Weyl's 1918 Theory Revisited

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11 December 2009
Revised June-July 2015

Author's Comments

This is a revision and expansion of an earlier (2009) paper that contained a calculational error in the derivation of Equation 3.1, which was pointed out to me by an observant graduate student. The mathematics in this revision is now correct, while providing a better overview of Hermann Weyl's 1918 attempt to unify the forces of gravity and electromagnetism.

Introduction

Einstein's general theory of relativity appeared in November 1915, and in spite of the theory's largely unfamiliar and obtuse (at the time) mathematical formalism, within a few years many researchers had mastered the theory and were actively involved in its further development. One of its most passionate champions was the German mathematical physicist Hermann Weyl (1885-1955), who investigated many of the theory's applications in physics and cosmology. More notably, Weyl attempted to generalize the theory's Riemannian basis in what turned out to be a failed search for a unification of the gravitational and electromagnetic fields.

Weyl's initial attempts at unification took place in early 1918, although he was undoubtedly building on earlier work. Einstein's then-new notion that gravitational dynamics was a geometrical consequence of the presence of matter and energy in a four-dimensional space-time continuum represented a tempting invitation to many others to see if the only other known force at the time, electrodynamics, could also be described as a geometrical construct. Weyl arguably took this idea further than anyone else at the time but his unification scheme, initially hailed by Einstein as "genius," contained a fatal flaw that was spotted by none other than Einstein himself. Weyl continued to explore the theory until 1921 but, unable to adequately defend it from Einstein's criticism, he abandoned the theory.

Although his theory was a failure, Weyl's work is important because in 1929 he resurrected his basic idea (which he called *gauge invariance*) and applied it to then still-emerging field of quantum theory. Gauge invariance is now recognized as one of the milestones of quantum physics, and is today a foundational aspect of all modern quantum theories.

The mathematical particulars of Weyl's 1918 theory are discussed in detail elsewhere on my website. Consequently, here I restrict myself to the derivation of the equations of motion from an expanded version of Weyl's action. I also present an interesting cosmological aspect of Weyl's theory, although I am under no illusions that it has anything to do with reality. Indeed, Weyl's theory is recognized today as a failure, but there are aspects of Weyl's attempt that are genuinely intriguing, particularly with respect to its possible cosmological implications.

1. Notation

Repeated indices are summed. All integrals are four-dimensional. Following Adler et al., partial derivatives are denoted with a single subscripted bar: ${F^{\mu\nu}}_{|\lambda} = \partial F^{\mu\nu}/\partial x^\lambda = \partial_\lambda F^{\mu\nu}$, etc., while covariant differentiation is denoted by a double bar. In Riemannian geometry the coefficient of affine connection is the Christoffel symbol

$$\left\{ {\alpha \atop \mu\nu} \right\} = \frac{1}{2} g^{\alpha\beta} \left(g_{\mu\beta|\nu} + g_{\beta\nu|\mu} - g_{\mu\nu|\beta} \right)$$

while in Weyl's geometry it is

$$\Gamma^\alpha_{\mu\nu} = \left\{ {\alpha \atop \mu\nu} \right\} - \delta^\alpha_\mu \phi_\nu - \delta^\alpha_\nu \phi_\mu + g_{\mu\nu} g^{\alpha\beta} \phi_\beta$$

where ϕ_μ is a vector field that Weyl identified with the electromagnetic 4-potential. Lastly, some covariant derivatives in Weyl's geometry are:

$$g_{\mu\nu||\alpha} = 2g_{\mu\nu}\phi \tag{1.1}$$

$$g^{\mu\nu}{}_{||\alpha} = -2g^{\mu\nu}\phi_\alpha \tag{1.2}$$

$$\left(\sqrt{-g}\right)_{||\alpha} = 4\sqrt{-g}\,\phi_\alpha \tag{1.3}$$

2. The Einstein and Weyl Actions

We will assume that the student is already familiar with the variational principle as it is applied to the so-called Einstein-Hilbert action which, complete with terms representing the electromagnetic field and its source, is

$$I = \int \sqrt{-g}\left(R + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)d^4x$$

Here $F_{\mu\nu} = \phi_{\mu|\nu} - \phi_{\nu|\mu}$, where ϕ_μ is the 4-potential. The Einstein-Hilbert action is thus based on a coordinate-independent Lagrangian density that is composed of the Ricci scalar R and the electromagnetic field. Restriction of the variation of this action with respect to the metric tensor $g^{\mu\nu}$ and the potential ϕ_μ will result in two independent terms: the coefficient of the variation $\delta g^{\mu\nu}$ and that of the variation $\delta\phi_\mu$. Setting each of these coefficients to zero will give us the field equations associated with the action. We anticipate that the coefficient of $\delta g^{\mu\nu}$ will correspond to gravitation, as in fact it does, while that of the variation $\delta\phi_\mu$ will correspond to electromagnetism. Thus,

$$\delta I = \int \sqrt{-g}\left(G_{\mu\nu}\delta g^{\mu\nu} + W^\mu \delta\phi_\mu\right)d^4x \tag{2.1}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \quad \text{and} \quad W^\mu = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}F^{\mu\nu}\right)_{|\nu} \tag{2.2}$$

Since $\delta I = 0$, both of these quantities must vanish; this gives Einstein's gravitational field equations $G_{\mu\nu} = T_{\mu\nu}$ (where $T_{\mu\nu}$ is the electromagnetic stress-energy tensor), along with $(\sqrt{-g}F^{\mu\nu})_{|\nu} = 0$. Note that the Einstein and stress-energy tensors are required to be divergenceless.

Weyl's 1918 theory was based on his assertion that the geometry of the world should not change when the metric tensor of Riemannian geometry $g_{\mu\nu}$ is *rescaled*; that is, if $g_{\mu\nu} \to \lambda g_{\mu\nu}$, where $\lambda(x)$ is an arbitrary scalar function of position, then the laws of Nature should not change. Weyl saw this regauging of the metric as a new, beautiful type of symmetry that should be reflected in the action Lagrangian. Weyl went on to develop a revised form of Riemannian geometry in which the associated connection term $\Gamma^\alpha_{\mu\nu}$ is itself invariant with regard to metric regauging. This observation considerably strengthened Weyl's belief that gauge or (conformal) invariance was a symmetry that Nature would use in shaping her laws. Since $\delta\Gamma^\alpha_{\mu\nu} = 0$ in Weyl's geometry, the Riemann-Christoffel tensor and its contracted form, the Ricci tensor, are both invariant: $\delta R^\lambda_{\mu\nu\alpha} = 0$ and $\delta R_{\mu\alpha} = 0$. Thus, the two most fundamental quantities of differential geometry are conformally invariant. Furthermore, Weyl knew that the electromagnetic tensor $F_{\mu\nu}$ was itself invariant with respect to a *gauge transformation* of the 4-potential. In view of these observations, Weyl set out to find an action Lagrangian that would reflect these symmetries. Let us do that now.

First, some preliminaries. Remember that in Weyl's theory the covariant derivative of the metric tensor is not zero: $g^{\mu\nu||\alpha} = -2g^{\mu\nu}\phi_\alpha$, while $(\sqrt{-g})_{||\alpha} = 4\sqrt{-g}\,\phi_\alpha$. You'll need these expressions to actually carry out the variations on the action, particularly those involving partial integration.

The simplest scale invariant action in Weyl's geometry utilizes the *square* of the Ricci scalar, and this is the action Weyl used to develop his original theory. In view of the Einstein-Hilbert action given above, we will write the Weyl action as

$$I = \int \sqrt{-g}\left(R^2 + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)d^4x \tag{2.3}$$

Einstein and others objected to the fact that the R^2 term is of fourth order in the metric tensor but, as we will see, this objection can be avoided.

3. Variation of the Weyl Action

Variation of Weyl's action with respect to ϕ_μ presents no problems. Variation with respect to $g^{\mu\nu}$ is straightforward but tedious, although it can be considerably simplified by utilizing Palatini's method along with *local coordinates*, in which all terms involving partial derivatives of the metric and its determinant vanish (except for $\delta g_{\mu\nu|\alpha}$ and $\delta(\sqrt{-g})_{|\alpha}$ and their covariant derivative variants, which require integration by parts). In the end we find that

$$\delta I = \int \sqrt{-g}\left(W_{\mu\nu}\delta g^{\mu\nu} + W^\mu \delta\phi_\mu\right) d^4x$$

where

$$W_{\mu\nu} = 2R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) + \frac{1}{2}\left(g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) + 8R\phi_\mu\phi_\nu + 8R_{|\mu}\phi_\nu - 2g_{\mu\nu}g^{\alpha\beta}R_{|\alpha||\beta}$$
$$+ 2R_{|\mu||\nu} + 4R\phi_{\mu||\nu} - 4g_{\mu\nu}g^{\alpha\beta}R\phi_{\alpha||\beta} - 8g_{\mu\nu}g^{\alpha\beta}R_{|\alpha}\phi_\beta - 8g_{\mu\nu}g^{\alpha\beta}R\phi_\alpha\phi_\beta = 0 \tag{3.1}$$

and

$$W^\mu = 24g^{\mu\nu}\left(R\phi_\nu + \frac{1}{2}R_{|\nu}\right) - \frac{1}{\sqrt{-g}}\left(\sqrt{-g}F^{\mu\nu}\right)_{|\nu} = 0 \tag{3.2}$$

or

$$\sqrt{-g}\,S^\mu = 24\sqrt{-g}\,g^{\mu\nu}\left(R\phi_\nu + \frac{1}{2}R_{|\nu}\right) \tag{3.3}$$

where we have set $\sqrt{-g}S^\mu = (\sqrt{-g}F^{\mu\nu})_{|\nu}$. Thus, the electromagnetic source vector appears in Weyl's theory in terms of the Ricci scalar and its first derivative.

The electromagnetic source vector density is a conserved quantity, meaning that its covariant divergence vanishes. We therefore demand that

$$\left(\sqrt{-g}S^\mu\right)_{||\mu} = \left(\sqrt{-g}F^{\mu\nu}\right)_{|\nu||\mu} = 0$$

Taking the divergence of (3.3), we have

$$\sqrt{-g}\,g^{\mu\nu}\left(2R\phi_\mu\phi_\nu + 2R_{|\mu}\phi_\nu + R\phi_{\mu||\nu} + \frac{1}{2}R_{|\mu||\nu}\right) = 0$$

Dropping the $\sqrt{-g}\,g^{\mu\nu}$ term, we then have the convenient identity

$$R\phi_{\mu||\nu} = -2R\phi_\mu\phi_\nu - 2R_{|\mu}\phi_\nu - \frac{1}{2}R_{|\mu||\nu} \tag{3.4}$$

This identity has no obvious interpretation, but it can be used to simplify the $W_{\mu\nu}$ term. Inserting the identity (3.4) into (3.1), most terms drop out and we are left simply with

$$W_{\mu\nu} = 2R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) + \frac{1}{2}\left(F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{3.5}$$

Contraction with $g^{\mu\nu}$ then gives the trace of $W_{\mu\nu}$, which vanishes as required by virtue of the tracelessness of the two terms in (3.5).

4. Cosmological Aspects of Weyl's Theory

All of this is very suggestive. Weyl believed that his geometry had produced a variant of Einstein's equations in which electromagnetism is embedded into the geometry.

In the absence of an electromagnetic field Weyl's geometry reduces to Riemannian geometry, but the Weyl field equations remain intact:

$$R\left(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right) = 0$$

So here we have two possible solutions,

$$R = 0,$$

$$R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R = 0 \tag{4.1}$$

If we reject the solution $R = 0$, the Ricci scalar can be divided out and we are left with the set of purely *second-order* equations in (4.1), which are easily solved. Let us assume the standard Schwarzschild line element for a radially symmetric gravitational field:

$$ds^2 = e^{\nu}\left(dx^0\right)^2 - e^{\lambda}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$

where ν and λ are functions of the radial parameter r alone. From any general relativity text, we find that the Ricci terms are

$$R_{00} = e^{\nu-\lambda}\left[-\frac{1}{2}\nu'' + \frac{1}{4}\nu'\lambda' - \frac{1}{4}\left(\nu'\right)^2 - \frac{1}{r}\nu'\right]$$

$$R_{rr} = \frac{1}{2}\nu'' - \frac{1}{4}\nu'\lambda' + \frac{1}{4}\left(\nu'\right)^2 - \frac{1}{r}\lambda'$$

$$R_{\theta\theta} = e^{-\lambda}\left[1 + \frac{1}{2}r\nu' - \frac{1}{2}r\lambda\right] - 1$$

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta}$$

where the primes refer to partial differentiation with respect to r. Similarly, from $R = g^{\mu\nu}R_{\mu\nu}$ we find that

$$R = e^{-\lambda}\left[-\nu'' + \frac{1}{2}\nu'\lambda' - \frac{1}{2}\left(\nu'\right)^2 - \frac{2}{r}\nu' + \frac{2}{r}\lambda' - \frac{2}{r^2}\right] + \frac{2}{r^2}$$

From these expressions it is easy to show that $\lambda' = -\nu'$ (a result that is also obtained from the Schwarzschild solution of Einstein's equations). Ignoring the integration constant, we then have $\lambda = -\nu$. With this result it is then easy to show that

$$e^{\nu} = 1 - \frac{2m}{r} - \lambda r^2 \tag{4.2}$$

$$e^{\lambda} = \left(1 - \frac{2m}{r} - \lambda r^2\right)^{-1} \tag{4.3}$$

$$R = 12\lambda \tag{4.4}$$

where $m = GM/c^2$ is the usual reduced gravitational mass and λ is a constant. These solutions are the same as those obtained from the Einstein field equations but with the addition of a term involving r^2 (obviously, this prevents the Weyl solution from being Minkowskian at great distances).

This is a most interesting result in view of the following considerations. Cartan showed that the most general form of the Einstein free-space field equations is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 \tag{4.5}$$

where Λ is Einstein's cosmological constant. Contraction of this expression shows that we can identify the Ricci scalar with this constant via $R = 4\Lambda$. But if we insert this result back into (4.5), we have precisely the Weyl result (4.1). We therefore see that Weyl's field equations automatically relate the Ricci scalar to the cosmological constant. This also justifies our assumption that R is a non-zero constant.

It is obvious that if the Ricci scalar R is sufficiently small its effects on the field equations will be negligible, and Weyl's theory becomes indistinguishable from Einstein's ($R_{\mu\nu} = 0$). Consequently, all the usual tests of general relativity—gravitational red shift, radar delay, perihelion shift of Mercury, bending of light—are satisfied to the same extent using Weyl's approach.

There is one more surprise. It is well known that, to first order, the usual Newtonian result for gravitation can be approximated by $g_{00} = 1 + 2\varphi/c^2$, where $\varphi = -GM/r$ is the Newtonian gravitational potential, the classical quantity responsible for the attraction between two massive bodies. But in Weyl's theory we have an additional term due to the non-vanishing of the cosmological constant:

$$\varphi = -\frac{GM}{r} - \frac{1}{2}\lambda c^2 r^2$$

Depending on the sign of λ, the effect of this extra term is to either strengthen or weaken the acceleration of a test particle, but here this additional acceleration is *independent* of any nearby mass. The extra acceleration arises from the peculiarity of the Weyl geometry itself, but it implies that the origin of the coordinate system used is special in some way; if r is a distance, then distance from *what*? Evidently, the origin plays a singular role here, in contradiction to the demands of relativity.

Cosmologists have known for some time that galactic rotation rates do not obey Newtonian gravity—the estimated mass of most galaxies is too small to support the observed high velocities of stars far from their galactic cores. This has resulted in the *dark matter* hypothesis, which posits the existence of some kind of exotic, unseen matter that permeates galaxies and provides the required gravitational boost. At the other end of the cosmological spectrum is the observation that the expansion of the universe appears to be *accelerating*, not slowing down as originally thought. This has given rise to the *dark energy* hypothesis, which assumes the existence of some kind of *repulsive* energy field permeating the universe that serves to speed up the expansion. Could these observations be somehow due to a non-zero cosmological constant, which we have shown appears in the Weyl equations of motion? (Adler has suggested a way to actually measure Λ. Take a large, very heavy, evacuated spherical shell and set it out in deep space. Then place a small test particle at the very center and see if the particle moves. If all effects such as external gravitational and electromagnetic fields can be eliminated, then a displacement of the particle might provide conclusive evidence of a non-zero cosmological constant.)

5. Final Comments

One may rightly wonder why Weyl's theory can be considered a unified field theory, considering the fact that we were compelled to utilize the electromagnetic tensor $F_{\mu\nu}$ in the Weyl action Lagrangian from the start. After all, the Einstein-Hilbert action also contains the electromagnetic part, and it was never considered "unified" in any substantive sense of the term. But what makes Weyl's theory different is Equation (3.3), which effectively ties electromagnetism to the metric tensor and the Ricci scalar R, both of which are fundamental geometric quantities. In this sense alone, the Weyl theory can indeed be considered a unified theory of the gravitational and electromagnetic fields.

References

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