Weyl's 1918 Theory Revisited

December 11 2009, William O. Straub

Einstein's general theory of relativity appeared in November 1915, and in spite of the theory's largely unfamiliar and obtuse (at the time) mathematical character, within a few years many researchers had mastered the theory and were actively involved in its further development. One of its most passionate champions was the German mathematical physicist Hermann Weyl (1885–1955), who investigated many of the theory's applications in physics and cosmology. More notably, Weyl attempted to generalize the theory's Riemannian basis in what turned out to be a failed search for a unification of the gravitational and electromagnetic fields.

Weyl's initial attempts at unification took place in early 1918, although he was undoubtedly building on earlier work. Einstein's then-new notion that gravitational dynamics was a geometrical consequence of the presence of matter and energy in a four-dimensional space-time continuum represented a tempting invitation to many others to see if the only other known force at the time, electrodynamics, could also be described as a geometrical construct. Weyl arguably took this idea further than anyone else at the time but his unification scheme, initially hailed by Einstein as "genius," contained a fatal flaw that was spotted by none other than Einstein himself. Weyl continued to explore the theory until 1921 but, unable to adequately defend it from Einstein's criticism, he then abandoned the theory.

Although his theory was a failure, Weyl's work is important because in 1929 he resurrected his basic idea (which he called *gauge invariance*) and applied it to then still-emerging field of quantum theory. Gauge invariance is now recognized as one of the milestones of quantum physics, and is today a foundational aspect of all modern quantum theories.

Weyl's 1918 theory is discussed in detail elsewhere on my website, but I also wanted to provide a shorter introduction that avoids many of the mathematical details that go beyond the grasp of non-experts. So here I will simply sketch Weyl's derivation of his field equations, which result from a variation of Weyl's gauge invariant (perhaps more properly called *conformally* invariant) action Lagrangian. I will also present an interesting cosmological aspect of Weyl's theory, although I am under no illusions that it has anything to do with reality.

Notation

Repeated indices are summed. All integrals are four-dimensional. Following Adler et al., I denote partial derivatives with a single subscripted bar: $F^{\mu\nu}_{|\lambda} = \partial F^{\mu\nu}/\partial x^\lambda = \partial_\lambda F^{\mu\nu}$, etc., while covariant differentiation is denoted by a double bar. In Riemannian geometry the coefficient of affine connection is the Christoffel symbol

$$\left\{ {\alpha \atop \mu\nu} \right\} = \frac{1}{2} g^{\alpha\beta} \left(g_{\mu\beta|\nu} + g_{\beta\nu|\mu} - g_{\mu\nu|\beta} \right)$$

while in Weyl's geometry it is

$$\Gamma^\alpha_{\mu\nu} = -\left\{ {\alpha \atop \mu\nu} \right\} + \delta^\alpha_\mu \phi_\nu + \delta^\alpha_\nu \phi_\mu - g_{\mu\nu} g^{\alpha\beta} \phi_\beta$$

where ϕ_μ is a vector field that Weyl identified with the electromagnetic 4-potential.

The Einstein and Weyl Action Lagrangians

I will assume that the reader is already familiar with the variational principle as it is applied to the so-called Einstein-Hilbert action, which includes a term representing the source-free electromagnetic field:

$$I = \int \sqrt{-g}\, \left(R + F_{\mu\nu} F^{\mu\nu} \right) d^4x$$

Here $F_{\mu\nu} = A_{\mu|\nu} - A_{\nu|\mu}$, where A_μ is the 4-potential. The electromagnetic source vector S^μ can also be included in the Lagrangian by adding the scalar $S^\mu A_\mu$, but I'm going to leave it out. The Einstein-Hilbert action is thus based on a coordinate-independent Lagrangian density that is composed of the Ricci scalar R and the source-free electromagnetic density. Variation of this action will result in two independent terms: the coefficient of the variation $\delta g^{\mu\nu}$ and that of the variation δA_μ (note that the δ-notation stands for *any* variation: coordinate change, translation, scale/gauge transformation, etc.). Setting these coefficients to zero will give us the field equations associated with the action. We anticipate that the coefficient of $\delta g^{\mu\nu}$ will correspond to gravitation, as in fact it does, while that of the variation δA_μ will correspond to the electromagnetic source vector. Thus,

$$\delta I = \int \sqrt{-g}\, \left(G_{\mu\nu}\, \delta g^{\mu\nu} + W^\mu \delta A_\mu\right) d^4x$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad \text{and} \quad W^\mu = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} F^{\mu\nu}\right)_{||\nu}$$

Since $\delta I = 0$, both of these quantities must vanish; this gives Einstein's gravitational field equations for free space, $G_{\mu\nu} = 0$, along with the condition for a vanishing electromagnetic source S^μ, which is $\sqrt{-g} S^\mu = (\sqrt{-g} F^{\mu\nu})_{||\nu} = 0$. Note that both equations exhibit vanishing divergence: $(\sqrt{-g} G^{\mu\nu})_{||\nu} = 0$, $(\sqrt{-g} S^\mu)_{||\mu} = 0$. While the divergence property of the source vector is tied to the conservation of electric charge, that of the Einstein tensor $G^{\mu\nu}$ can only *tentatively* be related to the conservation of mass-energy. This is because the source conservation law can also be written using an ordinary partial derivative, $(\sqrt{-g} S^\mu)_{|\mu} = 0$ (which is still fully covariant and expresses true conservation of charge) while the Einstein tensor exhibits no such property. The observation that the conservation of gravitational mass-energy cannot be expressed using ordinary partial derivatives remains a mystery, in spite of the fact that mass-energy is certainly conserved. In short, the covariant divergence of a tensor density of rank two in general relativity has no clear physical meaning.

Weyl's 1918 theory was based on his assertion that the geometry of the world should not change when the metric tensor of Riemannian geometry $g_{\mu\nu}$ is *rescaled*. That is, $g_{\mu\nu} \to \lambda g_{\mu\nu}$, where $\lambda(x)$ is a completely arbitrary scalar function of position, should change nothing. Weyl saw this regauging of the metric as a new, beautiful type of symmetry that should be reflected in the action Lagrangian. Weyl went on to develop a revised form of Riemannian geometry in which the associated connection term $\Gamma^\alpha_{\mu\nu}$ is itself invariant with regard to metric regauging. This observation considerably strengthened Weyl's belief that gauge or (conformal) invariance was a symmetry that Nature would use in shaping her laws. Since $\delta\Gamma^\alpha_{\mu\nu} = 0$ in Weyl's geometry, the Riemann-Christoffel tensor and its contracted form, the Ricci tensor, are both invariant: $\delta R^\lambda_{\mu\nu\alpha} = 0$ and $\delta R_{\mu\alpha} = 0$. Thus, the two most fundamental quantities of differential geometry are conformally invariant. Furthermore, Weyl knew that the electromagnetic tensor $F_{\mu\nu}$ was itself invariant with respect to a *gauge transformation* of the 4-potential. In view of these observations, Weyl set out to find an action Lagrangian that would reflect these symmetries.

First, some preliminaries. In Weyl's theory the covariant derivative of the metric tensor is not zero: $g_{\mu\nu||\alpha} = 2 g_{\mu\nu}\phi_\alpha$, while $\sqrt{-g}_{||\alpha} = 4\sqrt{-g}\phi_\alpha$. You'll need these expressions to actually carry out the variations on the action. The derivation of these quantities is provided in my more comprehensive write-up on the Weyl theory, which also explains the very important topic known as *gauge weight*.

The simplest invariant action in Weyl's geometry utilizes the square of the Ricci scalar, and this is the action Weyl used to develop his theory. It is

$$I = \int \sqrt{-g}\, \left(R^2 + F_{\mu\nu} F^{\mu\nu}\right) d^4x \tag{1}$$

where k is some constant. Einstein and others objected to the fact that the R^2 term is of fourth order in the metric tensor but, as we will see, this objection can be averted.

The Weyl Variational Principle

Variation of Weyl's action with respect to $g^{\mu\nu}$ is straightforward, but it is considerably simplified by utilizing Palatini's method along with *local coordinates*, in which all terms involving partial derivatives of the metric and its determinant are zero. Integration by parts is required numerous times, but in the end we find that

$$\delta I = \int \sqrt{-g}\,(W_{\mu\nu}\,\delta g^{\mu\nu} + W^{\mu}\delta\phi_{\mu})\,d^4x \tag{2}$$

where

$$\begin{aligned} W_{\mu\nu} = &-2\left(g^{\alpha\beta}F_{\mu\alpha}F_{\beta v} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) + 2R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) + 16R\phi_{\mu}\phi_{\nu} + 8R_{|\mu}\phi_{\nu} \\ &- g_{\mu\nu}g^{\alpha\beta}R_{|\alpha||\beta} - 6g_{\mu\nu}g^{\alpha\beta}R_{|\alpha}\phi_{\beta} - 2g_{\mu\nu}g^{\alpha\beta}R\phi_{\alpha||\beta} - 8g_{\mu\nu}g^{\alpha\beta}R\phi_{\alpha}\phi_{\beta} \end{aligned} \tag{3}$$

and

$$W^{\mu} = 24\sqrt{-g}g^{\mu\nu}\left(R\phi_{\nu} + \frac{1}{2}R_{|\nu}\right) - 4\frac{1}{\sqrt{-g}}\left(\sqrt{-g}F^{\mu\nu}\right)_{|\nu} \tag{4}$$

Setting these quantities to zero, we get

$$\begin{aligned} g^{\alpha\beta}F_{\mu\alpha}F_{\beta\nu} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} = R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) + 8R\phi_{\mu}\phi_{\nu} + 4R_{|\mu}\phi_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}R_{|\alpha||\beta} \\ - 3g_{\mu\nu}g^{\alpha\beta}R_{|\alpha}\phi_{\beta} - g_{\mu\nu}g^{\alpha\beta}R\phi_{\alpha||\beta} - 4g_{\mu\nu}g^{\alpha\beta}R\phi_{\alpha}\phi_{\beta} \end{aligned} \tag{5}$$

and

$$\left(\sqrt{-g}F^{\mu\nu}\right)_{|\nu} = 6\sqrt{-g}g^{\mu\nu}\left(R\phi_{\nu} + \frac{1}{2}R_{|\nu}\right) \tag{6}$$

The left side of (5) is the familiar energy tensor $T_{\mu\nu}$ of the electromagnetic field, while the Ricci term is reminiscent of Einstein's gravitational field equations; the fact that both quantities are *traceless* is interesting. The remaining terms all involve the 4-potential and its covariant derivative. Equation (6) is perhaps the most notable result of Weyl's theory; it indicates that electromagnetism is an intrinsic part of the Weyl geometry.

The $(\sqrt{-g}F^{\mu\nu})_{|\nu}$ term is the electromagnetic source density, and its divergence must vanish. This results in the condition

$$\begin{aligned} \left(\sqrt{-g}F^{\mu\nu}\right)_{|\nu||\mu} = 6\left[\sqrt{-g}g^{\mu\nu}\left(R\phi_{\nu} + \frac{1}{2}R_{|\nu}\right)\right]_{||\mu} \quad \text{or} \\ 0 = g^{\mu\nu}R_{|\mu||\nu} + 2g^{\mu\nu}R\phi_{\mu||\nu} + 4g^{\mu\nu}R_{|\mu}\phi_{\nu} + 4g^{\mu\nu}R\phi_{\mu}\phi_{\nu} \end{aligned} \tag{7}$$

By calculating the trace of W $(= g^{\mu\nu}W_{\mu\nu})$ and setting it to zero, we recover this same condition (this is not a miracle or even a coincidence, but simply a consequence of Noether's theorem). We can now use (7) to simplify (5) somewhat; eliminating the $g^{\mu\nu}R\phi_{\mu||\nu}$ term we get, finally,

$$T_{\mu\nu} = R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) + 8R\phi_{\mu}\phi_{\nu} + 4R_{|\mu}\phi_{\nu} - g_{\mu\nu}g^{\alpha\beta}R_{|\alpha}\phi_{\beta} - 2g_{\mu\nu}g^{\alpha\beta}R\phi_{\alpha}\phi_{\beta} \tag{8}$$

which is still traceless.

Cosmological Aspects of Weyl's Theory

All of this is very suggestive. Weyl believed that his geometry had produced a variant of Einstein's equations in which electromagnetism is embedded into the geometry. However, (8) is a bit of a mess, and its interpretation is open to question (if in fact it has any meaning at all). In view of this, we can ask what happens when the Weyl vector ϕ_μ is set to zero. In that case the energy tensor $T_{\mu\nu}$ also vanishes, and we are then left with

$$R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) = 0 \tag{9}$$

(Note that this is obtained from a variation of the $\sqrt{-g}R^2$ Lagrangian in ordinary Riemannian geometry.) Rejecting the trivial solution $R = 0$ (which corresponds to one version of Einstein's free-space field equations), we have the *second-order* expression

$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = 0, \qquad (R \neq 0) \tag{10}$$

which is now remarkably similar to Einstein's equations.

The set of equations in (10) is easily solved. First let us assume the standard Schwarzschild line element

$$ds^2 = e^\nu \left(dx^0\right)^2 - e^\lambda dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

where ν and λ are functions of the radial parameter r alone. From any general relativity text, we find that the Ricci terms are

$$\begin{aligned}
R_{00} &= e^{\nu-\lambda}\left[-\frac{1}{2}v'' + \frac{1}{4}\nu'\lambda' - \frac{1}{4}\left(\nu'\right)^2 - \frac{1}{r}\nu'\right] \\
R_{rr} &= \frac{1}{2}v'' - \frac{1}{4}\nu'\lambda' + \frac{1}{4}\left(\nu'\right)^2 - \frac{1}{r}\lambda' \\
R_{\theta\theta} &= e^{-\lambda}\left[1 + \frac{1}{2}r\nu' - \frac{1}{2}r\lambda\right] - 1 \\
R_{\phi\phi} &= \sin^2 R_{\theta\theta}
\end{aligned}$$

where the primes refer to partial differentiation with respect to r. Similarly, from $R = g^{\mu\nu}R_{\mu\nu}$ we find that

$$R = e^{-\lambda}\left[-v'' + \frac{1}{2}\nu'\lambda' - \frac{1}{2}\left(\nu'\right)^2 - \frac{2}{r}\nu' + \frac{2}{r}\lambda' - \frac{2}{r^2}\right] + \frac{2}{r^2} \tag{11}$$

It is easy to show from any two of the equations in (10) that $\lambda' = -\nu'$ (a result that is also obtained from Einstein's equations). Ignoring the integration constant, we then have $\lambda = -\nu$. Plugging this result back into any of the equations in (10), it is straightforward to show that

$$\begin{aligned}
e^\nu &= 1 - \frac{2m}{r} - \Lambda r^2 \\
e^\lambda &= \left(1 - \frac{2m}{r} - \Lambda r^2\right)^{-1}
\end{aligned} \tag{12}$$

where $m = G/c^2$ is the usual reduced gravitational mass and Λ is another constant. The two expressions in (12) are the same as those obtained from the Einstein field equations, but with the addition of a term involving r^2 (obviously, this result prevents the theory from being Lorentzian at great distances).

This is a most interesting result in view of the following considerations. Cartan showed that the most general form of the Einstein free-space field equations is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \bar{\Lambda}g_{\mu\nu} = 0 \tag{13}$$

where $\bar{\Lambda}$ is Einstein's cosmological constant. Contraction of this expression shows that we can identify the Ricci scalar with this constant via $R = 4\bar{\Lambda}$. But if we insert this result back into (13), we have precisely the Weyl result in (10)! We therefore see that Weyl's field equations automatically relate the Ricci scalar to the cosmological constant. By inserting the expressions in (12) into (11), we easily find that $R = 4\Lambda$, so that R is indeed a constant. This justifies our earlier demand in (9) that R be a non-zero quantity. Incidentally, this value for R trivially satisfies (7), where $g^{\mu\nu}R_{|\mu|\nu} = \Box^2 R = 0$ in a flat space with $\phi_\mu = 0$. Otherwise, we'd be tempted to think that R is a solution to the wave equation.

It is obvious that if the Ricci scalar R is sufficiently small, its effects on the field equations will be negligible, and Weyl's theory becomes indistinguishable from Einstein's ($R_{\mu\nu} = 0$). Consequently, all the usual tests of general relativity—gravitational red shift, radar delay, perihelion shift of Mercury, bending of light—are satisfied to the same extent using Weyl's approach.

There is one more surprise. It is well known that, to first order, the usual Newtonian result for gravitation can be approximated by $g_{00} = 1 + 2\varphi/c^2$, where $\varphi = -GM/r$ is the Newtonian gravitational potential, the classical quantity responsible for the attraction between two massive bodies. But in Weyl's theory we have an additional term due to the non-vanishing of the cosmological constant:

$$\varphi = -\frac{GM}{r} - \frac{1}{2}\Lambda c^2 r^2 \tag{14}$$

Depending on the sign of Λ, the effect of this extra term is to either strengthen or weaken the acceleration of a test particle in the field of the mass M; indeed, this additional acceleration is *independent* of any nearby mass! The extra acceleration arises from the peculiarity of the Weyl geometry itself, but it implies that the origin of the coordinate system used is special in some way; if r is a distance, then distance from *what*? Evidently, the origin plays a singular role here, in contradiction to the demands of relativity.

Cosmologists have known for some time that galactic rotation rates do not seem to obey Newtonian gravity—the estimated mass of many galaxies is too small to support the observed high velocities of stars far from the galactic centers. This has resulted in the *dark matter* theory, which posits the existence of some kind of exotic, unseen matter that permeates galaxies and provides the required gravitational boost. At the other end of the cosmological spectrum is the observation that the expansion of the universe appears to be *accelerating*, not slowing down as originally thought. This has given rise to the *dark energy* theory, which also posits the existence of some kind of *repulsive* energy field permeating the universe that serves to speed up the expansion. Could both of these observations be somehow due to a non-zero cosmological constant?

Adler has suggested a way to actually measure Λ. Take a large, very heavy, evacuated spherical shell and set it out in deep space. Then place a small test particle at the very center and see if the particle moves. If all effects such as external gravitational and electromagnetic fields can be eliminated, then a displacement of the particle might provide conclusive evidence of a non-zero cosmological constant.

Note to high-school physics and cosmology buffs: if you ever discover such evidence, prepare yourself for a wintertime visit to Stockholm.

Last Thoughts

Obviously, a glance at Equation (8) is sufficient to cast doubt on the Weyl field equations; it looks odd, but more importantly it doesn't predict anything. But by simply setting the Weyl vector ϕ_μ to zero, we avoid this problem, along with many others—the geometry becomes Riemannian, the covariant derivative of the metric tensor vanishes, and all is well again. But this still leaves $T_{\mu\nu} = R(R_{\mu\nu} - 1/4 g_{\mu\nu} R)$ as the set of Weyl field equations for non-empty space, and it has its own problems, one being the fact that it is of fourth order.

If you have the energy, you can try applying this simplified variant of Weyl's equations to a number of non-trivial problems, like the interior Schwarzschild metric (Tolman-Oppenheimer-Volkoff equation), or to any of the simpler cosmological models (Friedman, Robertson-Walker), or you can even apply it to the electromagnetic field itself (Weyl was apparently the first to do this, but with the traditional Einstein equations). If you do, please let me know what you find.

References

1. R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity.* McGraw-Hill, 2nd Edition, 1975.

2. W. Straub, *Weyl's 1918 Theory.* http://www.weylmann.com.