t]$; naturally, the time step will now go like $\Delta t \longrightarrow dt$. I regard the fact that we can carry this out in a (mostly) mathematically unambiguous manner as nothing short of a miracle.

As indicated earlier, the physics in all this lies in the Hamiltonian H, which describes just about any problem, from the free particle to the hydrogen atom and beyond. The exponential terms containing H in the last expression disappeared, but only so that the propagator could be expressed as a product term of many smaller propagators. Now we will see what Feynman discovered back in his days as a young graduate student at Princeton.

Let us pick a typical term in the above integral:

$$\langle x_{j+1} t_{j+1} | x_j t_j \rangle = \langle x_{j+1} t | \exp[-i/\hbar H (t_{j+1} - t_j)] \, | x_j t \rangle$$

We're going to express H in its most familiar form, $\hat{H} = \hat{p}^2/2m + \hat{V}(x)$, where the little hats as usual mean that the quantities are operators. This is not relativistic, of course, but it will serve our purposes for the time being. Let's deal with the momentum operator term first. Since $|x_j \, t\rangle$ is a position eigenket, we need something for the momentum operator $\hat{p}$ to "hit," so we insert a momentum closure relation (again, all limits are $\pm$ infinity) and rewrite this as the integral

$$\langle x_{j+1} t_{j+1} | x_j t_j \rangle = \int dp \;\; \langle x_{j+1} t | \exp[-i/\hbar H (t_{j+1} - t_j)] \, | p \rangle \langle p | x_j t \rangle$$

But this is just

$$\int dp \;\; \langle x_{j+1} t | \exp[-i/\hbar H (t_{j+1} - t_j)] \, | p \rangle \langle p | x_j t \rangle = \int dp \;\; \langle x_{j+1} t | p \rangle \langle p | x_j t \rangle \exp[-i/\hbar p^2/2m \, (t_{j+1} - t_j)]$$

where the p term in the exponential is no longer an operator. From any basic quantum mechanics text, we learn that

$$\langle x_{j+1} | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp[i/\hbar \, p x_{j+1}]$$

and

$$\langle p | x_j \rangle \;\; = \frac{1}{\sqrt{2\pi\hbar}} \exp[-i/\hbar \, p x_j]$$

both which hold for any t. We then have

$$\int dp \;\; \langle x_{j+1} t | p \rangle \langle p | x_j t \rangle \exp[-i/\hbar p^2/2m \, (t_{j+1} - t_j)] = \frac{1}{2\pi\hbar} \int dp \;\; \exp\left\{ i/\hbar \left[p(x_{j+1} - x_j) - i/\hbar p^2/2m \, (t_{j+1} - t_j) \right] \right\}$$

This is a Gaussian integral, and it can be solved exactly once we complete the square in the exponential term (you should be happy to know that the high school algebra exercise of "completing the square" actually has a practical use). To save time, I'll just write down the answer:

$$\langle x_{j+1} t | \exp[-i/\hbar \hat{H} (t_{j+1} - t_j)] \, | x_j t \rangle = \left[\frac{m}{2\pi i \hbar \Delta t} \right]^{1/2} \exp\left[i/\hbar \, \frac{1}{2} m \left(\frac{\Delta x}{\Delta t} \right)^2 \Delta t \right]$$

where $\Delta x = x_{j+1} - x_j$ and $\Delta t = t_{j+1} - t_j$. For the potential energy operator $\widehat{V}(x)$ in the Hamiltonian, it is a relief to get the simpler result

$$\widehat{V}(x)|x_j t\rangle = V(\widehat{x})|x_j t\rangle = V(x_j)|x_j t\rangle$$

Putting all this together, we have

$$\langle x_{j+1} t| \exp[-i/\hbar \hat{H}(t_{j+1} - t_j)]\, |x_j t\rangle = \left[\frac{m}{2\pi i\hbar \Delta t}\right]^{1/2} \exp\left\{ i/\hbar \left[\frac{1}{2} m \left(\frac{\Delta x}{\Delta t} \right)^2 - V(x_j) \right] \Delta t \right\}$$

This rather messy expression results from just one infinitesimal element in the total path integral! But the bulk of the work is done. There are a total of n such elements (remember, we broke the time translation operator into n pieces), while $n-1$ closure relations have been utilized. The total path integral is therefore

$$\langle x_n t_n | x_0 t_0 \rangle = \left[\frac{m}{2\pi i\hbar \Delta t}\right]^{n/2} \int \mathcal{D}x \, \exp\left\{ i/\hbar \sum_{j=0}^{n-1} \left[\frac{1}{2} m \left(\frac{x_{j+1} - x_j}{t_{j+1} - t_j} \right)^2 - V(x_j) \right] \Delta t \right\}$$

where $\int \mathcal{D}x$ is shorthand for $\int dx_1\, dx_2 .... dx_{n-1}$. As n goes to infinity, the summation in the integral becomes an integral over dt, giving

$$\begin{aligned} \langle x_n t_n | x_0 t_0 \rangle &= \lim_{n\to\infty} \left[\frac{m}{2\pi i\hbar \Delta t}\right]^{n/2} \int \mathcal{D}x \, \exp\left\{ i/\hbar \int_{t_0}^{t_n} \left[\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - V(x) \right] dt \right\} \\ &= \lim_{n\to\infty} \left[\frac{m}{2\pi i\hbar \Delta t}\right]^{n/2} \int \mathcal{D}x \, \exp\left\{ i/\hbar \int_{t_0}^{t_n} L\, dt \right\} \end{aligned} \tag{2.2}$$

This at last is the Feynman path integral. The exponential term in the first expression should look familiar – it's the Lagrangian L of classical dynamics, and the integral itself is called the action S:

$$S = \int L\, dt$$

Thus, we have found the source of Dirac's mysterious $\exp[i/\hbar\, S]$ term!

Note that, as $n \longrightarrow \infty$, the coefficient $[m/2\pi i\hbar \Delta t]^{n/2}$ blows up. However, transition amplitudes are always normalized, so we won't worry too much about this (the coefficient is usually sucked up into the definition of $\int \mathcal{D}x$).

3. Classical Limit of the Path Integral

The quantity $\langle x_n t_n | x_0 t_0 \rangle$ represents an infinite set of paths that a particle can take from one point to another over a finite time. However, in classical mechanics there is only one path that the particle can take, the so-called classical path. While it is also defined by a Lagrangian, the difference between one path and an infinite number of paths is obviously very confusing. How can this be explained?

In the classical scheme, Planck's constant $\hbar$ is, for all practical purposes, zero; however, when we set $\hbar = 0$ the quantity S in the path integral oscillates violently and becomes meaningless. There is only one way out of this – if the Lagrangian S is also set to zero, in the limit the indefinite quantity $S/\hbar = 0/0$ might somehow leave something finite behind. Unfortunately, while S may have some minimum value, setting it exactly to zero is usually not valid. Nevertheless, minimizing S is precisely the route one takes to get the classical path, and this is the only option available to us. One way of looking at this is that even in the classical world, $\hbar$ is not exactly zero, either, so the ratio remains finite. This singles out one unique path, which is the classical path.

Remarkably, at the quantum level there is no unique path – all possible paths contribute to the transition amplitude. And even more amazingly, each path is just as important as any other. It's only when $\hbar$ is

comparatively small that the paths begin to interfere destructively, leaving a large propagation amplitude only in the vicinity of the classical path. When you studied the electron double-slit experiment, your professor no doubt informed you that each electron in reality passes through both slits on the screen on its way to the detector and, in doing so, interferes with itself, which is why the detector shows an interference pattern. For a triple slit, the electron has three possible routes, and there is a corresponding interference pattern. We can in fact make an infinite number of slits in an infinite number of sequential screens (leaving empty space!), and the electron will then be describable by Feynman's path integral formalism. Fantastic as this may seem, the formalism appears to be a correct description of reality.

4. The Free Particle Propagator

The real power of the path integral lies in the fact that it includes the interaction term $V(x)$. We can therefore (in principle) compute the amplitude of a particle that interacts with external fields and other particles as it propagates from one place to another, *an infinite number of times*, if necessary (which is normally the case). Another advantage lies in the fact that the Lagrangian can accommodate any number of particles, because we can always write

$$L = \sum_k \left[\frac{1}{2} m_k \left(\frac{dx_k}{dt} \right)^2 - V(x_k) \right] \tag{4.1}$$

Consequently, the path integral can be applied to many-particle systems, making it a good candidate for a quantum field theoretic approach (in my opinion, it's unequivocally the best and most natural approach). You may recall that quantum mechanics normally deals with only one or several particles, and gets progressively more difficult as the number of particles becomes large. In quantum field theory, large numbers of particles are par for the course.

What assurance do we have that the path integral in (2.2) represents reality? Well, we might try to actually compute one, hopeless though this appears at first glance. After all, a single integral may not be a problem, but computing an infinite-dimensional integral might become tedious after a while. It turns out that, for the case $V(x) = 0$, the path integral can be obtained in closed form. This leads us to the *free particle propagator*.

First let's derive this quantity using ordinary quantum mechanics. It is simplicity itself. We write

$$\begin{aligned} \langle x_n t_n | x_0 t_0 \rangle &= \langle x_n t | \exp\left[-i/\hbar\, \hat{p}^2/2m\,(t_n - t_0) \right] \, | x_0 t \rangle \\ &= \int dp\, \langle x_n t | \exp\left[-i/\hbar\, \hat{p}^2/2m\,(t_n - t_0) \right] \, |p\rangle \langle p | x_0 t \rangle \\ &= \frac{1}{2\pi\hbar} \int dp \;\; \exp\left[i/\hbar\, p(x_n - x_0) - i/\hbar p^2/2m\,(t) \right] \end{aligned}$$

This is the same Gaussian integral we evaluated earlier. The integration over p is elementary, and the free particle propagator turns out to be

$$\langle x_n t_n | x_0 t_0 \rangle = \sqrt{\frac{m}{2\pi i\hbar (t_n - t_0)}} \, \exp\left[\frac{im}{2\hbar (t_n - t_0)} (x_n - x_0)^2 \right] \tag{4.2}$$

Well, that was easy enough. Can we reproduce this using the path integral? We can indeed, although the algebra is a bit more involved. Let's rewrite (2.2) as

$$\langle x_n t_n | x_0 t_0 \rangle = \left[\frac{m}{2\pi i\hbar \Delta t} \right]^{n/2} \int \mathcal{D}x \, \exp\left[-\sum_{j=0}^{n-1} a\,(x_{j+1} - x_j)^2 \right]$$

where

$$a = -\frac{im}{2\hbar \Delta t}$$

Why did I write this so that a would have a negative sign? It's because we want a decreasing exponential to evaluate the Gaussian integrals that we will introduce next (the fact that a is pure imaginary completely

voids this argument, but what the hell). Now let's focus on the first integral we'll have to evaluate, which is

$$I_1 = \int \exp\left[-a(x_2 - x_1)^2 - a(x_1 - x_0)^2\right]\, dx_1$$

(we're starting from the far right-hand side of the integral string). This is a Gaussian integral, as promised, although the integration variable x_1 is coupled with x_0 and x_2. Holding the latter two variables constant, the integration is straightforward, and we get

$$I_1 = \sqrt{\frac{\pi}{2a}} \exp\left[-\frac{1}{2}a(x_2 - x_0)^2\right]$$

Now we need to do the next integral, which looks like

$$I_2 = \int \exp\left[-a(x_3 - x_2)^2 - \frac{1}{2}a(x_2 - x_0)^2\right]\, dx_2$$

Again, this is just another Gaussian integral, though slightly different than the one we evaluated for I_1. Holding x_0 and x_3 constant, this time around we get

$$I_2 = \sqrt{\frac{2\pi}{3a}} \exp\left[-\frac{1}{3}a(x_3 - x_0)^2\right]$$

I can keep doing this, but hopefully you've already spotted the pattern, which is

$$I_k = \sqrt{\frac{k\pi}{(k+1)a}} \exp\left[-\frac{1}{k+1}a(x_{k+1} - x_0)^2\right]$$

As a result, we get a chain of leading square root terms in the complete integration, which goes like

$$\sqrt{\frac{1\pi}{2a}}\sqrt{\frac{2\pi}{3a}}\sqrt{\frac{3\pi}{4a}}\cdots \longrightarrow \frac{1}{\sqrt{n}}\left[\frac{\pi}{a}\right]^{(n-1)/2}$$

(The $n-1$ term results from the fact that we're doing a total of $n-1$ integrals.) Putting everything together (including the $(m/2\pi i\hbar\Delta t)^{n/2}$ term), we have finally

$$\begin{aligned}\langle x_n t_n | x_0 t_0\rangle &= \left[\frac{m}{2\pi i\hbar\Delta t}\right]^{n/2}\frac{1}{\sqrt{n}}\left[\frac{\pi}{a}\right]^{(n-1)/2}\exp\left[-\frac{1}{n}a\left(x_n - x_0\right)^2\right] \\ &= \left[\frac{m}{2\pi i\hbar n\Delta t}\right]^{1/2}\exp\left[\frac{im}{2\hbar n\Delta t}(x_n - x_0)^2\right]\end{aligned}$$

where we have inserted the definition for a into the second expression. Lastly, we set $n\Delta t = t_n - t_0$, leaving us with

$$\langle x_n t_n | x_0 t_0\rangle = \sqrt{\frac{m}{2\pi i\hbar(t_n - t_0)}}\exp\left[\frac{im}{2\hbar(t_n - t_0)}(x_n - x_0)^2\right]$$

which is the free particle propagator again! Although the effort was enormous, the path integral has duplicated a result from quantum mechanics (let this simple example be a lesson to you – computing path integrals is no fun). This happy outcome indicates that the path integral approach seems to work, even though it is an entirely different way of looking at things. Furthermore, we now have greater confidence that the path integral approach will be valid even when the potential term $V(x)$ is retained. Indeed, for simple systems like the harmonic oscillator, the path integral approach exactly replicates the results of quantum mechanics (I won't do that one here, as this discussion is already long enough). We are now ready to make the big leap from quantum mechanics to quantum field theory using Feyman's path integral.

5. The Path Integral Approach to Quantum Field Theory

The transition from quantum mechanics to quantum field theory is straightforward, but the underlying concept is a little difficult to grasp (at least it was for me). Basically, three issues must be dealt with.

One, we demand that the theory be dynamically relativistic (in other words, $E = p^2/2m + V$ must be given the axe). Two, space and time variables must share equal billing; this is just another relativistic demand. In quantum mechanics, time is just a parameter whereas position is an operator (that's why we see things like $|x\rangle$, whereas the object $|t\rangle$ is nonsensical). And third, quantum mechanics is primarily a one-particle theory, while a quantum field theory we must somehow accommodate many particles (to account for particle creation and destruction). The path integral fulfills all of these requirements admirably.

Now here's the big leap in a nutshell: quantum field theory replaces the position coordinate x with a field $\phi(x)$, where the quantity x is now shorthand for x, y, z, t. That is, dimensional coordinates are downgraded from operators to parameters, just like t, so everything's on the same footing (in relativity, space and time are conjoined into *spacetime*). This process of coordinate reassignment is known as *second quantization*. To reiterate (this is very important), we must have a quantity whose functions are x, y, z and t, something like the wave function $\Psi(\overrightarrow{x}, t)$. In quantum field theory, we assume the existence of a *quantum field* $\phi(x)$ which may also include specifications concerning particle spin, particle number, angular momentum, etc. In what is known as *canonical field theory*, the field itself is an operator (path integrals thankfully avoid this complication). If all of this makes sense to you (and even if it doesn't), then it shouldn't surprise you that we can write the path integral in quantum field theory as

$$Z = \int_{-\infty}^{\infty} \mathcal{D}\phi \, \exp\left[\frac{i}{\hbar}\int_{-\infty}^{\infty} L(\phi, x^\mu, \partial_\mu\phi)\, d^4x\right] \tag{5.1}$$

where we assume that any and all coefficients (nasty or otherwise) are now lumped into the $\mathcal{D}\phi$ notation, which goes like $\phi(x_1)\phi(x_2)\ldots$. Why it's called Z is just convention. You might want to think of this quantity as the transition amplitude for a field to propagate from the vacuum at $t = -\infty$ to the vacuum again at $t = \infty$, but I'm not sure that this prescription really describes it. A field can be just about anything, but you can look at it in this situation as a quantity that might describe a population of particles, energy fields and/or force carriers at every point in spacetime. Also, it is no longer appropriate to call (5.1) a path integral, since it does not describe the situation in terms of paths in spacetime anymore. It is now called the *Z functional integral*.

You might now be wondering what the boundaries of the field are in terms of its possible values. Well, we can single out one very special field – the so-called vacuum state – in which the energy density of spacetime in the vicinity of the system being considered is a minimum (usually zero), so that $Z \sim |0\rangle$. By this we mean a state such that the modulus of the quantity Z cannot possibly assume any smaller value. By convention, we consider a vacuum state which arises at $t = -\infty$, then propagates along as something other than a vacuum state before returning to a vacuum field at $t = \infty$. In propagator language, we say that $Z = \langle 0,\, \infty | 0,\, -\infty\rangle$. In between these times, the field interacts with particles and other fields (and even creates them) in a manner prescribed by the Lagrangian. Thus, the field is born at $t = -\infty$, enjoys a "life" of some sort, and then dies at $t = \infty$ (that's why both integrals in (5.1) go from minus to plus infinity). Very simplistic, perhaps, but it seems to work alright in practice. By the way, this business of selecting the vacuum state as a starting point is fundamental to what is to follow. Because the path integral with interaction terms cannot be evaluated directly, a perturbative approach must be used. Selection of the vacuum or "ground" state ensures that the perturbation method will not "undershoot" the vacuum and give sub-vacuum results, which are meaningless. If the true vacuum state is not assigned from the beginning, then the system may jump to states of even lower energy. In QFT this would be a disaster, because (as we will see shortly) the method of solving for Z uses successive approximations (perturbation theory), and if we have a false vacuum, this method fails utterly. In fact, the Higgs process (which you may have read about) absolutely depends on fixing the true vacuum under a gauge transformation of the bosonic Lagrangian.

The form of the Lagrangian for a field depends on what kind of particles and force carriers are going to be involved. Consequently, there are Lagrangians for scalar (spin zero) particles (also called *bosons*), spinors (spin 1/2 particles, also called *fermions*), and vector (spin one) particles. There's even a messy one for spin-2 gravitons. The simplest of these is the scalar or bosonic Lagrangian, and it is the one we will use here. The scalar Lagrangian for relativistic fields is given by

$$L = \frac{1}{2}\left[\,\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right] - V$$

(for a derivation, see any intermediate quantum mechanics text).

Just like the ordinary propagator in quantum mechanics, we're going to experience problems evaluating Z when the potential term V is not a linear or quadratic function of its argument. As God would have it, the simplest interaction term for a scalar particle in quantum field theory turns out to be $V \sim \lambda\phi^4$, where λ is called a *coupling constant*. This gives rise to what is called a *self-interacting* field theory; that is, the field interacts with itself and with any particles that are created along the way. As a result, the integral for Z cannot be obtained in closed form, and we will have to resort to perturbation theory, as previously indicated. This leads to a very interesting interpretation of particle creation and propagation as a consequence of this model – at every order in the perturbative expansion (including zero order), particles appear and begin to propagate about the spacetime stage. Since in principle there is an infinite number of spacetime points where interaction can occur, the number of particles involved can also be infinite. However, the total number of all interactions is fixed by the number of λ that enter the perturbative expansion of Z.

So the problem comes down to solving the integral

$$Z(\lambda) = \int \mathcal{D}\phi \, \exp\left\{ \frac{i}{\hbar} \int \left[\frac{1}{2} \left(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2 \right) - \lambda \phi^4 \right] d^4x \right\} \tag{5.3}$$

Alas, I will tell you right now that this integral cannot be attacked in this form. The main problem is the infinite-dimensional integral; it is simply too unwieldy. We will have to make some changes before a perturbative solution can be employed.

6. Modifying the Z Functional Integral

Consider the free-space ($\lambda = 0$) form of Z with a "source" term $J(x)$:

$$Z(J) = \int \mathcal{D}\phi \, \exp\left\{ i \int \left[\frac{1}{2} \left(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2 \right) + J(x)\phi \right] d^4x \right\} \tag{6.1}$$

which we will set to zero later (note that from here on, in keeping with the fashion standard in physics, I'm setting $\hbar = c = 1$ so I won't have to carry them around everywhere). Although the introduction of $J(x)$ into the integral is a standard mathematical artifice, there is some physical justification for it, but I won't bore you with the details. Integrating by parts over the $\partial_\mu \phi \, \partial^\mu \phi$ term, we get

$$\int \frac{1}{2} \left[\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2 \right] d^4x = - \int \left[\frac{1}{2} \phi \, \partial^2 \phi + \frac{1}{2} m^2 \phi^2 \right] d^4x$$

Now assume that the field ϕ can be written as

$$\phi(x) = \phi_0(x) + \varphi(x)$$

where ϕ_0 is the so-called "classical" solution to the heterogeneous equation

$$- \left[\partial^2 + m^2 \right] \phi_0 = J \tag{6.2}$$

and φ is the corresponding "$\infty - 1$" field. The classical solution is unique (it corresponds to the classical "path"), but it's just one of the infinite identities the field can assume. Please don't worry about that leading minus sign; you can ignore it if you want, as it's really not critical (I'm just following convention). Now, the solution to (6.2) can be obtained using the usual method of Green's functions:

$$- \left[\partial^2 + m^2 \right] G(x - x') = \delta^4(x - x') \tag{6.3}$$

where $G(x - x')$ is the four-dimensional Green's function associated with the operator $- \left[\partial^2 + m^2 \right]$. Then

$$\phi_0(x) = \int G(x - x') \, J(x') \, d^4x'$$

is the desired solution (after all, this is what Green's functions do for a living). To solve (6.3) for $G(x - x')$, we assume that it can be expressed as a Fourier transform

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4} \, G(k) \, e^{ik_\mu (x^\mu - x^{\mu'})} \tag{6.4}$$

where k is the momentum four-vector $(E/c, \overrightarrow{p})$. By hitting (6.4) with the operator $-\left[\partial^2 + m^2\right]$, you should be able to show that (6.3), along with the definition for the Dirac delta function

$$\delta^4(x - x') = \int \frac{d^4k}{(2\pi)^4}\; e^{ik_\mu(x^\mu - x^{\mu\prime})}$$

leads to

$$G(k) = \frac{1}{k^2 - m^2}, \qquad \text{so that} \tag{6.5}$$

$$G(x - x') = \int \frac{d^4k}{(2\pi)^4}\, \frac{e^{ik_\mu(x^\mu - x^{\mu\prime})}}{k^2 - m^2} \qquad \text{and} \tag{6.6}$$

$$\phi_0(x) = \int G(x - x')\, J(x')\, d^4x' \tag{6.7}$$

In scalar QFT, it is conventional to rename the Green's function in (6.6) as the Feynman propagator $\Delta_F(x - x')$. At the same time, the momentum-space propagator (6.5), which is not an integral quantity, will be of use later when we define the so-called Feynman rules for a scattering process. [The above definition for $G(x-x')$ normally includes a "fudge factor" in the denominator (that is, $k^2 - m^2 + i\epsilon$) to help with convergence, but I will not use it just yet.] Anyway, we now have

$$\phi_0(x) = \int \Delta_F(x - x')\, J(x')\, d^4x', \quad \text{where}$$

$$\Delta_F(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2 - m^2}$$

where kx means $k_\mu x^\mu$. That done, we can then write (6.1) as

$$Z(J) = \int \mathcal{D}\varphi \exp\left[\frac{i}{2}\int \left[\partial_\mu\varphi\, \partial^\mu\varphi - m^2\varphi^2\right]\, d^4x - \iint J(x')\Delta_F(x'' - x')J(x'')\, d^4x'\, d^4x''\right]$$

Okay, now here's the trick: $J(x)$ appears under the integral, but it is an explicit function of the spacetime coordinates x, *and not a function of* φ, so the J integral term *can be taken out of the infinite-dimensional integral altogether*:

$$Z(J) = \exp\left[-\frac{i}{2}\iint J(x')\Delta_F\left(x' - x''\right) J(x'')\, d^4x'\, d^4x''\right] \int \mathcal{D}\varphi\, \exp\left[\frac{i}{2}\int \left(\partial_\mu\varphi\, \partial^\mu\varphi - m^2\varphi^2\right)\right] d^4x$$

So just what is the residual integral over $\mathcal{D}\varphi$? Who knows, and who cares; *it's just some number*, and you can call it N if you want (like most textbooks), but I will set $N = 1$ because we'll be using normalized amplitudes later on. We then have, finally,

$$Z(J) = \exp\left[-\frac{i}{2}\int J(x)\Delta_F\left(x - x'\right) J(x')\, dx\, dx'\right] \tag{6.8}$$

where I'm now using one integral sign and dx for brevity. Believe it or not, this is an enormous achievement, for we have successfully rid ourselves of that infinite-dimensional integral and replaced it with two four-dimensional integrals. From here on out, everything we do will involve taking successive derivatives of Z with respect to the $J(x)$. This is the main reason why Z was "simplified" in this way – it provides a parameter, $J(x)$, with which the solution of $Z(\lambda)$ can now be straightforwardly developed.

7. Power Series Representation of the Z Functional Integral; Green's Functions

Because we will have to resort to approximation to solve Z when the interaction term is included, it will be helpful to see how this quantity can be expressed as a power series expansion (more importantly, it serves as a means of introducing a form of Green's function that is critical to the approximation scheme). Recall

that the series expansion of any two-variable function $F(x,y)$ about zero can be written as

$$\begin{aligned} F(x,y) &= F(0,0) + x\,\frac{\partial F(x,y)}{\partial x}|_{x,y=0} + y\,\frac{\partial F(x,y)}{\partial y}|_{x,y=0} + \frac{1}{2!}xy\,\frac{\partial^2 F(x,y)}{\partial x\,\partial y}|_{x,y=0} + \ldots \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{n!}x^m y^{n-m}\frac{\partial^n F(x,y)}{\partial x^m \partial y^{n-m}}|_{x,y=0} \end{aligned}$$

The extension of this formula to n variables is straightforward (but you'll need n summation symbols!). By definition, a *functional* is a function of one or more functions. For a functional, the variables x and y become functions which we can expand out to a string of n quantities [say, $s(x_1), s(x_2), \ldots, s(x_n)$] and the summations become integrals over $dx_1 dx_2 \ldots dx_n$, so the analogous expression for a functional looks like

$$F[s(x_1), s(x_2)\ldots s(x_n)] = \sum_{n=0}^{\infty}\int \frac{1}{n!}\,dx_1 dx_2 \ldots dx_n\; s(x_1)\,s(x_2)\ldots s(x_n)\,R_n(x_1, x_2 \ldots x_n)$$

where

$$R_n(x_1, x_2 \ldots x_n) = \left[\frac{\delta}{\delta s(x_1)}\,\frac{\delta}{\delta s(x_2)}\,\cdots\frac{\delta}{\delta s(x_n)}\right] F[s]|_{s=0}$$

The operator δ is what is known as the *functional derivative* operator. I'll discuss this operator a little later on, but for now all you need to know is that it more or less does to functionals (which are almost always integrals containing one or more functions of the integrating argument) what the ordinary partial derivative operator does to functions, except that:

$$\begin{aligned} \frac{\partial x_i}{\partial x_j} &= \delta_{ij} \;\text{ (the Kronecker delta)} \\ \frac{\delta F(x)}{\delta F(y)} &= \delta^4(x-y) \;\text{ (the Dirac delta)} \end{aligned}$$

Believe it or not, the quantity $R_n(x_1, x_2 \ldots x_n)$ is a kind of Green's function, but in QFT it is called the *n-point function*. We will see shortly that the n-point function is nonzero only for even n.

In view of this, the functional $Z(J)$ can be written as

$$Z(J) = \sum_{n=0}^{\infty}\int \frac{i^n}{n!}\,dx_1 dx_2 \ldots dx_n\; J(x_1)J(x_2)\ldots J(x_n)\,G(x_1, x_2 \ldots x_n)$$

where

$$G(x_1,x_2 \ldots x_n) = \frac{1}{i^n}\left[\frac{\delta}{\delta J(x_1)}\,\frac{\delta}{\delta J(x_2)}\,\cdots\frac{\delta}{\delta J(x_n)}\right] Z(J)|_{J=0}$$

The G functions pretty much define the mathematical problem at hand, so if we know them then we know Z. Therefore, knowing how to calculate them efficiently is very important. Physicists have learned (or at least they believe) that the quantities $G(x_1,x_2 \ldots x_n)$ are the amplitudes for particles going from place to place. For example, using (6.8) we can calculate the two-point function

$$\begin{aligned} G(x_1,x_2) &= \frac{1}{i^2}\left[\frac{\delta}{\delta J(x_1)}\,\frac{\delta}{\delta J(x_2)}\right] Z(J)|_{J=0} \\ &= i\Delta_F\,(x_1 - x_2) \end{aligned}$$

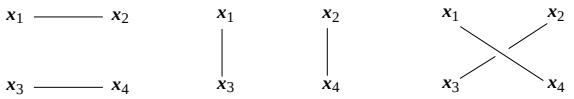
This is taken to represent the 2-point connected graph

$$x_1 \;\text{———————}\; x_2$$

Similarly, it is a simple matter to calculate $G(x_1,x_2,x_3,x_4)$, which turns out to be

$$\begin{aligned} G(x_1,x_2,x_3,x_4) &= -\Delta_F\,(x_1-x_2)\Delta_F\,(x_3-x_4) \\ &\quad -\Delta_F\,(x_1-x_3)\Delta_F\,(x_2-x_4) \\ &\quad -\Delta_F\,(x_1-x_4)\Delta_F\,(x_2-x_3) \end{aligned}$$

This represents the three 4-point graphs



(Note that there are only three ways to connect the four points. This is basically what all these graphs involve – permutations of diagrams, connected up in every possible way.) I strongly urge you to calculate $G(x_1,x_2,x_3,x_4)$ for yourself. Z is an exponential, so we can never run out of functional derivatives, no matter how many of them we take. You will see that although it is straightforward, it's somewhat tedious. You will learn that QFT can involve the calculation of $Z(J)$ to many orders of x_i, so we will have to find a way of determining them without actually doing the calculations. Fortunately, there is a simple formula for this that you will learn later on.

Try to think of the G quantities as associated with particles that get created at some point (say, x_i), propagate along a connecting line for a while [the line is represented by the quantity $\Delta_F(x_f - x_i)$], and then get annihilated at a terminating point (x_f). The Feynman propagator Δ_F is therefore the basic building block of the n-point functions. Because the mathematics is described basically by points sitting on opposites ends of lines represented by Δ_F, it should come as no surprise that these little points and lines are themselves the building blocks of what are known as *Feynman diagrams*. We will see that when the interaction term λ gets involved, the lines will get attached only at points at which λ occurs.

8. Interpretation of Z as a Generator of Particles

Feynman recognized that this business of multiple Z differentiations brings down terms that combine in ways that are describable by simple graphs. Each term has a coefficient associated with it that comes from the topology of the associated diagram, and each term has a multiplicity that depends on the number of ways that the diagram can be drawn. These amplitudes and multiplicities can be expressed mathematically using combinatorial algebra. We'll get to that later. But first let's have a look at those diagrams.

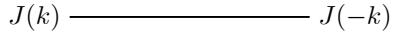
I'm going to show you how physicists interpret Z and how they associate diagrams with its expansion. Let's start with the free-space version,

$$Z(J) = \exp\left[-\frac{i}{2}\iint dx'dx''J(x')\Delta_F(x'-x'')\,J(x'')\right] \tag{8.1}$$

Now replace each term in the integral with its Fourier counterpart (I'm retaining all the integral signs here for accounting purposes):

$$\begin{aligned} Z(J) &= \exp\left[-\frac{i}{2}\frac{1}{(2\pi)^{12}}\iint dx'dx''\iiint dp'dp''dk\,J(p')\,J(p'')\,\frac{e^{ix'p'}e^{ix''p''}e^{ik(x'-x'')}}{k^2-m^2}\right] \\ &= \exp\left[-\frac{i}{2}\frac{1}{(2\pi)^4}\iiint dp'dp''dk\,J(p')\,J(p'')\,\frac{1}{k^2-m^2}\,\delta^4(p'+k)\,\delta^4(p''-k)\right] \\ &= \exp\left[-\frac{i}{2}\frac{1}{(2\pi)^4}\int dk\,J(k)\,\frac{1}{k^2-m^2}\,J(-k)\right] \end{aligned} \tag{8.2}$$

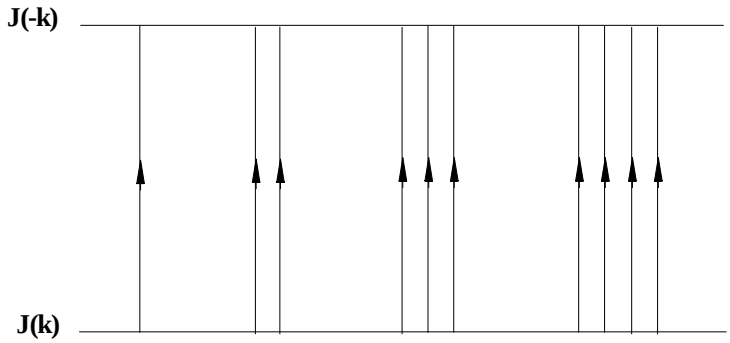
The form of this quantity motivates the following interpretation: *A particle with momentum k and mass m is created at x' by the source $J(k)$, propagates freely to another point x'', and then is destroyed by the negative source $J(-k)$.* Pictorially, this process can be described by the graph



We can also expand the exponential in (8.1) directly to get

$$Z \sim 1 - \int J\Delta J + \frac{1}{2\cdot 2!}\left[\int J\Delta J\right]^2 - \frac{1}{2^2\cdot 3!}\left[\int J\Delta J\right]^3 + \ldots$$

(For brevity, I've taken some liberties with the notation, but you know what I mean.)



What all of this is supposed to mean is this: a particle of mass m is created by a source J located at some point and propagates to another source (actually, a "sink") at yet another point, where it is destroyed. It also means that two other particles are created and destroyed with relative amplitude $1/(2{\cdot}2!)$, and so on. To all orders of the exponential, an infinite number of particles can be created, but the amplitudes get progressively smaller, making large numbers of particles less and less likely. The interpretation of Z as a particle generator should be obvious; also, you should be able to see that the n-point functions $G(x_1, x_2, \cdots, x_n)$ are associated with each expansion term.

9. The Z Functional Integral with the Interaction Term

We have now only to figure out a way of getting the interaction term λ back into the problem and solving it. It is conventional to write $V = \lambda\varphi^4/4!$, where the factorial term is a convenience whose inclusion will become clear later on. Restoring this term into our old definition for Z, we have

$$\begin{aligned} Z &= \int \mathcal{D}\varphi \, \exp i \int \left[\frac{1}{2} \left(\partial_\mu \varphi \, \partial^\mu \varphi - m^2 \varphi^2 - \frac{\lambda}{4!} \varphi^4 \right) \right] d^4x \\ &= \int \mathcal{D}\varphi \, \exp i \int \left[\frac{1}{2} \left(\partial_\mu \varphi \, \partial^\mu \varphi - m^2 \varphi^2 \right) \right] \exp \left[-\frac{i\lambda}{4!} \varphi^4 \right] d^4x \end{aligned} \tag{9.1}$$

I said earlier that I would return to the concept of functional differentiation. Let's look at an example of how we're going to use this technique in the problem at hand. You should already know that

$$I(a) = \int e^{-ax^2} \, dx = \sqrt{\frac{\pi}{a}} = \sqrt{\pi} a^{-\frac{1}{2}} \tag{9.2}$$

A somewhat more complicated integral is

$$\int x^6 e^{-x^2} \, dx$$

How can this be evaluated? Simple – we just differentiate both sides of (9.2) three times with respect to the parameter a, and we find that

$$\begin{aligned} \frac{d^3 I(a)}{da^3} &= -\int x^6 e^{-ax^2} \, dx = -\frac{15}{8} \sqrt{\pi} a^{-\frac{7}{2}} \quad \text{so} \\ \int x^6 e^{-x^2} \, dx &= \frac{15}{8} \sqrt{\pi} \end{aligned} \tag{9.3}$$

Obviously, the presence of the parameter a came in pretty handy, even though we set it equal to one when we were finished. Now let's take a more complicated integral, one that cannot be evaluated in closed form:

$$I = \int e^{-x^2 - bx^4} \, dx \tag{9.4}$$

Expanding the exponential in bx^4gives

$$I = \int e^{-x^2} \left[1 - bx^4 + \frac{(bx^4)^2}{2!} - \ldots \right] dx \, ,$$

so one approach to evaluating this is to use integral identities like (9.3) for all powers of x^4. This will give a solution involving an infinite number of terms, but if b is small we can always truncate the series at some point and obtain a solution that is as accurate as we want. But there's another way of looking at this basic approach. Let us write (9.4) without the x^4 term but with another that is proportional to x:

$$I(c) = \int e^{-x^2+cx}\,dx \tag{9.5}$$

Let us now differentiate this four times with respect to the new parameter c, multiply by b, and then resolve the integral at $c = 0$:

$$b\,\frac{d^4 I}{dc^4} \;= \int bx^4\,e^{-x^2}\,dx$$

Subtracting this from $I(0)$, we get the quantity

$$\int e^{-x^2}\,\left[1 - bx^4\right]\,dx$$

The quantity in brackets represents the first two terms in the expansion of $\exp[-bx^4]$. You should be able to convince yourself that, by taking appropriate differentiations of the integral I with respect to c, we can build up all the terms we need to put $\exp[-bx^4]$ under the integral. In essence, what we are doing is constructing the exponential operator

$$\exp\left[-b\frac{d^4}{dc^4}\right]$$

which will now act on the "generating" integral (9.5). We then have

$$\int e^{-x^2-bx^4}\,dx = \exp\left[-b\frac{d^4}{dc^4}\right]\int e^{-x^2+cx}\,dx$$

(remember to take $c = 0$ at the end). I have to admit that I was really quite impressed the first time I saw this little trick, although it is a rather common mathematical device. We're going to use this same basic approach to introduce the exponential term involving λ into the $Z(J)$ generating function (6.8). We will therefore have

$$Z(\lambda) = \exp\left[-\frac{i\lambda}{4!}\int dx\frac{\delta^4}{i^4\delta J(x)^4}\right]\exp\left[-\frac{i}{2}\int dx'dx''J(x')\Delta_F(x'-x'')\,J(x'')\right] \tag{9.6}$$

(Since $i^4 = 1$, I'm going to just drop this term from here on.) This time, there's an integration that has to be performed following the quadruple derivative. The problem should now be clear. First, by expanding the operator exponential in (9.6), we'll have to deal with the multiple differential operators

$$\exp\left[-\frac{i\lambda}{4!}\int dx\frac{\delta^4}{\delta J(x)^4}\right] = 1 - \frac{i\lambda}{4!}\int dx\frac{\delta^4}{\delta J(x)^4} - \frac{\lambda^2}{2!(4!)^2}\int dx\frac{\delta^4}{\delta J(x)^4}\int dy\frac{\delta^4}{\delta J(y)^4}\,\ldots \tag{9.7}$$

Note the change from ordinary to functional derivative operators. Note also that for increasing orders of λ, each integral operator gets a different dummy integration variable. But that's just the beginning. To see the effect of the interaction on particle creation and annihilation, we'll have to take even more differentials as required by the n-point functions (did you forget about them?). Therefore, to solve the problem to just second order for the 4-point function, we have to perform a total of 12 differentiations on Z. The good news is that, since Z is an exponential, the operations are relatively easy. Even better, there are simple formulas you can use that will eliminate the need to do anything (well, hardly anything). But first, let's make sure you understand functional differentiation and how it will be used on Z.

When the integral (9.5) involves functions $c(x)$ and not scalar parameters, it is known as a functional integral, and the same mathematical approach outlined above is known as *functional integration*. In the Feynman path integral, the source term $J(x)$ is a function of the four coordinates x, so it is a function, not a parameter. When dealing with functional parameters, we cannot use plain old partial differentiation anymore. However,

this complication is easily fixed by formally defining the process of functional differentiation. Recall the definition for an ordinary partial derivative:

$$\frac{\partial F(x,y,z...)}{\partial x} = \lim_{\Delta x \longrightarrow 0} \frac{F(x,y,z...+\Delta x) - F(x,y,z...)}{\Delta x}$$

We define functional differentiation as

$$\frac{\delta F[J(x)]}{\delta J(y)} = \lim_{\epsilon \longrightarrow 0} \frac{F[J(x)+\epsilon\delta^4(x-y)] - F[J(x)]}{\epsilon}$$

However, in practice the distinction between the two definitions is hardly even noticeable, and you will find that functional differentiation and ordinary differentiation look and act pretty much the same.

When you functionally differentiate the double integral in Z with respect to J taken at some specific spacetime point x_1, you get

$$\begin{aligned} & \frac{\delta}{\delta J(x_1)} \left[-\frac{i}{2} \int J(x') \Delta_F(x'-x'')\, J(x'')\, dx'\, dx'' \right] \\ = & -\frac{i}{2} \int \delta^4(x_1-x') \Delta_F(x'-x'')\, J(x'')\, dx'\, dx'' - \frac{i}{2} \int J(x') \Delta_F(x'-x'')\, \delta^4(x_1-x'')\, dx'\, dx'' \\ = & -i \int J(x') \Delta_F(x_1-x')\, dx' \end{aligned}$$

Consequently,

$$\frac{\delta Z}{\delta J(x_1)} = \left[-i \int J(x') \Delta_F(x_1-x')\, dx' \right] Z \tag{9.8}$$

(Very important – note that this quantity vanishes for $J = 0$.) A second differentiation works on both the integral and on Z again, giving

$$\frac{\delta^2 Z}{\delta J(x_1)\delta J(x_2)} = [-i\Delta_F(x_1-x_2)]\, Z - \left[i \int J(x') \Delta_F(x_2-x')\, dx' \right] \frac{\delta Z}{\delta J(x_2)} \tag{9.9}$$

It is convenient to adopt a shorthand for these operations. I use $Z_1 = \delta Z/\delta J(x_1)$, $Z_{12} = \delta^2 Z/\delta J(x_1)\, \delta J(x_2)$, and so on, along with $\Delta_{12} = \Delta(x_1 - x_2)$. Using (9.8), we can eliminate the integral term in brackets and write (9.9) as

$$Z_{12} = -i\Delta_{12}\, Z + \frac{1}{Z}\, Z_1 Z_2$$

Remember that everything will eventually be evaluated at $J = 0$, so that the only terms that will survive are those proportional to Z (which goes to unity), while any Z term with a subscript goes to zero. As we will be taking multiple derivatives of Z (at least four to accommodate each order of λ), it is also very important to note that terms with odd numbers of derivatives (like Z_{12345}) will go to zero; only even-numbered terms survive (so that $Z_{12}|_{J=0} \longrightarrow -i\Delta_{12}$, etc.).

10. Problem Definition

Let us (finally) write down the Z functional integral with the interaction term in the form that we'll use:

$$\overline{Z}(\lambda) = \frac{\exp\left[-\frac{i\lambda}{4!} \int dx \frac{\delta^4}{\delta J(x)^4} \right] \exp\left[-\frac{i}{2} \int dx' dx'' J(x') \Delta_F(x'-x'')\, J(x'') \right]}{\exp\left[-\frac{i\lambda}{4!} \int dx \frac{\delta^4}{\delta J(x)^4} \right] \exp\left[-\frac{i}{2} \int dx' dx'' J(x') \Delta_F(x'-x'')\, J(x'') \right] |_{J=0}} \tag{10.1}$$

Notice that this is the same as (9.6), but here Z has been normalized using the denominator term (this is why I'm calling it $\overline{Z}$). Recalling (8.1), we can also write this as

$$\overline{Z}(\lambda) = \frac{\exp\left[-\frac{i\lambda}{4!} \int dx \frac{\delta^4}{\delta J(x)^4} \right] Z}{\exp\left[-\frac{i\lambda}{4!} \int dx \frac{\delta^4}{\delta J(x)^4} \right] Z\, |_{J=0}} \tag{10.2}$$

You can see from this that the very first thing we have to do is apply the integral operator, where x is the point of interaction. But that's just the start. To get the n-point functions $G(x_i)$, we'll have to perform additional differentiations on $\overline{Z}$ at the labelled "starting" and "ending" spacetime points $x_1, x_2, \ldots x_n$. For example, the first-order 4-point function will be $\overline{Z}_{xxxx1234}$, while the second-order 4-point function would go like $\overline{Z}_{xxxxyyyy1234}$, all evaluated at $J = 0$. Obviously, taking all these differentials is going to be a pain in the neck. What we need are those formulas I promised earlier for calculating these quantities.

11. Differentiation Formulas and Symmetry Factors

In a nutshell, to find $\overline{Z}$ we have to determine the n-point functions $G(x_1, x_2, \ldots x_n)$, and to find them we have to take n functional derivatives of Z, all evaluated at $J = 0$. Let's do a few and see what we get:

$$\begin{aligned}
Z_a|_{J=0} &= 0 \\
Z_{ab}|_{J=0} &= -i\Delta_F(a-b) \\
Z_{abc}|_{J=0} &= 0 \\
Z_{abcd}|_{J=0} &= -\Delta_F(a-b)\Delta_F(c-d) - \Delta_F(a-c)\Delta_F(b-d) \\
&\quad -\Delta_F(a-d)\Delta_F(b-c) \\
Z_{abcde}|_{J=0} &= 0 \\
Z_{abcdef}|_{J=0} &= i\Delta_F(a-b)\Delta_F(c-d)\Delta_F(e-f) \quad + 14 \text{ other terms}
\end{aligned}$$

Do a few more and you'll see the pattern for n differentiations: when n is odd, we get zero; when n is even we get a total of $(n-1)!!$ terms (for odd m we define $m!! = 1 \cdot 3 \cdot 5 \cdot 7 \ldots m$), where each term is a $1/2\,n$-multiple of the Feynman propagator Δ_F, along with a prefactor like i or -1 (you should to able to clearly see the heavy hand of permutation at work in these formulas). Thus, you can almost automatically write down the derivative of Z to any order. Each non-zero differential term can be viewed as a connected graph. For example, there are three ways to connect the points a, b, c, d (we already did this in the graph on page 14). That's why odd-numbered differentiations go to zero: every connecting line must have two and only two points. This is no big deal, but it gets more interesting when the differential arguments are the same (for example, when $a = b$).

To see this, let's calculate $Z_{abcd} = Z_{xxxx}$ the hard way – by just doing the differentiations. It's not too bad, and you should have no trouble getting

$$Z_{xxxx} = -3\Delta_{xx}^2\, Z - \frac{6}{Z}\, i\, \Delta_{xx}\, Z_x^2 + \frac{1}{Z^3} Z_x^4 \tag{11.1}$$

where Δ_{xx} is shorthand for $\Delta_F(x-x)$ [Most textbooks write this as $\Delta_F(0)$. Ryder expresses it as a circle, $\bigcirc$, to signify that a particle is created at x, propagates for a while, and then gets annihilated at the same x. It thus goes around in a little loop, and the analogy makes a lot of sense. But I'm going to leave it as Δ_{xx}, for a reason that will become apparent later.].

From (11.1) we have $Z_{xxxx}|_{J=0} = -3\Delta_{xx}^2$ (Ryder expresses this as $-3\bigcirc\bigcirc$, which also makes sense). The factor -3 represents a weighting factor, and in fact it is known as the *symmetry factor* for the term Z_{xxxx}. The symmetry factor reflects the number of ways that a graph can be drawn. In a sense, the problem of doing the differentiations and figuring out these symmetry factors is one and the same. Obviously, when many differentials are involved, the required calculations can become exceedingly laborious. Is there any way to get these quantities directly? The answer is yes, and it's really quite simple.

Let's look at the problem from a combinatoric point of view. Because non-zero results are obtained only for an even number of differentiations, it makes sense to consider the number of ways we can *pair* these operations. Let each pair be represented schematically by brackets, i.e., $Z_{xx} = [x\,x]$. Try think of this as putting the two identical "objects" x and x into a single grouping (I'd use a box to group these quantities, but my word processor is not quite up to the task). The combinatoric formula for this combination is just $x!/(2^1 1!)$, where $x = 2$. Therefore, $x!/2 = 1$ Direct calculation shows that $Z_{xx}|_{J=0} = -i\Delta_{xx}$. Setting $[x\,x] = \Delta_{xx}$ and ignoring, the $-i$ factor for the moment, we have agreement. Now let's go with two more operations. We now have $Z_{xxxx} = [x\,x][x\,x] = [x\,x]^2 = \Delta_{xx}^2$, and the number of ways this can be expressed is

$x!/(2^2 2!) = 3$, where $x = 4$. Again, ignoring the -1 factor, we have agreement with (11.1). A little inductive reasoning reveals that the prefactor is just $(-i)^{x/2}$, so the symmetry factor C for any term can be expressed as

$$C = (-i)^{x/2} \frac{x!}{2^{S_{xx}} S_{xx}!}$$

where S_{xx} represents the number of xx pairs that are involved. You should try this formula on a few examples to convince yourself that it works.

But what happens when we differentiate with respect to specific spacetime points? To see this, let us take Z_{xx12}, which schematically is $Z_{xx12}|_{J=0} = [x\,x][1\,2] = \Delta_{xx}\Delta_{12}$ (where $1 = x_1$, etc.). However, rearrangement of the terms allows this to also be written as $Z_{xx12} = [x\,1][x\,2] = \Delta_{x1}\Delta_{x2}$. The addition of the points allows us to "spread out" or "share" the x points, giving an additional set of pairs. Combinatorially, the number of ways each of these terms can be written is $x!/2^1 1! = 1$ and $x! = 2$, respectively. Direct calculation gives $Z_{xx12}|_{J=0} = -\Delta_{xx}\Delta_{12} - 2\Delta_{x1}\Delta_{x2}$, so we're on the right track. If the number of specific spacetime points is p, then the prefactor is $(-i)^{1/2(x+p)}$ and the combinatoric formula for the terms can be written as

$$C = (-i)^{1/2(x+p)} \frac{x!}{2^{S_{xx}} S_{xx}!} \tag{11.2}$$

If your combinatoric algebra is rusty, you'll just have to take my word for it that these formulas are correct.

It is easy to see that the number of xx pairs S_{xx} is given by $n_s = p/2 + 1$ up to a maximum of $x/2 + 1$. For $Z_{xxxx1234}$, we'll have $S_{xx} = 2, 1, 0$; using (11.2), I get

$$\begin{aligned} Z_{xxxx1234}|_{J=0} &= \frac{4!}{2^2 2!}[x\,x]^2[1\,2][3\,4] + \frac{4!}{2^1 1!}[x\,x][x\,1][x\,2][3\,4] + \frac{4!}{2^0 0!}[x\,1][x\,2][x\,3][x\,4] \\ &= 3\Delta_{xx}^2\Delta_{12}\Delta_{34} + 12\Delta_{xx}\Delta_{x1}\Delta_{x2}\Delta_{34} + 24\Delta_{x1}\Delta_{x2}\Delta_{x3}\Delta_{x4} \end{aligned}$$

However, actual calculation of this quantity gives

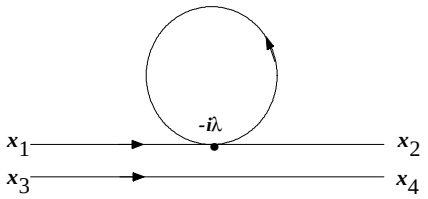
$$\begin{aligned} Z_{xxxx1234}|_{J=0} &= 3\Delta_{xx}^2\Delta_{12}\Delta_{34} + 12\Delta_{xx}\Delta_{x1}\Delta_{x2}\Delta_{34} + 24\Delta_{x1}\Delta_{x2}\Delta_{x3}\Delta_{x4} \\ &\quad +3\Delta_{xx}^2\Delta_{23}\Delta_{14} + +3\Delta_{xx}^2\Delta_{13}\Delta_{24} \\ &\quad +12\Delta_{xx}\Delta_{x2}\Delta_{x4}\Delta_{13} + 12\Delta_{xx}\Delta_{x2}\Delta_{x3}\Delta_{14} \\ &\quad +12\Delta_{xx}\Delta_{x3}\Delta_{x4}\Delta_{12} + 12\Delta_{xx}\Delta_{x1}\Delta_{x4}\Delta_{23} \\ &\quad +12\Delta_{xx}\Delta_{x1}\Delta_{x3}\Delta_{24} \end{aligned}$$

Hmm ... there is a 3-fold multiplicity in the Δ_{xx}^2 term and a 6-fold multiplicity in the Δ_{xx}. How did that happen? It's because the positions of the point labels can be *permuted* to give equivalent graphs (see figure below). For example, the spacetime points in the term $\Delta_{xx}\Delta_{x2}\Delta_{x4}\Delta_{13}$ can be rearranged to give $\Delta_{xx}\Delta_{x2}\Delta_{x3}\Delta_{14}$, etc., and this rearrangement can be performed a total of different six ways. Notice that the number p has no effect on the symmetry factor (with the possible exception of some power of i), and the most it can do is produce "copies" of equivalent terms. Combinatoric analysis shows that the *multiplicity* M of any term can be expressed by

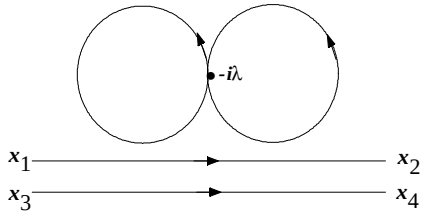
$$M = \frac{p!}{2^t t!(p-2t)!} \tag{11.3}$$

where

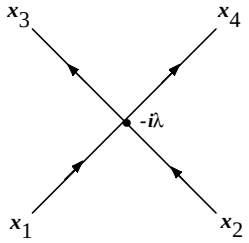
$$t = S_{xx} + \frac{1}{2}(p - x) \tag{11.4}$$



Symmetry Factor: 12
Multiplicity: 6



Symmetry Factor: 3
Multiplicity: 3



Symmetry Factor: 24
Multiplicity: 1

From (9.7) you can see that taking derivatives higher than 4 with respect to the interaction point requires a new dummy integration variable (for example, to second order in λ we'll have $Z_{xxxxyyyy}$). The above formulas for C and t then become

$$\begin{aligned} C &= (-i)^{1/2(x+y+p)} \frac{x!\,y!}{2^{S_{xx}} 2^{S_{yy}} S_{xx}! S_{yy}! S_{xy}!} \\ t &= \frac{1}{2}(p-x-y) + S_{xx} + S_{yy} + S_{xy} \end{aligned}$$

where S_{yy} and S_{xy} are the exponents in Δ_{yy} and Δ_{xy} terms, respectively (the definition for M is unchanged). The extension of these expressions to higher orders of λ should be obvious.

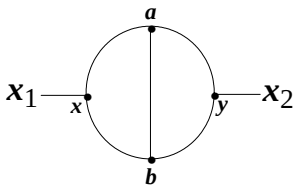
For n orders of interaction, the prefactor term will look like $(-i)^{1/2(p+4n)}$. However, since amplitudes are always squared, the prefactor will square to unity, so most authors don't even bother with it; consequently, I will dispense with it from here on.

It is important to note that the quantities x, y, etc. have 4 as their maximum value (that is, you must take four differentiations "per interaction"). Thus, for n-order problems the numerator in C will go like $(4!)^n$. This explains the reason why we wrote the interaction term as $V \sim \lambda/4!$; as the interaction exponential is expanded, the 4! terms will cancel one another to all orders of λ. In view of this, we can dispense with these numerator factorials altogether and write the symmetry factors and multiplicities using the simple but rather ugly combinatorial expressions

$$\begin{aligned} C &= \prod_{i,j=1(i\le j)}^{n} \frac{1}{2^{S_{ii}}\, S_{ij}!} \quad \text{and} \\ M &= \frac{p!}{2^t t!(p-2t)!} \qquad \text{where} \\ t &= \frac{1}{2}\,(p-4n) + \sum_{i,j=1(i\le j)}^{n} S_{ij} \end{aligned} \tag{11.5}$$

and where n is the order of interaction.

At the risk of being obsessively complete, I need to tell you that there is one more symmetry that can enter into the above definition of C. Earlier, I noted that the symmetry factor is just a number that reflects the number of ways that a diagram can be labelled (and by this I mean the interaction labels), so it is essentially a combinatoric quantity. The expressions in (11.5) are the result of algebraic combinatorics, but there's another, purely topological, symmetry that resists being put into any formula (that is to say, I haven't found any). To see this, consider the following graph:



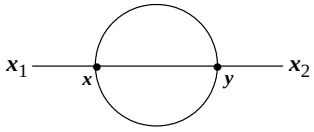
This corresponds to $n = 4$, $p = 2$, $S_{xx} = S_{yy} = S_{aa} = S_{bb} = 0$, and all $S_{ij} = 1$. According to (11.5), we should then have

$$C = \frac{1}{(2^0)^4(1!)^5} = 1$$

However, the correct value is $C = 1/2$. Notice that the interaction indices a, b can be interchanged without changing the signatures of any of the internal propagators. This interchangeability introduces an additional factor of two into the denominator of C. For really complicated diagrams, the topological symmetry factor can be very difficult to determine, and even seasoned quantum field theorists can get flummoxed. In practice, you should use (11.5) first, then look to see if the graph has this type of exchange symmetry.

Lastly, please don't confuse the symmetry operations that lie behind C and M with one another. The symmetry factor C always deals with permutations of the interaction labels, while M involves permutations resulting from the relabelling of the external points p.

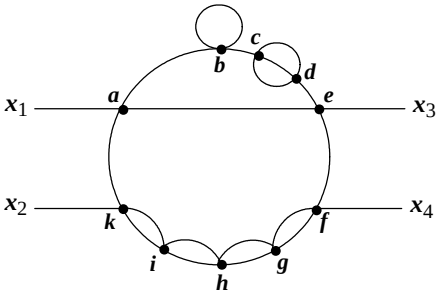
Let's do a few examples to practice what (I hope) you've learned. The graph



has $n = 2$, $p = 2$, $S_{xx} = S_{yy} = 0$, $S_{xy} = 3$, $t = 0$ and $M = 1$. There's no topological symmetry to worry about, so

$$C = \frac{1}{3!} = \frac{1}{6}$$

One more, this time a bit more complicated to make sure you've got the hang of it:



Here, $n = 10$, $p = 4$, $S_{bb} = 1, S_{cd} = 3, S_{ki} = 2, S_{ih} = 2, S_{hg} = 2, S_{gf} = 2$, with all other S terms equal to zero or 1; $t = 0$ and $M = 1$, and so

$$C = \frac{1}{2^1 1!3!2!2!2!2!} = \frac{1}{192}$$

You will note from these examples that the only S terms you need to deal with come from internal lines (that is, lines that are connected from one interaction to another). With a little practice, you can figure out C by just looking at the graphs.

As an exercise, let's now calculate $\overline{Z}$ for the first-order, 4-point problem. To start, we expand (10.2) to first order in λ:

$$\overline{Z}(\lambda) = \frac{\left[1 - \frac{i\lambda}{4!}\int dx \frac{\delta^4}{\delta J(x)^4}\right] Z}{\left[1 - \frac{i\lambda}{4!}\int dx \frac{\delta^4}{\delta J(x)^4}\right] Z\,|_{J=0}}$$

Using the notation we've developed, this is

$$\begin{aligned}
\overline{Z}(\lambda) &= \frac{Z - \frac{i\lambda}{4!}\int Z_{xxxx}\,dx}{\left[Z - \frac{i\lambda}{4!}\int Z_{xxxx}\,dx\right]\,|_{J=0}} \\
&= \frac{Z - \frac{i\lambda}{4!}\int Z_{xxxx}\,dx}{1 - \frac{i\lambda}{4!}\int(-3\Delta_{xx}^2)\,dx} \\
&= \frac{Z - \frac{i\lambda}{4!}\int Z_{xxxx}\,dx}{1 + \frac{i\lambda}{8}\int \Delta_{xx}^2\,dx}
\end{aligned}$$

The 4-point Green's function is then given by

$$\begin{aligned}
G(x_1, x_2, x_3, x_4) &= \overline{Z}_{1234} \\
&= \frac{1}{i^4}\frac{Z_{1234} - \frac{i\lambda}{24}\int Z_{xxxx1234}\,dx}{1 + \frac{i\lambda}{8}\int \Delta_{xx}^2\,dx}
\end{aligned} \tag{11.6}$$

(The numerator will be evaluated at $J = 0$ after the differentiations have been performed.) Using (11.5), we see that

$$Z_{1234}|_{J=0} = -\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} - \Delta_{14}\Delta_{23}$$

while

$$\begin{aligned}
\int Z_{xxxx1234}\,dx|_{J=0} &= 3\Delta_{xx}^2\Delta_{12}\Delta_{34} + 3\Delta_{xx}^2\Delta_{13}\Delta_{24} + 3\Delta_{xx}^2\Delta_{14}\Delta_{23} \\
&\quad +24\int \Delta_{x1}\Delta_{x2}\Delta_{x3}\Delta_{x4}\,dx \\
&\quad +12\Delta_{xx}\int \Delta_{x1}\Delta_{x2}\Delta_{34}\,dx + 12\Delta_{xx}\int \Delta_{x2}\Delta_{x4}\Delta_{13}\,dx \\
&\quad +12\Delta_{xx}\int \Delta_{x2}\Delta_{x3}\Delta_{14}\,dx + 12\Delta_{xx}\int \Delta_{x3}\Delta_{x4}\Delta_{12}\,dx \\
&\quad +12\Delta_{xx}\int \Delta_{x1}\Delta_{x4}\Delta_{23}\,dx + 12\Delta_{xx}\int \Delta_{x1}\Delta_{x3}\Delta_{24}\,dx
\end{aligned}$$

Now, the denominator in (11.6) can be binomially inverted to first order in λ, giving

$$\frac{1}{1 + \frac{i\lambda}{8}\int \Delta_{xx}^2\,dx} = 1 - \frac{i\lambda}{8}\int \Delta_{xx}^2\,dx$$

Multiplying this into the numerator in (11.6), we see that all of the double vacuum terms Δ_{xx}^2 cancel each other, and we're left with

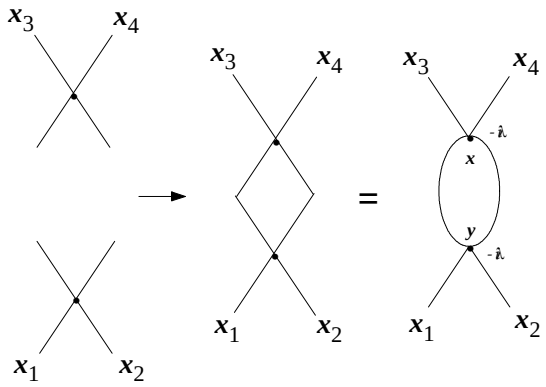
$$\begin{aligned}
G(x_1, x_2, x_3, x_4) &= -\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} - \Delta_{14}\Delta_{23} \\
&\quad -i\lambda\int \Delta_{x1}\Delta_{x2}\Delta_{x3}\Delta_{x4}\,dx \\
&\quad -\frac{1}{2}i\lambda\left[\Delta_{xx}\int \Delta_{x1}\Delta_{x2}\Delta_{34}\,dx + \text{ the 5 other terms}\right]
\end{aligned} \tag{11.7}$$

The elimination of the pure vacuum term ◯◯ is a neat characteristic of normalization, and it can be shown to persist to all orders of the perturbation process. The first three terms in (11.7) do not participate in the interaction and so can be ignored. Schematically, G represents the processes shown in the figure on page 19.

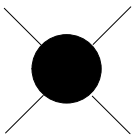
Only the $\times$ term is fully "connected" in the sense that all propagators in the process are connected to one another. The point of connection is of course the interaction point x. Because of this, it is the only term in the entire process that we're really interested in; the other terms are all disconnected to some extent, as they include propagators involving particles that go from one point to another without interacting with anything. Notice that the coefficient of $\times$ is $-i\lambda$. The vertex is the point x, but remember that this actually involves integration over all of spacetime, $\int \Delta_{x1}\Delta_{x2}\Delta_{x3}\Delta_{x4}\,dx$.

Please note that we've focussed on a 4-point function for a good reason. It clearly represents a process in which two scalar particles are created at the lower end of $\times$ (points x_1 and x_2), propagate along until they interact with each other at some point x, then move away from each other until they are annihilated at the upper end of $\times$ at the points x_3 and x_4. If we look at the vertical and horizontal directions of these diagrams as representing time and space, respectively, then this is an ideal way of representing a scattering process for two scalar particles (such as mesons).

If we now go to a second order process, we'll have the generalized diagram $\times\times$, which now involves two interactions with an overall coefficient of $(-i\lambda)^2$. But we still have just two particles going in and coming out, so the diagram must be collapsed to account for this. One way is to draw it as



which represents one of the possible 4-point, second-order diagrams (there are seven). Zee likens the action of each interaction operator $\int dx\,\delta^4/\delta J(x)^4$ as that of a machine that grabs the four free ends and ties them together – a very fitting description. Notice that if you mentally "block out" the central portions of the first- and second-order $\times$ graphs, they look the same – two particles go in, and two particles come out. In fact, no matter how many orders of λ are included, the graphs will always retain this characteristic. In a way, the exact details of the interaction are hidden from view. The situation is often denoted pictorially by putting a "blob" where all the action takes place:



Notice that in the second order graph there are two kinds of lines: external lines, like $\Delta_F(x_1 - x)$, and internal ones, like $\Delta_F(x - y)$. For all 4-point graphs, the propagators associated with external lines would occur for any scattering process, so it is conventional to ignore them.

12. Feynman Rules

To calculate the amplitude $\overline{Z}$ for an actual process, Feynman developed a set of rules that now carry his name. At the risk of being overly terse, I will just summarize them here because they should make sense to you based on what you should have learned by now. Recall (6.5), which is the Feynman propagator in

transform space. This is actually much more convenient than using the space form Δ_{xy}, and we can use (6.5) to associate every line in a Feynman diagram with a four-momentum k. The Feynman rules for ϕ^4 QFT are then:

1. Draw all possible diagrams corresponding to the desired number of interactions and spacetime points, using time as the vertical axis and space as the horizontal axis. For each graph, label each internal and external line with a momentum k_n and give it an arrow indicating direction (the direction can be completely arbitrary). For each *internal* line, write the integral/propagator combination

$$\int \frac{dk_n}{(2\pi)^4} \frac{i}{k_n^2 - m^2}$$

2. For each interaction vertex, write down a factor $-i\lambda$.

3. For each interaction vertex, write a Dirac delta function that expresses the conservation of momentum about that vertex:

$$(2\pi)^4 \, \delta^4 \left[\sum_i k_i \right]$$

   where k_i is positive if the line is entering the vertex and negative if it is exiting the vertex.

4. For each graph, there will be a residual delta function of the form $(2\pi)^4 \delta^4(k_1 + k_2 + \ldots)$ that expresses overall conservation of momentum in the diagram. Cancel this term.

5. If there are any integrals remaining, you'll have to do them. However, the delta functions you encountered in Step 3 simplify things enormously, and you may not even have to do any integrals.

6. Calculate the symmetry factor for each graph using (11.5) and multiply this by the result obtained in Step 5.

7. Determine the total amplitude by taking the products of all the graphs. By convention, the total amplitude is called $\mathcal{M}$.

That's it. If any of the integrals diverge (as often happens for certain loop diagrams), then you'll have to consult a more advanced resource than this one – you've encountered the *divergence problem* (see the next section).

Note that for any physical process there can be a huge number of possible diagrams depending on how many interaction orders you're willing to consider. As the interaction order n grows, the number and complexity of possible diagrams increases rapidly. However, the interaction term λ is generally a small quantity; in quantum electrodynamics it is numerically equal to about 1/137), so the smallness of the $(-i\lambda)^n$ term for large n effectively reduces the probability that a complicated process will actually occur. This is why the perturbation approach works – you need only consider the most likely processes to get an accurate result. Even so, processes involving more than just a couple of orders in the interaction term can be a real pain to calculate (Feynman used to joke that this is why we have graduate students). In the path-integral approach to the strong force (gluons), the interaction term λ is relatively large, requiring the calculation of many terms to get decent convergence. The observed magnetic moment of the proton, for example, is approximately 2.79275, but gluonic QFT gives us at best a figure of 2.7, with a rather big margin of error. It seems nothing comes easily in QFT!

Maybe some day a really bright young physicist will come along and discover a way to do the Z functional integral in closed form (if she does happen along, though, I hope she will turn her attention first to some of humankind's more pressing problems, like the need for an environmentally friendly sustainable energy source).

Examples

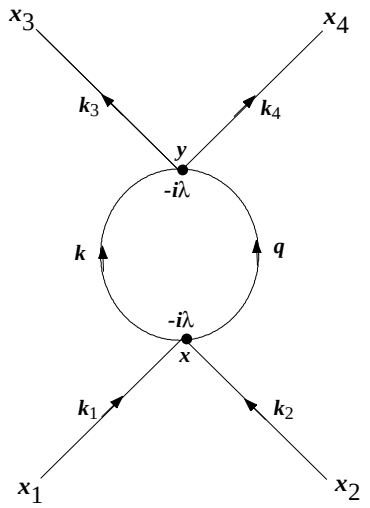
To see how these rules work, let's try two examples (for brevity, I'll do this for specific graphs and not entire processes). For the $\times$ diagram, all the lines are external, so we don't have to do very much. From (11.5), we have $C = 1$ and $M = 1$, and the amplitude is just $\overline{Z}_{xxxx1234} = -i\lambda$.

Now let's calculate the amplitude for $\overline{Z}_{xxxxyyyy1234}$. We label it up as indicated below. We have two interaction terms, which contribute an overall factor $(-i\lambda)^2$ to the amplitude. We have two internal lines, so we write

$$\overline{Z} = (-i\lambda)^2 \int\int \frac{d^4k}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{i}{k^2-m^2}\frac{i}{q^2-m^2}(2\pi)^4\delta^4(k_1+k_2-k-q)\,(2\pi)^4\delta^4(k+q-k_3-k_4)$$

This simplifies to

$$\overline{Z} = \lambda^2 \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2-m^2)}\,\frac{1}{(k_1+k_2-k)^2-m^2}\,(2\pi)^4\delta^4(k_1+k_2-k_3-k_4)$$



where, as promised, there is a residual delta function expressing overall conservation of momentum. We cancel this term and move on to the symmetry factor. For this diagram, $\overline{Z}_{xxxxyyyy1234} = [x\,x_1][x\,x_2][y\,x_3][y\,x_4][x\,y]^2$; therefore, $S_{xx} = S_{yy} = 0$, $S_{xy} = 2$. Using (11.5), we have

$$C = \frac{1}{2!} = \frac{1}{2}$$

This leaves

$$\overline{Z} = \frac{1}{2}\lambda^2 \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2-m^2)}\,\frac{1}{(k_1+k_2-k)^2-m^2}$$

Does this integral converge? Well, for large momenta k the integral will go like

$$\int \frac{d^4k}{k^4} = \int \frac{k^3dk}{k^4} \sim \log k \longrightarrow \infty$$

Well, damn – the integral diverges. This is the famous "ultraviolet divergence" that bedeviled physicists for 20 years. Like I said, you'll have to consult another resource if you want to find the amplitude for this particular second order process. (Zee gives the answer in Chapter 3 of his book. If you're anything like me, you won't be particularly happy with the solution approach, which is called *renormalization*).

13. Interpretation of Feynman's Propagator for a Scalar Particle

In the Feynman path integral, we have seen that the Feynman propagator $\Delta_F(x-x')$ plays a central role in the perturbative expansion of the functional integral $Z(J)$ in the presence of the interaction term $\lambda\phi^4$. Here we'll take a closer look at the propagator and provide an interpretation of its physical significance. We have

$$\Delta_F(x^\mu - x'^\mu) = \int \frac{d^4k}{(2\pi)^4}\,\frac{e^{ik_\mu(x^\mu-x'^\mu)}}{k^2-m^2}$$

where $d^4k = dk_0\, dk_x\, dk_y\, dk_z$ and $k^2 = g^{\mu\nu}k_\mu k_\nu = k_0^2 - k_x^2 - k_y^2 - k_z^2 = k_0^2 - \overrightarrow{k}^{\,2}$(remember that we're using units in which $c = \hbar = 1$, while k_x is the momentum in the x-direction, etc.). Thus, the propagator expresses the probability amplitude that a particle of mass m will move from the spacetime point x'^μ to some other point x^μ. Now let's expand this in terms of the time variable k_0:

$$\Delta_F(x - x') = \int \frac{d^3k\ \ e^{i\,\overrightarrow{k}\cdot\overrightarrow{x}}}{(2\pi)^4} \int \frac{dk_0\ \ e^{ik_0x^0}}{k_0^2 - \overrightarrow{k}^{\,2} - m^2}$$

If we now try to integrate this improper Fourier integral over dk_0, we're going to run into trouble because the pole occurs at $k_0^2 = \overrightarrow{k}^{\,2} + m^2$, which inconveniently lies along the real axis of the complex plane. In complex analysis, this is a disaster, because we cannot use the theory of residues to resolve the integral. To get around this, we resort to the usual artifice of introducing a small imaginary term $i\epsilon$ into the denominator:

$$\Delta_F(x - x') = \int \frac{d^3k\ \ e^{i\,\overrightarrow{k}\cdot\overrightarrow{x}}}{(2\pi)^4} \int \frac{dk_0\ \ e^{ik_0x^0}}{k_0^2 - \omega^2 + i\epsilon} \tag{13.1}$$

where $\omega^2 = \overrightarrow{k}^{\,2} + m^2$ (I've always hated this trick, because in reality there's no way to avoid the real axis, but everybody does it, and it seems to work, so what the hell). We can then write

$$\Delta_F(x - x') = \int \frac{d^3k\ \ e^{-i\,\overrightarrow{k}\cdot\overrightarrow{x}}}{(2\pi)^4} \int \frac{dk_0\ \ e^{ik_0x^0}}{(k_0 - \omega + i\epsilon)(k_0 + \omega - i\epsilon)} \tag{13.2}$$

I know what you're thinking – the expanded denominator in (13.2) doesn't match the one in (13.1). But remember that the number $i\epsilon$ is generic: if we multiply it by any positive real number, it's still of order $i\epsilon$, so nothing has changed (sometimes it's written as $i\delta$ to show that $i\epsilon$ has been multiplied by something). Anyway, we now have two complex poles, and evaluation of the integral is a snap. When $x^0 = ct$ is positive, we must use an integration contour in the upper complex plane; this surrounds only one pole ($k_0 = -\omega + i\epsilon$), and the residue is therefore $2\pi i\, \exp[-i\omega x^0]/(-2\omega)$ (I've let $\epsilon = 0$, so we can now pretend that we did nothing amiss when we stuck that $i\epsilon$ term in). For $x^0 < 0$ we have to close in the lower plane, where the pole is $k_0 = \omega - i\epsilon$; this gives us the residue $-2\pi i\, \exp[+i\omega x^0]/(2\omega)$. Putting this all together, we can now write the propagator as

$$\Delta_F(x - x') = \frac{-i}{(2\pi)^3} \int \frac{d^3k}{2\omega}\, [e^{-i(\omega x^0 + \overrightarrow{k}\cdot\overrightarrow{x})}\theta(x^0) + e^{i(\omega x^0 - \overrightarrow{k}\cdot\overrightarrow{x})}\theta(-x^0)]$$

where $\theta(\pm x^0)$ is the unit step function for forward or backward time, which is either zero or one depending on the sign of its argument. To make the terms in $\overrightarrow{k}\cdot\overrightarrow{x}$ symmetric, notice that we can replace k with $-k$ in the integral without changing anything :

$$d^3k = k^2 dk\, \sin\theta_k\, d\theta_k\, d\phi_k = d^3(-k)$$

So we then have, equivalently,

$$\Delta_F(x - x') = \frac{-i}{(2\pi)^3} \int \frac{d^3k}{2\omega}\, [e^{-i(\omega x^0 - \overrightarrow{k}\cdot\overrightarrow{x})}\theta(x^0) + e^{i(\omega x^0 - \overrightarrow{k}\cdot\overrightarrow{x})}\theta(-x^0)] \tag{13.3}$$

Notice that ω has units of energy; that is, if we put back the speed of light term c in $\omega^2 = \overrightarrow{k}^{\,2} + m^2$, we recover the familiar relativistic energy relation $E^2 = c^2\overrightarrow{p}^{\,2} + m^2c^4$, where $E = \hbar\omega$. You may recall from elementary quantum mechanics that a wave corresponding to a particle having positive energy propagates like $\psi \sim \exp[-iEt/\hbar]$. We can see this quantity in (13.3), where it is constrained to move in the forward time direction by $\theta(x^0)$. This is a good thing – we interpret this as positive-energy particles moving forward in time in the Feynman propagator. However, there is that other term $\exp[iEt/\hbar]\,\theta(-x^0)$ in (13.3) which, by the same logic, must be interpreted as *negative-energy* particles propagating *backward* in time. How do we make sense of this? Simple – if E is negative and t is running backward, the conjugate pair Et can also be viewed as positive E and positive t. We therefore have a prescription for positive-energy *antiparticles* moving forward in time, since both views are mathematically equivalent. The ability of the propagator to

accommodate scalar (bosonic) antiparticles into the same theory is just one of the fantastic successes that Feynman's calculus has achieved. This same prescription holds when we go from a scalar theory to a spin 1/2 fermionic theory, which can also be developed using the same basic path integral approach.

Incidentally, sooner or later you're going to have to evaluate 4-dimensional integrals "all the way," not just through dx^0. When you do, you may initially have trouble (like I did) dealing with the notation "d^4k." Since many integrands will have terms like $e^{i\,\vec{k}\cdot\vec{x}} = e^{ikx\cos\theta}$, it makes sense to consider doing the integrals in spherical coordinates, where $d^3k = k^2 dk\,\sin\theta\,d\theta\,d\phi$ (just think of the momentum 3-vector $\vec{k}$ as behaving like the radial vector $\vec{r}$). In four dimensions, we have $k^2 = k_0^2 - k_x^2 - k_y^2 - k_z^2 = k_0^2 - \vec{k}^{\,2}$, so you'll have to mind the occasional confusion between k^2 (which is 4-dimensional), and $\vec{k}^{\,2}$, which is also usually denoted as k^2 as well.

There's yet another way of looking at d^4k. Consider a 90° rotation of the time coordinate in the complex plane so that $t \longrightarrow it$. Then $k^2 = -k_0^2 - k_x^2 - k_y^2 - k_z^2$, and we say that space is "euclidean." In this space, any direction is mathematically the same as any other, so that $d^4k = k^3 dk$ (that is, it depends only on the magnitude of the four-vector k). Why k^3 and not k^2? In regular polar coordinates, one has $dA = r\,dr\,d\theta$ and $dV = r^2 dr\,\sin\theta\,d\theta\,d\phi$, so the exponent on r is always one less than number of dimensions. In tensor language, we have $dA = \sqrt{g}\,dx^1 dx^2$ and $dV = \sqrt{g}\,dx^1 dx^2 dx^3$, where $\sqrt{g}$ is the metric determinant (exactly how the tensor prescription produces $d^4k = k^3 dk$ in four dimensions escapes me for the moment).

As you've already seen, you're going to encounter Feynman diagrams (loop diagrams, in particular) that give divergent results. This complication (sometimes called the "problem of the infinities") plagued quantum field theory until the rather dubious (in my opinion) mathematical technique known as *renormalization* arrived on the scene to bail out the theory. I'm certainly not going to discuss renormalization in any detail in this simple write-up, but I can give you a taste of what the approach involves.

Consider the 2-point function to first order in λ, which is

$$\begin{aligned} G(x_1,x_2) &= \overline{Z}_{12} = \frac{1}{i^2}\frac{Z_{12} - \frac{i\lambda}{24}\int Z_{xxxx12}\,dx}{1 + \frac{i\lambda}{8}\int \Delta_{xx}^2\,dx} \\ &= i\Delta_{12} - \frac{1}{2}\lambda\Delta_{xx}\int \Delta_{x1}\Delta_{x2}\,dx \end{aligned}$$

Expanding the integral term, we have

$$\begin{aligned} (\ ) &= -\frac{1}{2}\lambda\Delta_{xx}\int \frac{dp\,dq\,dx}{(2\pi)^8}\frac{e^{ipx}e^{-ipx_1}e^{iqx}e^{-iqx_2}}{(p^2-m^2)(q^2-m^2)} \\ &= -\frac{1}{2}\lambda\Delta_{xx}\int \frac{dp\,dq\,\delta^4(p+q)}{(2\pi)^4}\frac{e^{-ipx_1}e^{-iqx_2}}{(p^2-m^2)(q^2-m^2)} \\ &= -\frac{1}{2}\lambda\Delta_{xx}\int \frac{dp}{(2\pi)^4}\frac{e^{ip(x_1-x_2)}}{(p^2-m^2)^2} \end{aligned}$$

Now, since

$$i\Delta_{12} = i\int \frac{dp}{(2\pi)^4}\frac{e^{ip(x_1-x_2)}}{(p^2-m^2)}$$

we have

$$G(x_1,x_2) = i\int \frac{dp\,e^{ip(x_1-x_2)}}{(2\pi)^4(p^2-m^2)}\left[1 + \frac{i\lambda\Delta_{xx}}{2(p^2-m^2)}\right]$$

Binomial expansion allows us to write

$$1 + \frac{i\lambda\Delta_{xx}}{2(p^2-m^2)} = \left[1 - \frac{i\lambda\Delta_{xx}}{2(p^2-m^2)}\right]^{-1}$$

so that

$$\begin{aligned} G(x_1,x_2) &= i\int \frac{dp\,e^{ip(x_1-x_2)}}{(2\pi)^4(p^2-\widetilde{m}^2)} \\ &= i\widetilde{\Delta}_{12} = \widetilde{G}(x_1,x_2) \end{aligned}$$

where $\widetilde{m}^2 = m^2 + i\lambda\Delta_{xx}/2$. Thus, we have a new version of the 2-point function, $\widetilde{G}(x_1, x_2)$, that looks just like $G(x_1, x_2)$ except that it contains the revised mass term $\widetilde{m}$. Since Δ_{xx} diverges, $\widetilde{m}$ is an infinite quantity – the effect of the interaction term is to make the particle mass infinite! This is annoying, but in the laboratory we always measure the *physical mass* m, which is not necessarily the same quantity that starts out in the Lagrangian (at least when the interaction term is present). The mass is then said to be *renormalized*.

Along the same line, you may have noticed that the denominator in the Feynman propagator, $k^2 - m^2$, is normally zero (it's just the relativistic mass-energy relation $E^2 - c^2p^2 - m^2c^4 = 0$). But in QFT, when $k^2 - m^2 \neq 0$ we say that the particle is *off mass shell*, so the integral doesn't blow up. Particle 4-momentum conservation has to hold at each interaction vertex, but not when the particle is just "propagating along."

If you can swallow all of this, I congratulate you!

14. Is Quantum Field Theory Real?

Let's wrap up this overly long discussion by asking the question – Is there any reason to believe that quantum field theory is not an accurate depiction of reality after all? In spite of its impressive ability to correctly predict a wide variety of experimental results, we cannot rigorously prove that all those infinite-dimensional integrals are convergent. To see this, let's take an exceedingly simple problem: a single Gaussian integral with an interaction term but no kinetic term:

$$Z(m, \lambda) = \int e^{-\frac{1}{2}m^2\phi^2 - \lambda\phi^4} d\phi \tag{14.1}$$

where λ is the coupling constant. You can think of this as a vastly simplified version of what we're trying to do with QFT. Certainly, (14.1) must have a solution if QFT is to have any chance of being a valid theory! Although this integral cannot be evaluated in closed form, we can do it term by term by expanding the exponential in $\lambda\phi^4$. We then have

$$Z(m, \lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{n!} \int \phi^{4n}\, e^{-\frac{1}{2}m^2\phi^2} d\phi$$

The integral can be performed by noticing that repeated differentiation of the Gaussian integral

$$Z(m, 0) = \int e^{-\frac{1}{2}m^2\phi^2} d\phi = \sqrt{2\pi} m^{-1}$$

with respect to m will introduce enough ϕ^2 terms into the integral for us to build up the required ϕ^{4n} term (this is just like using the source term $J(x)$ in the functional integral to "bring down" the expansion terms associated with $\lambda\phi^4$ in the full theory). The answer is

$$Z(m, \lambda) = \frac{\sqrt{2\pi}}{m} + \frac{\sqrt{2\pi}}{m} \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^n (4n-1)!!}{n!\, m^{4n}}$$

The double factorial can be written as

$$(4n-1)!! = \frac{2(4n-1)!}{4^n (2n-1)!}$$

so that

$$Z(m, \lambda) = \frac{\sqrt{2\pi}}{m} + \frac{\sqrt{2\pi}}{m} \sum_{n=1}^{\infty} (-1)^n \frac{2\lambda^n (4n-1)!}{n! 4^n m^{4n} (2n-1)!}$$

Does this quantity converge? For large factorials, we can use Sterling's approximation $N! \approx \sqrt{2\pi N}\,[N/e]^N$ to simplify things. Plugging this into the above, I get

$$Z(m, \lambda) \sim \left[\frac{-16\lambda n}{m^4 e}\right]^n$$

Well, this is a disaster – as n approaches infinity, the integral Z blows up. This is quite discouraging, particularly in view of the fact that (14.1) is the simplest possible integral we can imagine that is QFTish. Can we ever hope that the infinite-dimensional integrals in interacting quantum field theory have anything to do with reality?

In my opinion, there are few satisfactory ways out of this dilemma. It is known in QFT that renormalization forces us to abandon the idea that the actual values for mass and charge are really what we think they are. The renormalized mass of a particle, as you have seen, is infinite. Is it possible that an infinite mass somehow cancels the infinity in $Z(m, \lambda)$ for large values of n? Very unlikely, because the actual value for the coupling constant λ isn't what we think it is, either! Most physicists believe that QFT is valid only within a certain range of n which they call the *radius of convergence*. Within this range, QFT provides the fantastic accuracy it has shown for quantum electrodynamics and other applications, while outside this range it gives nonsensical infinities.

As little more than a neophyte myself, I would like to offer another possible answer, and that is this: Much of the mathematics that physicists use is non-rigorous (it may look impressive at first glance, but it is often very sloppy and makes mathematicians cringe), and it's possible that we've just pushed things further than we should have in order to obtain a solution. In my opinion, physicists are not above resorting to dirty tricks to make things work, especially when improper integrals are lurking about. A case in point is the integral expression for $Z(J, \lambda)$ in (9.1). I won't go into it, but physicists usually insert the term $\frac{1}{2}i\epsilon\phi^2$ into the Lagrangian (where ϵ is a small, positive real constant) which ensures that the integral will converge; afterwards, they set $\epsilon = 0$, and nobody is any the wiser. They can even rotate the spacetime coordinates to make time imaginary, which also makes the Z integral converge. I'll give you an example that may be easier to understand. Consider the much simpler quantity

$$I(\omega) = \int_0^\infty e^{i\omega x}\, dx$$

which occurs occasionally in quantum mechanical problems (ω is taken to be real and positive). This integrates straightforwardly to

$$I(\omega) = \frac{1}{i\omega}\left[e^{i\omega\,\infty} - 1\right]$$

At $x = \infty$, the exponential oscillates violently, so $I(\omega)$ doesn't converge. However, if we replace ω with $\omega + i\epsilon$ in the integrand, this becomes

$$I(\omega) = \frac{1}{i(\omega + i\epsilon)}\left[e^{i\omega x}e^{-\epsilon\,\infty} - 1\right]$$

The $e^{-\epsilon\,\infty}$ term now kills off the offending harmonic factor, giving us $I(\omega) = i/\omega$ after we cancel the remaining ϵ in the denominator. As an engineer, this kind of mathematical chicanery has always bothered me, but I guess it beats not having a solution.

15. Final Thoughts

1. You should look at all this as a very elementary beginning to QFT. In the end, it's not all that interesting because what we've been discussing is *scalar* QFT; much more interesting is *fermionic* QFT, especially quantum electrodynamics. It is here that you will find the kind of Feynman diagrams you've no doubt seen in books, with the squiggly photon lines and such, along with some pretty amazing calculations. But the scalar version is a good place to get your feet wet.

2. Anthony Zee has written a wonderful book (see references) in which he has tried to encapsulate the gist of QFT "in a nutshell." He bemoans the fact that what started out as an introductory overview ended up covering over 500 pages. Now I know how he feels, because it is impossible to cover this topic in a few dozen pages or so. I strongly encourage you to get his book and look at how rich QFT is, even at a very elementary level.

3. Remember that the square of the amplitude $\mathcal{M}$ is proportional to the probability that the process in a particular Feynman diagram will occur. The proportionality constant comes from what is known as *Fermi's Golden Rule* (this is a simple formula, but I will not go into it any further here). Putting all this together, you can accurately calculate things like particle decay rates and event cross sections.

4. The Feynman rules for quantum electrodynamics (QED) are the same as those for scalar particles, but the terms are a little different. For example, the photon propagator is proportional to i/k^2, which might be expected since the photon is massless. While the divergence problem persists in QED, physicists have developed powerful renormalization techniques which, while perhaps aesthetically unpleasing, have resulted in calculations whose predictive accuracy is nothing less than fantastic (the calculated gyromagnetic ratio of the electron, for example, matches the experimental value to within 10 decimal places).

5. Lastly, think of how amazing all this is – in spite of our pathetic model-building and mathematical hand-waving over particles and things that we can never actually observe (or truly understand), quantum field theory is nothing less than a glimpse into the mind of God. Although the path integral is a beautiful mathematical edifice, imagine how incredibly beautiful the true reality behind the curtain of existence must be!

And with that rather philosophical remark, I bid you adieu and God bless.

References

1. P.A.M. Dirac, *The Principles of Quantum Mechanics*. Clarendon Press, 4th ed., Oxford, 1967. Dirac is just about my favorite physicist, perhaps in part because he started out in life as an electrical engineer! Born in Bristol, England on 8 August 1902 to a Swiss father and English mother, Dirac was already a giant in the field of quantum theory by the time he was 25. Aloof (but not unfriendly) and ultra-taciturn (look up *taciturn* in the dictionary and you'll find his picture), Dirac preferred solitary walks and thinking about physics to just about everything else. He discovered the relativistic electron equation in 1927 – to me, the most profound and beautiful achievement of the human mind.

   A brilliant physicist, Dirac was an equally brilliant mathematician. The Dirac delta function is an indispensible tool to physicists and mathematicians alike, while his introduction of creation and annihilation operators has made the harmonic oscillator problem transparent to generations of physics students. Dirac's mind also gave us antimatter, the graviton, the magnetic monopole, quantum electrodynamics, and the four-component spinor formalism. He is said to have been particularly happy with his development of the bra and ket "language" of modern quantum mechanics, which is more commonly (and rightly) called Dirac notation.

   Elected Lucasian Professor of Mathematics at St. John's College in Cambridge in 1932 (Isaac Newton's old post), Dirac won the Nobel Prize in physics in 1933, sharing it with Erwin Schrödinger. In 1937, Dirac married the sister of fellow physics Nobel laureate Eugene Wigner. In 1969, Dirac left Cambridge and in 1972 joined the faculty of Florida State University in Tallahassee! Dirac died there on 20 October 1984 and was buried at Roselawn Cemetery, not far from the school. In November 1995, a plaque was placed in his honor at Westminster Abbey in London, close to the tomb of Newton (Stephen Hawking, the current Lucasian Professor, delivered the dedication address via voice synthesizer). On the plaque is engraved "$i\gamma \cdot \partial\Psi = m\Psi$," Dirac's relativistic electron equation. In a saner and more orderly world, Dirac's name would be as recognizable as Einstein's or Newton's. The fact that it is not is a great mystery to the world's physicists and mathematicians, who owe so much to this quiet genius from Bristol.

   Dirac's book on quantum mechanics appeared in 1930 and became an instant classic (Dirac's path integral "analogy" remark appears in this book). However, I have had one hell of a time understanding it, and I simply don't know why – maybe it's the language Dirac uses, or maybe it's the book's typesetting; it just doesn't speak to me. Maybe you'll have better luck.

2. D. Griffiths, *Introduction to Elementary Particles*. Wiley & Sons, New York, 1987. In my opinion, the best book of its kind – cleverly written and lucid, with much of Griffiths' wonderful humor thrown in to liven things up. I highly recommend it.

3. F. Halzen and A. Martin, *Quarks and Leptons*: *An Introductory Course in Modern Particle Physics*. Wiley & Sons, New York, 1984. Kind of like a companion book to Griffiths' text. A recommended introduction.

4. M. Kaku, *Quantum Field Theory: A Modern Introduction*. Oxford University Press, New York, 1993. Highly readable for the beginner by an author who is also an expert on superstring theory. The book includes a chapter on this topic, along with supergravity.

5. P. Ramond, *Field Theory*: *A Modern Primer*. Addison-Wesley, 2nd ed., Redwood City, California, 1989. Written by one of the giants of string theory, this book is somewhat more advanced and detailed than Kaku's text but is very readable nevertheless. I was disappointed to find that some of the formulas I thought I had discovered were in this book! Also highly recommended.

6. L. Ryder, *Quantum Field Theory*. Cambridge University Press, 2nd ed., Cambridge, 1996. The most comprehensive and readable book on the subject that I have seen to date. Alas, this second edition includes a chapter on supersymmetry, and is far over my head.

7. J.J. Sakurai, *Modern Quantum Mechanics*. Addison-Wesley, Redwood City, California, 1985. Tragically, Sakurai died in an accident at the age of 49, and this book was finished by a colleague. The chapters written by Sakurai include the most lucid expositions on quantum physics I have ever read, and I cannot recommend it too highly.

8. J.J. Sakurai, *Advanced Quantum Mechanics*. Addison-Wesley, Redwood City, California, 1967. Oddly enough, Sakurai wrote this book many years before the previously-cited text. In the preface, he writes that the purpose of the book was to present QFT "in a manner that cannot be made any simpler." Well, I can't make head or tails out of this book, and if you're just starting out, I suggest you stay away from it.

9. J. Schwinger (ed.), *Selected Papers on Quantum Electrodynamics*. Dover Publications, 1958. Here you'll find the collected works of many great physicists on QED, including Dirac's 1933 paper on the quantum mechanical action principle and Feynman's 1948 classic on the path integral.

10. R. Shankar, *Principles of Quantum Mechanics*. Yale University, 2nd ed., New York, 1994. This textbook is an undergraduate favorite, and includes two chapters on the path integral and applications. If your basic quantum mechanics is a little rusty, this is a great book to seek out.

11. A. Zee, *Quantum Field Theory in a Nutshell*. Princeton University Press, Princeton, New Jersey, 2003. Years ago I wondered why nobody had ever written a book along the lines of "Quantum Field Theory for Dummies." Well, Dr. Zee has done it, and done it well. Even if your math is not quite up to QFT standards, you will benefit from his insightful, comprehensive, enthusiastic and very understandable approach to the subject. The book includes relatively easy mathematical treatments of many related topics, including renormalization, quantum gravity and cosmology, symmetry breaking, condensed matter theory, grand unification and strings. This is the book I wish I had 25 years ago. If you buy just one book on QFT, get this one.