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Weyl's Geometry and Physics

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It is proposed to remove the difficulty of nonintegrability of length in the Weyl geometry by modifying the law of parallel displacement and using "standard" vectors. The field equations are derived from a variational principle slightly different from that of Dirac and involving a parameter σ. For $\sigma = 0$ one has the electromagnetic field. For $\sigma < 0$ there is a vector meson field. This could be the electromagnetic field with finite-mass photons, or it could be a meson field providing the "missing mass" of the universe. In cosmological models the two natural gauges are the Einstein gauge and the cosmic gauge. With the latter the universe has a fixed size, but the sizes of small systems decrease with time and their masses and energies increase, thus producing the Hubble effect. The field of a particle in this gauge is investigated, and it leads to an interesting solution of the Einstein equations that raises a question about the Schwarzschild solution.

1. INTRODUCTION

After Einstein[(1)] put forth his general theory of relativity, which provided a geometrical description of gravitation, Weyl[(2)] proposed a more general theory that also included a geometrical description of electromagnetism. In the case of general relativity one has a Riemannian geometry with a metric tensor $g_{\mu\nu}$. If a vector undergoes a parallel displacement, its direction may change, but not its length. In Weyl's geometry, in addition to the metric tensor, there is also a vector ϕ_μ. For a vector undergoing parallel displacement, not only the direction, but also the length, may change, and the change of length depends on ϕ_μ. If the vector is carried from point A to point B, its length at B will in general depend on the path between the two points, i.e., length is not integrable. One can introduce an arbitrary standard of length, or gauge, at each point. Under a transformation of the gauge the

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vector ϕ_μ is changed by the addition of the gradient of a scalar function. This vector is identified with the potential vector describing the electromagnetic field.

Weyl obtained the field equations with the help of a variational principle. These equations included the Maxwell equations for the electromagnetic field. However, there was a difficulty: in order to get satisfactory equations for the gravitational field, Weyl had to choose a special gauge, one in which the curvature scalar was constant.

Weyl's theory was rejected by most physicists because of the fact that, in the presence of an electromagnetic field, length is nonintegrable. This implies that some properties of elementary particles should depend on their past histories, which is in contradiction to the observed uniformity of their properties.

Not long ago Dirac[3] revived Weyl's theory in connection with his large-number hypothesis. There exist certain large, dimensionless numbers in nature of about the same order of magnitude, 10^{39}, which are difficult to explain, such as the ratio of the electric to the gravitational force between an electron and a proton, the ratio of the age of the universe to the characteristic atomic time e^2/mc^3 (e and m are the electronic charge and mass), and the square root of the number of baryons in the universe. Dirac's large-number hypothesis asserts that such numbers are connected. In particular, since the first one is proportional to $1/G$, where G is Newton's gravitational constant, and the second one is proportional to t, the age of the universe, Dirac concluded that G varies as $1/t$. Since Einstein's general relativity theory gives a constant value for G, Dirac made use of Weyl's theory in order to have G vary with the time in accordance with his large-number hypothesis.

As for the difficulty with the nonintegrability of lengths, Dirac assumed that in practice one makes use of two different intervals between neighboring events in spacetime. Measurements in the laboratory with the help of apparatus fixed by the atomic properties of matter give an interval ds_A. This interval refers to atomic units and does not depend on an arbitrary gauge. On the other hand, there is an interval ds_E which is associated with the Einstein equations. This is modified by the Weyl theory so that it is nonintegrable under parallel displacement, and one must therefore refer it to an arbitrary gauge in order to get a definite value. The metric ds_E cannot be measured directly, but reveals itself through its influence on the motion of heavenly bodies.

Dirac modified the variational principle that Weyl had used. He introduced a Lagrangian multiplier involving a scalar β into the variational principle and was thus able to get satisfactory field equations without having to impose a special gauge. This scalar, or gauge function, β changes under a gauge transformation. Pietenpol *et al.*[4] criticized Dirac's work by showing that, with the help of β, one can get field variables that are independent of the gauge, so that the field equations are just the Einstein–Maxwell equations. According to them, the Weyl gauge invariance of the theory thus seems to be devoid of physical constent. However, Gregorash and Papini[5] pointed out that there is nevertheless a fundamental difference between the Einstein–Maxwell theory and that of Weyl in that in the latter the electromagnetic field is part of the geometry.

The Weyl–Dirac formalism has also been used in the absence of an electromagnetic field and has been referred to as a scale-covariant theory of gravitation. As an example, one might mention the work by Canuto *et al.*,[6] who chose the gauge so as to have G vary inversely to the cosmological time, in accordance with Dirac's large-number hypothesis, and then applied the field equations and equations of motion to problems in cosmology, astronomy, and astrophysics.

The purpose of the present work is to try to obtain a deeeper understanding of Weyl's geometry and its implications for physics.

2. WEYL GEOMETRY

In Riemannian geometry, which forms the basis of Einstein's general theory of relativity, one has a spacetime, the points of which are labeled by means of coordinates x^μ in a given coordinate system, and at each point there is a metric tensor $g_{\mu\nu}$ which determines the length of a vector at that point. Thus, for the infinitesimal displacement vector dx^μ one gets the interval ds according to the relation

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{1}$$

and for any other vector ξ^μ the length ξ is given by

$$(\xi)^2 = g_{\mu\nu}\xi^\mu\xi^\nu \tag{2}$$

Under a coordinate transformation $x^\mu \to x'^\mu$, the metric tensor is transformed, $g_{\mu\nu} \to g'_{\mu\nu}$, with

$$g'_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} \tag{3}$$

but the length of the vector is invariant.

Suppose that the vector ξ^μ is carried by parallel displacement from a point P to a point Q along a given curve. By this is meant that the change in the component ξ^μ, $d\xi^\mu$, is given by

$$d\xi^\mu = -\xi^\alpha \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} dx^\nu \tag{4}$$

when the vector is carried along the curve from x^μ to $x^\mu + dx^\mu$. Here, as usual, $\{ {}^{\lambda}_{\mu\nu} \}$ is the Christoffel 3-index symbol,

$$\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} = \frac{1}{2} g^{\lambda\alpha}(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}) \tag{5}$$

with a comma denoting a partial derivative. From (4) it follows that, in general, the components ξ^μ at Q will differ from those at P and will depend on the curve along which the vector is carried between the two points. On the other hand, it is easy to show from (4) and (5) that

$$d(\xi)^2 = 0 \tag{6}$$

so that the length of the vector at Q is equal to that at P whatever the curve may be. It should be remarked that the condition for parallel displacement given by (4) can also be written in terms of the covariant components of the vector

$$d\xi_\mu = \xi_\alpha \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} dx^\nu \tag{7}$$

In the Weyl geometry one also has the interval ds and the length of the vector ξ given by (1) and (2), but the metric tensor $g_{\mu\nu}$ is changed, not only under a coordinate transformation (CT) as in (3), but also under a gauge transformation (GT), $g_{\mu\nu} \to g'_{\mu\nu}$, with

$$g'_{\mu\nu} = e^{2\lambda} g_{\mu\nu} \tag{8}$$

where λ is an arbitrary (differentiable) function of the coordinates. It follows that $ds \to ds'$, with

$$ds' = e^{\lambda}\, ds \tag{9}$$

This possibility of a GT corresponds to the idea that one can take at each point an arbitrary standard of length, instead of taking a fixed standard for the whole spacetime as in tacitly done in Riemannian geometry. Putting it into more physical terms, if one has at each point a measuring rod and a clock (together a "four-dimensional measuring rod"), one can change the scales of both of them arbitrarily, but in the same ratio, so as to keep the speed of light unchanged. For we see that the equation of the light cone, $ds = 0$, remains invariant under the GT (9).

In the case of the vector ξ^μ, if we assume that under the GT the components ξ^μ remain unchanged, then the length changes, $\xi \to \xi'$, with

$$\xi' = e^\lambda \xi \tag{10}$$

In the Weyl geometry, corresponding to (4) one defines the parallel displacement of the vector ξ^μ by the relation

$$d\xi^\mu = -\xi^\alpha \Gamma^\mu_{\alpha\nu}\, dx^\nu \tag{11}$$

where $\Gamma^\mu_{\alpha\nu}$ is a 3-index symbol, or affine connection, which is in general different from that in (5). A distinguishing feature of this geometry is that, in general, (6) no longer holds. The length of the vector ξ changes in accordance with the relation

$$d\xi = \xi\phi_\mu\, dx^\mu \tag{12}$$

where ϕ_μ is a given vector which, together with $g_{\mu\nu}$, characterizes the geometry. It should be noted that, if ϕ_μ is the gradient of a function, then it follows from (12) that the change in the length ξ in going from one point to another is independent of the path followed.

After the GT one hs a corresponding relation

$$d\xi' = \xi'\phi_\mu{}'\, dx^\mu \tag{13}$$

If one makes use of (10) and (12), one finds

$$\phi_\mu{}' = \phi_\mu + \lambda_{,\mu} \tag{14}$$

as the GT for ϕ_μ. This is the familiar transformation that one has in the Maxwell theory for the electromagnetic potential vector, and it led Weyl to identify ϕ_μ with this vector, so that one has a connection between electromagnetism and geometry.

In order for (12) to be a consequence of (2) and (11) for arbitrary ξ^μ and dx^ν, one finds that

$$g_{\mu\alpha}\Gamma^\alpha_{\nu\lambda} + g_{\nu\alpha}\Gamma^\alpha_{\mu\lambda} = g_{\mu\nu,\lambda} - 2g_{\mu\nu}\phi_\lambda \tag{15}$$

If we assume that

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} \tag{16}$$

we get

$$\Gamma^{\sigma}_{\mu\nu} = \left\{ {\sigma \atop \mu\nu} \right\} + g_{\mu\nu}\phi^{\sigma} - \delta^{\sigma}_{\mu}\phi_{\nu} - \delta^{\sigma}_{\nu}\phi_{\mu} \tag{17}$$

Carrying out the GT (8) gives

$$\left\{ {\sigma \atop \mu\nu} \right\}' = \left\{ {\sigma \atop \mu\nu} \right\} - g_{\mu\nu}\lambda_{,\alpha} g^{\alpha\sigma} + \delta^{\sigma}_{\mu}\lambda_{,\nu} + g^{\sigma}_{\nu}\lambda_{,\mu} \tag{18}$$

If one takes into account the GT (14) for ϕ_{μ}, one sees that $\Gamma^{\sigma}_{\mu\nu}$ is invariant under the GT.

In the Weyl geometry one is interested in quantities which are invariant or covariant with respect to the GT (8). These will be indicated by the prefixes in- or co-, as proposed by Eddington.[7] A quantity is covariant under a GT if it is multiplied by $e^{n\lambda}$ $(n \neq 0)$. In that case it is said to have the power n.[3] Following Canuto *et al.*,[6] we will denote the power by Π, so that, according to (8), $\Pi[g_{\mu\nu}] = 2$, $\Pi[g^{\mu\nu}] = -2$.

If we have a co-scalar ψ with $\Pi = n$, so that under a GT $\psi \to \psi' = e^{n\lambda}\psi$, then

$$\psi'_{,\mu} = e^{n\lambda}(\psi_{,\mu} + n\psi\lambda_{,\mu}) \tag{19}$$

Making use of (14) we can define the co-covariant derivative

$$\psi_{*\mu} = \psi_{,\mu} - n\psi\phi_{\mu} \tag{20}$$

so that

$$\psi'_{*\mu} = e^{n\lambda}\psi_{*\mu} \tag{21}$$

This can be extended to co-tensors of any order. Let us define a covariant derivative with the help of $\Gamma^{\sigma}_{\mu\nu}$, to be denoted by a bar (|). For example,

$$A_{\mu|\nu} = A_{\mu,\nu} - A_{\alpha}\Gamma^{\alpha}_{\mu\nu} = A_{\mu;\nu} - g_{\mu\nu}A_{\alpha}\phi^{\alpha} + A_{\mu}\phi_{\nu} + A_{\nu}\phi_{\mu} \tag{22}$$

where $A_{\mu;\nu}$ is the Riemannian covariant derivative formed with the Christoffel symbol (5). If $\Pi(A_{\mu}) = n$, we can take as the co-covariant derivative

$$A_{\mu*\nu} = A_{\mu|\nu} - nA_{\mu}\phi_{\nu} \tag{23}$$

In the case of $g_{\mu\nu}$ one gets

$$g_{\mu\nu|\sigma} = 2g_{\mu\nu}\phi_{\sigma}, \qquad g_{\mu\nu*\sigma} = 0 \tag{24}$$

We have seen that a vector undergoing parallel displacement in general has a length at a given point depending on the path by which the vector reached this point. From the physical standpoint this seems to imply that the mass of a particle or the energy levels of an atom depend on their past history, and this is hard to reconcile with the fact that particle masses and atomic energy levels appear to have definite values in nature. This question will be considered in the next section.

3. PARALLEL DISPLACEMENT AND STANDARD VECTORS

Let us go back to the consideration of the parallel displacement of a vector. Suppose that at a given point we have ξ^μ and ξ_μ, these being related through the metric tensor, as usual. One can define parallel displacement either in terms of ξ^μ or in terms of ξ_μ, corresponding to (4) or (7) in the Riemannian case. However, whereas (4) and (7) are equivalent, in the Weyl case the two definitions lead to different results.

Let us consider ξ^μ, and let us therefore define parallel displacement by the condition (11). If we denote the change of the vector in this case by $d_1\xi^\mu$, we have

$$d_1\xi^\mu = -\xi^\alpha \Gamma^\mu_{\alpha\nu}\, dx^\nu \tag{25}$$

The corresponding change in the covariant vector is then

$$d_1\xi_\mu = d_1(g_{\mu\sigma}\xi^\sigma) = g_{\mu\sigma} d_1\xi^\sigma + \xi^\sigma g_{\mu\sigma,\nu}\, dx^\nu \tag{26}$$

Making use of (25), (17), and (5), one gets

$$d_1\xi_\mu = \xi_\alpha \Gamma^\alpha_{\mu\nu}\, dx^\nu + 2\xi_\mu \phi_\nu\, dx^\nu \tag{27}$$

From (25) and (27) it follows that

$$d_1(\xi)^2 = d_1(\xi_\mu \xi^\mu) = 2(\xi)^2 \phi_\nu\, dx^\nu \tag{28}$$

in agreement with (12).

Now let us carry out a parallel displacement of the vector in terms of its covariant components ξ_μ. Denoting the change in this case by $d_2\xi_\mu$, we take as the condition for parallel displacement, in analogy with (7),

$$d_2\xi_\mu = \xi_\alpha \Gamma^\alpha_{\mu\nu}\, dx^\nu \tag{29}$$

From this one readily finds

$$d_2\xi^\mu = -\xi^\alpha \Gamma^\mu_{\alpha\nu}\, dx^\nu - 2\xi^\mu \phi_\nu\, dx^\nu \tag{30}$$

and in consequence one gets

$$d_2(\xi)^2 = -2(\xi)^2\phi_\nu\, dx^\nu \tag{31}$$

We see that the two different kinds of parallel displacements give opposite signs to the change in the length of the vector. This suggests defining a third kind of parallel displacement by requiring the change to be $d\xi^\mu$ given by

$$d\xi^\mu = \tfrac{1}{2}(d_1\xi^\mu + d_2\xi^\mu) \tag{32}$$

that is,

$$d\xi^\mu = -\xi^\alpha \Gamma^\mu_{\alpha\nu}\, dx^\nu - \xi^\mu \phi_\nu\, dx^\nu \tag{33}$$

Similarly, let us take

$$d\xi_\mu = \tfrac{1}{2}(d_1\xi_\mu + d_2\xi_\mu) \tag{34}$$

so that

$$d\xi_\mu = \xi_\alpha \Gamma^\alpha_{\mu\nu}\, dx^\nu + \xi_\mu \phi_\nu\, dx^\nu \tag{35}$$

We can now define a new affine connection

$$\tilde{\Gamma}^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu} + \delta^\sigma_\mu \phi_\nu \tag{36}$$

and can write

$$d\xi^\mu = -\xi^\alpha \tilde{\Gamma}^\mu_{\alpha\nu}\, dx^\nu \tag{37}$$

$$d\xi_\mu = \xi_\alpha \tilde{\Gamma}^\alpha_{\mu\nu}\, dx^\nu \tag{38}$$

These two relations follow from each other, and they give

$$d(\xi)^2 = 0 \tag{39}$$

One can define a covariant derivative with the help of $\tilde{\Gamma}^\sigma_{\mu\nu}$, to be denoted by $\|$, as, for example,

$$\xi_{\mu\|\nu} = \xi_{\mu,\nu} - \xi_\alpha \tilde{\Gamma}^\alpha_{\mu\nu} \tag{40}$$

One then finds

$$g_{\mu\nu\|\sigma} = 0, \qquad g^{\mu\nu}{}_{\|\sigma} = 0 \tag{41}$$

It should be noted that $\xi_{\mu\|\nu}$ is co-covariant if $\Pi[\xi_\mu] = 1$. Similarly one finds that $\xi^\mu{}_{\|\nu}$ is co-covariant if $\Pi[\xi^\mu] = -1$. Let us call a vector with such powers a standard vector. It should be noted that, if ξ^μ is a standard vector, then its length ξ is gauge invariant. More generally, one can define a standard tensor as one which has a power equal to the number of covariant indices minus the number of contravariant ones.

In the case of a standard vector ξ_μ, one has

$$\xi_{\mu\|\nu} = \xi_{\mu_*\nu} \tag{42}$$

where the right side is in accordance with the definition given by (23). A corresponding equality holds for any standard tensor.

We see that, if elementary particles and atoms can be characterized by standard vectors and if their equations of motion are such that these vectors are carried from point to point by parallel displacements defined by (37) or (38), then the lengths of these vectors will remain constant. The intrinsic properties of these particles will remain unchanged, and their vectors will provide single-valued gauges at each point, corresponding to the atomic gauge of Dirac.[3]

The above can obviously be generalized to the case in which the standard vector is displaced in such a way that in (37) or (38) there is an additional (infinitesimal) terms on the right that is orthogonal to the vector. In this case, too, the length will remain constant. For example, let us associate with the particle a standard vector V^μ the motion of which is given by

$$DV^\mu/Ds \equiv V^\mu{}_{\|\alpha} u^\alpha = P^\mu \tag{43}$$

where $u_\alpha(=dx^\alpha/ds)$ is the particle velocity and

$$P^\mu V_\mu = 0 \tag{44}$$

Then the length V will be constant in the course of the motion. A very simple example of a standard vector is the velocity u^μ. A more interesting one is Φu^μ, where Φ is a gauge-invariant quantity characterizing the particle (its charge, for example).

Let us now consider the question: how does one determine the gauge at a given point? Let us discuss this with the help of an idealized procedure. Suppose we have a point P with coordinate x^μ and a metric tensor $g_{\mu\nu}$, together with its tangent space spanned by the coordinate differentials dx^μ. Let us imagine that in this tangent space the infinitesimal quantities dx^μ are magnified by multiplication with a large (constant) scale factor K to give finite quantities, and these are marked off on a rigid framework to give a finite local rectilinear coordinate system. Now we bring a measuring rod (in the four-dimensional sense), and we lay it off from P in some direction, so that it is represented by the vector S^μ. Then we can write

$$S^\mu = K\, dx^\mu \tag{45}$$

the dx^μ being given by the coordinate marks at the tip of the vector. Since we regard the measuring rod as having unit length, we have

$$g_{\mu\nu} S^\mu S^\nu = K^2 g_{\mu\nu}\, dx^\mu\, dx^\nu = 1 \tag{46}$$

We can give the vector S^μ various directions and in this way determine the metric $g_{\mu\nu}$, in principle.

To change the gauge we now bring a different measuring rod represented by the vector

$$S'^\mu = e^{-\lambda} S^\mu \tag{47}$$

When this is laid off from P, one gets

$$S'^\mu = K\, d'x^\mu \tag{48}$$

where $d'x^\mu$ is determined by the coordinate marks at the tip of the new vector, so that

$$d'x^\mu = e^{-\lambda}\, dx^\mu \tag{49}$$

If we now regard the new rod as having unit length, we get (for arbitrary directions)

$$g'_{\mu\nu} S'^\mu S'^\nu = K^2 g'_{\mu\nu}\, d'x^\mu\, d'x^\nu = 1 \tag{50}$$

where $g'_{\mu\nu}$ is the new metric tensor. Substituting (49) into (50) and comparing with (46), we get (8),

$$g'_{\mu\nu} = e^{2\lambda} g_{\mu\nu}$$

corresponding to our GT. According to (47), the vector S^μ has the power -1, and it is therefore a standard vector. If it is carried by parallel displacement according to (37) to various points, it provides a gauge at each point.

Of course we can have other vectors, behaving differently. Suppose, for example, we have at P a vector ξ^μ, the components of which are determined by the coordinate marks at its tip. When one changes the gauge, as described above, these components do not change. Hence $\Pi[\xi^\mu] = 0$ and therefore $\Pi[\xi_\mu] = 2$.

On the other hand, we can have a given scalar function ψ in the neighborhood of P independent of the gauge, and we can define a vector ζ_μ as the gradient of ψ,

$$\zeta_\mu = \psi_{,\mu} \tag{51}$$

Then $\Pi[\zeta_\mu] = 0$, $\Pi[\zeta^\mu] = -2$.

4. CURVATURE

In the previous section we dealt with two affine connections, $\Gamma^\sigma_{\mu\nu}$ and $\tilde{\Gamma}^\sigma_{\mu\nu}$. The first is invariant under a GT, but covariant derivatives formed with it are in general not co-covariant and require corrections to become so, as in (23). In the case of standard vectors (and, more generally, standard tensors) one gets co-covariant derivatives if one forms them with $\tilde{\Gamma}^\sigma_{\mu\nu}$ as in (40). It should be noted that

$$\tilde{\Gamma}^\sigma_{\mu\nu} - \tilde{\Gamma}^\sigma_{\nu\mu} = \delta^\sigma_\mu \phi_\nu - \delta^\sigma_\nu \phi_\mu \tag{52}$$

so that this affine connection describes a geometry with torsion. However, $\tilde{\Gamma}^\sigma_{\mu\nu}$ is not gauge-invariant, as is brought out by (52). For describing the geometry of spacetime it is convenient to have a gauge-invariant formalism; hence we will make use of $\Gamma^\sigma_{\mu\nu}$. Since the latter is symmetric, it describes a torsion-free geometry.

Going back to (22), if one calculates the second covariant derivatives of A_μ, one finds that

$$A_{\mu|\nu\sigma} - A_{\mu|\sigma\nu} = A_\lambda D^\lambda{}_{\mu\nu\sigma} \tag{53}$$

with

$$D^\lambda{}_{\mu\nu\sigma} = -\Gamma^\lambda_{\mu\nu,\sigma} + \Gamma^\lambda_{\mu\sigma,\nu} - \Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\alpha\sigma} + \Gamma^\alpha_{\mu\sigma}\Gamma^\lambda_{\alpha\nu} \tag{54}$$

the curvature in-tensor. Using (17), one can write

$$D^\lambda{}_{\mu\nu\sigma} = R^\lambda{}_{\mu\nu\sigma} + \Delta^\lambda{}_{\mu\nu\sigma} \tag{55}$$

where $R^\lambda{}_{\mu\nu\sigma}$ is the Riemann curvature tensor, given by an expression like that of (54), but with Christoffel symbols, and

$$\begin{aligned} \Delta^\lambda{}_{\mu\nu\sigma} = {} & \delta^\lambda_\mu F_{\nu\sigma} + g_{\mu\sigma}(\phi^\lambda{}_{;\nu} + \phi^\lambda \phi_\nu) - g_{\mu\nu}(\phi^\lambda{}_{;\sigma} + \phi^\lambda \phi_\sigma) \\ & + \delta^\lambda_\nu(\phi_{\mu;\sigma} + \phi_\mu \phi_\sigma - g_{\mu\sigma}\phi^\alpha \phi_\alpha) \\ & - \delta^\lambda_\sigma(\phi_{\mu;\nu} + \phi_\mu \phi_\nu - g_{\mu\nu}\phi^\alpha \phi_\alpha) \end{aligned} \tag{56}$$

On contracting, one gets

$$D_{\mu\nu} \equiv D^{\lambda}{}_{\mu\nu\lambda} = R_{\mu\nu} + F_{\nu\mu} - 2\phi_{\mu;\nu} - g_{\mu\nu}\phi^{\alpha}{}_{;\alpha} - 2\phi_{\mu}\phi_{\nu} + 2g_{\mu\nu}\phi^{\alpha}\phi_{\alpha} \quad (57)$$

where

$$F_{\mu\nu} = \phi_{\mu,\nu} - \phi_{\nu,\mu} \quad (58)$$

so that

$$D_{\mu\nu} - D_{\nu\mu} = 4F_{\nu\mu} \quad (59)$$

A further contraction gives

$$D \equiv g^{\mu\nu}D_{\mu\nu} = R - 6\phi^{\alpha}{}_{;\alpha} + 6\phi^{\alpha}\phi_{\alpha} \quad (60)$$

One also gets

$$D^{\lambda}{}_{\lambda\nu\sigma} = 4F_{\nu\sigma} \quad (61)$$

from which follows the useful relation

$$g^{\sigma\tau}(g_{\sigma\tau|\mu\nu} - g_{\sigma\tau|\nu\mu}) = 8F_{\mu\nu}$$

The symmetry properties of the curvature tensor are perhaps best seen if it is written in the covariant form,

$$\begin{aligned} D_{\rho\mu\nu\sigma} = R_{\rho\mu\nu\sigma} + g_{\rho\mu}F_{\nu\sigma} + g_{\mu\sigma}B_{\rho\nu} + g_{\rho\nu}B_{\mu\sigma} - g_{\mu\nu}B_{\rho\sigma} \\ - g_{\rho\sigma}B_{\mu\nu} + (g_{\rho\sigma}g_{\mu\nu} - g_{\rho\nu}g_{\mu\sigma})\phi^{\alpha}\phi_{\alpha} \end{aligned} \quad (62)$$

with

$$B_{\mu\nu} = \phi_{\mu;\nu} + \phi_{\mu}\phi_{\nu} \quad (63)$$

One has, for example,

$$D_{\rho\mu\nu\sigma} + D_{\mu\rho\nu\sigma} = 2g_{\rho\mu}F_{\nu\sigma} \quad (64)$$

from which (61) follows.

The curvature tensor characterizes the geometry of spacetime. Its contractions are important in connection with the field equations which determine the geometry and the physical phenomena taking place in it.

5. FIELD EQUATIONS AND EQUATIONS OF MOTION

Following Weyl[2] and Dirac,[3] let us derive the field equations from a variational principle

$$\delta I = 0 \tag{65}$$

with the in-invariant action

$$I = \int W(-g)^{1/2}\, d^4x \tag{66}$$

Since $\Pi[(-g)^{1/2}] = 4$, we must have $\Pi[W] = -4$. Let us then write, guided by Dirac's work but with some small modifications,

$$W = F_{\mu\nu}F^{\mu\nu} - \beta^2 D + k\beta_{*\mu}\beta_{*\underline{\mu}} + 2\Lambda\beta^4 + L_m \tag{67}$$

where an underlined index is to be raised with $g^{\mu\nu}$.

If we interpret the vector ϕ_μ as the electromagnetic potential, then $F_{\mu\nu}$ is the electromagnetic field tensor, and the first term on the right is the Maxwell scalar. On the other hand, one can regard it from the geometrical point of view as a scalar related to the curvature tensor, in view of (59) and (61). It has $\Pi = -4$.

The next term on the right contains the curvature scalar D given by (60). Now $\Pi[D] = -2$, and Weyl therefore worked with D^2 in order to get $\Pi = -4$. However, such a term leads to complicated field equations. In order to simplify matters, Weyl chose a gauge so as to make the Riemann curvature scalar R constant. This is not entirely satisfactory. Dirac therefore preferred to have a term which was more closely related to the term $-R$ that one has in the action of the Einstein general relativity theory. For this purpose he introduced a "Lagrangian multiplier" β^2, where β is a scalar function with $\Pi = -1$, so that $\Pi[\beta^2 D] = -4$, as required.

Once having introduced the scalar β, Dirac added the next two terms involving it. The first is a scalar with $\Pi = -4$ formed from the co-covariant derivative, corresponding to (20) with $n = -1$,

$$\beta_{*\mu} = \beta_{,\mu} + \beta\phi_\mu \tag{68}$$

the coefficient k being an arbitrary constant. The second involves β^4 (for which $\Pi = -4$) multiplied by an arbitrary constant which we write 2Λ. This term is regarded by Dirac as only of cosmological significance.

The term L_m has been added to describe the matter, which is the source of the fields.

Making use of (60) and (68), one can write (67) with $\sigma = k - 6$,

$$W = F_{\mu\nu}F^{\mu\nu} - \beta^2 R + (\sigma + 6)\beta_{,\mu}\beta_{,\underline{\mu}} + \sigma\beta^2\phi_\mu\phi^\mu + 2\sigma\beta\phi^\mu\beta_{,\mu} + 2\Lambda\beta^4 + L_m + 6(\beta^2\phi^\mu)_{;\mu} \tag{69}$$

where the last term can be discarded, since it gives a surface integral in (66).

If we now carry out the variation of (66) in the usual way, varying $g_{\mu\nu}$, ϕ_μ, and β (for details see Dirac[3]) and writing

$$\delta I = \int (V^{\mu\nu}\delta g_{\mu\nu} + Q^\mu\,\delta\phi_\mu + S\delta\beta)(-g)^{1/2}\,d^4x \tag{70}$$

we find

$$V^{\mu\nu} = \beta^2 G^{\mu\nu} + 8\pi M^{\mu\nu} + 8\pi T^{\mu\nu} + 2\beta(\beta_{;\underline{\mu}\underline{\nu}} - g^{\mu\nu}\beta_{;\alpha\underline{\alpha}}) + g^{\mu\nu}\beta_{,\alpha}\beta_{,\underline{\alpha}} - 4\beta_{,\underline{\mu}}\beta_{,\underline{\nu}} + \Lambda\beta^4 g^{\mu\nu} + \sigma(\tfrac{1}{2}g^{\mu\nu}\beta_{*\alpha}\beta_{*\underline{\alpha}} - \beta_{*\underline{\mu}}\beta_{*\underline{\nu}}) \tag{71}$$

$$Q_\mu = -4F^{\mu\nu}{}_{;\nu} + 2\sigma\beta\beta_{*\underline{\mu}} + 16\pi J^\mu \tag{72}$$

$$S = -2\beta R - 2(\sigma + 6)\beta_{;\alpha\underline{\alpha}} + 2\sigma\beta\phi_\alpha\phi^\alpha - 2\sigma\beta\phi^\alpha{}_{;\alpha} + 8\Lambda\beta^3 + \psi \tag{73}$$

Here $G^{\mu\nu}$ is the Einstein tensor and $M^{\mu\nu}$ is the Maxwell tensor, the energy–momentum density tensor of the electromagnetic field,

$$M^{\mu\nu} = (1/4\pi)(\tfrac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F^\mu{}_\alpha F^{\nu\alpha}) \tag{74}$$

The matter is described by the energy–momentum density tensor $T^{\mu\nu}$, the current-density vector J^μ, and the scalar ψ. These are defined in terms of L_m,

$$8\pi T^{\mu\nu} = (-g)^{-1/2}\,\delta[(-g)^{1/2}L_m]/\delta g_{\mu\nu} \tag{75}$$

$$16\pi J^\mu = \delta L_m/\delta\phi_\mu \tag{76}$$

$$\psi = \delta L_m/\delta\beta \tag{77}$$

It will be noted that $T^{\mu\nu}$ has been defined in such a manner that it appears in (71) with the same coefficient as that of $M^{\mu\nu}$. This seems reasonable since both tensors describe physical quantities of the same kind, e.g., energy and momentum density.

To get the field equations corresponding to the condition (65), we set $V^{\mu\nu}$, Q^μ, and S equal to zero. The equations can be written

$$G^{\mu\nu} = -\frac{8\pi}{\beta^2}(T^{\mu\nu} + M^{\mu\nu}) + \frac{2}{\beta}(g^{\mu\nu}\beta_{;\alpha\underline{\alpha}} - \beta_{;\underline{\mu}\underline{\nu}})$$

$$+ \frac{1}{\beta^2}(4\beta_{,\underline{\mu}}\beta_{,\underline{\nu}} - g^{\mu\nu}\beta_{,\alpha}\beta_{,\underline{\alpha}}) - \Lambda\beta^2 g^{\mu\nu}$$

$$+ \frac{\sigma}{\beta^2}\left(\beta_{*\underline{\mu}}\beta_{*\underline{\nu}} - \frac{1}{2}g^{\mu\nu}\beta_{*\alpha}\beta_{*\underline{\alpha}}\right) \tag{78}$$

$$F^{\mu\nu}{}_{;\nu} = \tfrac{1}{2}\sigma(\beta^2\phi_\mu + \beta\beta_{,\underline{\mu}}) + 4\pi J^\mu \tag{79}$$

$$R = -(\sigma + 6)\beta_{;\alpha\underline{\alpha}}/\beta + \sigma\phi_\alpha\phi^\alpha - \sigma\phi^\alpha{}_{;\alpha} + 4\Lambda\beta^2 + \psi/2\beta \tag{80}$$

From (78), since $G^{\alpha}_{\underline{\alpha}} = -R$, one gets

$$R = \frac{8\pi}{\beta^2}T - \frac{6}{\beta}\beta_{;\alpha\underline{\alpha}} + \frac{\sigma}{\beta^2}(\beta_{,\alpha}\beta_{,\underline{\alpha}} + 2\beta\phi^\alpha\beta_{,\alpha} + \beta^2\phi^\alpha\phi_\alpha) + 4\Lambda\beta^2 \tag{81}$$

Combining this with (80) gives

$$8\pi T + \sigma(\beta^2\phi^\mu + \beta\beta_{,\underline{\mu}})_{;\mu} - \tfrac{1}{2}\beta\psi = 0 \tag{82}$$

From (79) one gets

$$\sigma(\beta^2\phi_\mu + \beta\beta_{,\underline{\mu}})_{;\mu} + 8\pi J^\mu{}_{;\mu} = 0 \tag{83}$$

If $\sigma = 0$, this gives

$$J^\mu{}_{;\mu} = 0 \tag{84}$$

If $\sigma \neq 0$, one can assume that (84) holds, although one is not forced to. In that case

$$(\beta^2\phi^\mu + \beta\beta_{,\underline{\mu}})_{;\mu} = 0 \tag{85}$$

and (82) gives for any value of σ

$$16\pi T - \beta\psi = 0 \tag{86}$$

Let us now take the divergence of (78). After some calculation one finds

$$G^{\mu\nu}{}_{;\nu} = -(8\pi/\beta^2)(T^{\mu\nu}{}_{;\nu} - T\beta_{,\underline{\mu}}/\beta + F^{\mu\alpha}J_\alpha)$$
$$+ (\sigma/\beta^2)(\phi^\mu + \beta_{,\underline{\mu}}/\beta)(\beta^2\phi^\alpha + \beta\beta_{,\underline{\alpha}})_{;\alpha} \tag{87}$$

Since

$$G^{\mu\nu}{}_{;\nu} \equiv 0 \tag{88}$$

one gets, with the help of (83),

$$T^{\mu\nu}{}_{;\nu} - T\beta_{,\underline{\mu}}/\beta = J_\alpha F^{\alpha\mu} - (\phi^\mu + \beta_{,\underline{\mu}}/\beta)J^\alpha{}_{;\alpha} \tag{89}$$

If (84) holds, this gives

$$T^{\mu\nu}{}_{;\nu} - T\beta_{,\underline{\mu}}/\beta = J_\alpha F^{\alpha\mu} \tag{90}$$

These are the equations of motion for the matter.

From the form of the right side of (78) we see that we can assign to the nongravitational part of the field an energy–momentum density tensor

$$\begin{aligned} T_f^{\mu\nu} = M^{\mu\nu} + (\sigma/8\pi)[\tfrac{1}{2}g^{\mu\nu}(\beta\phi^\alpha + \beta_{,\underline{\alpha}})(\beta\phi_\alpha + \beta_{,\alpha}) \\ - (\beta\phi^\mu + \beta_{,\underline{\mu}})(\beta\phi^\nu + \beta_{,\underline{\nu}})] \end{aligned} \tag{91}$$

Making use of (79) and (83), one finds

$$(T_f^{\mu\nu})_{;\nu} - T_f\beta_{,\underline{\mu}}/\beta = F^{\mu\alpha}J_\alpha + (\phi^\mu + \beta_{,\underline{\mu}}/\beta)J^\alpha{}_{;\alpha} \tag{92}$$

Comparing this with (89), we have

$$(T^{\mu\nu} + T_f^{\mu\nu})_{;\nu} - (T + T_f)\beta_{,\underline{\mu}}/\beta = 0 \tag{93}$$

as the general energy–momentum relation.

Let us now consider the simple example of matter consisting of identical particles in the form of (pressureless) dust, so that

$$T^{\mu\nu} = \rho_m u^\mu u^\nu \tag{94}$$

where the scalar mass density ρ_m is given by

$$\rho_m = m\rho_n \tag{95}$$

with ρ_n the particle density and m the rest mass of a particle. The conservation of the number of particles is described by

$$(\rho_n u^\mu)_{;\mu} = 0 \tag{96}$$

or

$$[\rho_n(-g)^{1/2}u^\mu]_{,\mu} = 0 \tag{97}$$

Since this must be an in-invariant relation, it follows that

$$\Pi[\rho_n] = -3 \tag{98}$$

Let us suppose that the particles considered above each have an electric charge e, assumed to be gauge-invariant. Then, if we write

$$J^{\mu} = e\rho_n u^{\mu} = (e/m)\rho_m u^{\mu} \tag{99}$$

it follows from (96) that charge is conserved, i.e., that (84) holds. Furthermore, from (98) and (99) we have $\Pi[J^{\mu}] = -4$, and this agrees with the powers of the other terms appearing in (79).

Let us now substitute (94) and (99) into (90). We get

$$\rho_m u^{\mu}{}_{;\nu} u^{\nu} + (\rho_m u^{\nu})_{;\nu} u^{\mu} - \rho_m \beta_{,\underline{\mu}}/\beta = (e/m)\rho_m u_{\alpha} F^{\alpha\mu} \tag{100}$$

Multiplying by u_{μ} gives, with $u_{\mu}u^{\mu} = 1$,

$$(\rho_m u^{\nu})_{;\nu} - \rho_m u^{\nu}\beta_{,\nu}/\beta = 0 \tag{101}$$

or

$$\left(\frac{\rho_m}{\beta} u^{\mu}\right)_{;\mu} = 0 \tag{102}$$

Making use of (101) in (100) gives

$$u^{\mu}{}_{;\alpha} u^{\alpha} + (u^{\mu}u^{\alpha} - g^{\mu\alpha})\beta_{,\alpha}/\beta = (e/m)u_{\alpha}F^{\alpha\mu} \tag{103}$$

which can also be written

$$\frac{du^{\mu}}{ds} + \left\{ \begin{matrix} \mu \\ \alpha\beta \end{matrix} \right\} u^{\alpha}u^{\beta} + \frac{(u^{\mu}u^{\alpha} - g^{\mu\alpha})\beta_{,\alpha}}{\beta} = \frac{e}{m} u_{\alpha}F^{\alpha\mu} \tag{104}$$

as the equation of motion of a particle in the co-covariant form.

Let us go back to (102) and write it, with the help of (95),

$$\left(\frac{m}{\beta}\rho_n u^{\mu}\right)_{;\mu} = 0 \tag{105}$$

Comparing this with (96), we are led to write

$$m = m_0\beta \qquad (m_0 = \text{const}) \tag{106}$$

so that $\Pi[m] = -1$, and $\Pi[\rho_m] = -4$. The present approach differs from that of Dirac,[3] who took $\Pi[m] = 0$.

6. IDENTITIES

Since the action (66) is invariant both under a GT and under a CT, there must exist two corresponding identities among the field equations.

Let us consider the variation δI resulting from an infinitesimal GT. (In view of the invariance of I, δI should vanish identically.) If we take $|\lambda| \ll 1$, then we can write for the GT

$$\delta g_{\mu\nu} = 2\lambda g_{\mu\nu}, \qquad \delta\phi_\mu = \lambda_{,\mu}, \qquad \delta\beta = -\lambda\beta \tag{107}$$

Putting these into (70) and integrating the second term by parts give

$$\delta I = \int (2V - Q^\mu{}_{;\mu} - \beta S)\,\lambda(-g)^{1/2}\, d^4x \tag{108}$$

For this to vanish with λ arbitrary, we have the identity

$$2V - Q^\mu{}_{;\mu} - \beta S \equiv 0 \tag{109}$$

From (71) (with $G^\alpha_\alpha = -R$)

$$V = -\beta^2 R + 8\pi T - 6\beta\beta_{;\alpha\underline{\alpha}} + \sigma\beta_{*\alpha}\beta_{*\underline{\alpha}} + 4\Lambda\beta^4 \tag{110}$$

while (72) gives

$$Q^\mu{}_{;\mu} = 2\sigma(\beta\beta_{*\underline{\mu}})_{;\mu} + 16\pi J^\mu{}_{;\mu} \tag{111}$$

These, together with (73) for S, substituted into (109), give

$$16\pi T - 16\pi J^\mu{}_{;\mu} - \beta\psi = 0 \tag{112}$$

Comparing (112) with (82), we get (83). This means that (80) is not an independent equation, but rather follows from (78) and (79) if one requires gauge invariance of the field sources.

Now let us consider the infinitesimal CT given by

$$x^\mu = x'^\mu + \xi^\mu \tag{113}$$

where ξ^μ is an infinitesimal vector. As usual, we compare $g_{\mu\nu}$ at a given point before the transformation with $g'_{\mu\nu}$ after it at a point having the same coordinates, so that one finds

$$g'_{\mu\nu}(x) = g_{\mu\nu}(x) + g_{\mu\alpha}\xi^\alpha_{,\nu} + g_{\alpha\nu}\xi^\alpha_{,\mu} + g_{\mu\nu,\alpha}\xi^\alpha \tag{114}$$

This gives

$$\delta g_{\mu\nu} = g_{\mu\alpha}\xi^\alpha_{,\nu} + g_{\alpha\nu}\xi^\alpha_{,\mu} + g_{\mu\nu,\alpha}\xi^\alpha \tag{115}$$

Similarly one gets

$$\delta\phi_\mu = \phi_\alpha \xi^\alpha_{,\mu} + \phi_{\mu,\alpha}\xi^\alpha \tag{116}$$

and

$$\delta\beta = \beta_{,\alpha}\xi^\alpha \tag{117}$$

Putting these variations into (70) and carrying out integrations by parts, one gets a result which can be written

$$\delta I = \int (-2V^\nu_{\mu;\nu} - Q^\nu{}_{;\nu}\phi_\mu + Q^\nu F_{\nu\mu} + S\beta_{,\mu})\,\xi^\mu(-g)^{1/2}\,d^4x \tag{118}$$

For this to vanish with ξ^μ arbitrary, we have the identity

$$2V^{\mu\nu}{}_{;\nu} + \phi^\mu Q^\nu{}_{;\nu} - Q_\nu F^{\nu\mu} - S\beta_{,\underline{\mu}} \equiv 0 \tag{119}$$

If we make use of (71)–(73), this gives

$$16\pi(T^{\mu\nu}{}_{;\nu} + \phi^\mu J^\nu{}_{;\nu} - J_\nu F^{\nu\mu}) - \psi\beta_{,\underline{\mu}} = 0 \tag{120}$$

Eliminating ψ between this equation and (112), we get (89).

We see that the identities obtained on the basis of invariance under transformations do not give anything really new—they serve mainly as checks on the results obtained previously.

7. FIELDS AND GAUGES

Let us go back to the field equations (78)–(80). We have seen that (80) follows from the others, so that it is enough to work with (78) and (79).

The equations contain the arbitrary parameter σ. Dirac,[3] for the sake of simplicity, took $k = 6$, or $\sigma = 0$. In that case the equations have the form

$$G_{\mu\nu} = -(8\pi/\beta^2)(T_{\mu\nu} + M_{\mu\nu}) + (2/\beta)(g_{\mu\nu}\beta_{;\alpha\underline{\alpha}} - \beta_{;\mu\nu}) + (1/\beta^2)(4\beta_{,\mu}\beta_{,\nu} - g_{\mu\nu}\beta_{,\alpha}\beta_{,\underline{\alpha}}) - \Lambda\beta^2 g_{\mu\nu} \tag{121}$$

$$F^{\mu\nu}{}_{;\nu} = 4\pi J^\mu \tag{122}$$

From (122) one gets (84),

$$J^\mu{}_{;\mu} = 0$$

so that this is a consequence of the field equations and is not an assumption. From (121) one then gets (90) [see (87) and (88)]

$$T^{\mu\nu}{}_{;\nu} - T\beta_{,\underline{\mu}}/\beta = J_\alpha F^{\alpha\mu}$$

The function β, with $\Pi = -1$, is not determined by the equations and can be chosen arbitrarily. Since it is changed by a GT, one can regard it as determining the gauge, and one can call it the gauge function. It is always possible to carry out a GT with $e^\lambda = \beta$, so that after the transformation one has

$$\bar{g}_{\mu\nu} = \beta^2 g_{\mu\nu}, \qquad \bar{\phi}_\mu = \phi_\mu + \beta_{,\mu}/\beta, \qquad \bar{\beta} = 1 \tag{123}$$

The field equations after this GT then have the form

$$\bar{G}_{\mu\nu} = -8\pi(\bar{T}_{\mu\nu} + \bar{M}_{\mu\nu}) - \Lambda\bar{g}_{\mu\nu} \tag{124}$$

$$\bar{F}^{\mu\nu}{}_{;\nu} = 4\pi\bar{J}^\mu \tag{125}$$

where an overbar indicates the quantity in the new gauge. Equations (124) and (125) are the Einstein–Maxwell equations of general relativity with a cosmological term. The gauge with $\bar{\beta} = 1$ used here is often referred to as the Einstein gauge.

Other choices of gauge function are possible, of course. In view of the co-covariance of the field equations, one can say that their physical contents are independent of the choice of gauge function. However, the functional dependence of the fields on the coordinates will depend on β, and the physical interpretation may also depend on it. For example, as we shall see later, from the coefficient of $T_{\mu\nu}$ in (121) one can conclude that Newton's gravitational constant G can be written

$$G = G_0/\beta^2 \tag{126}$$

where G_0 is the value of G for $\beta = 1$. Hence, if β varies with time, the gravitational "constant" will also vary. Another example to be considered is connected with cosmology: by a suitable choice of gauge one can eliminate the "big bang" that one has in cosmological models based on the Einstein gauge.

Let us consider the fields in empty space, i.e., for $T^{\mu\nu} = 0$, $J^\mu = 0$. Since ϕ_μ appears in the field equations only in $F_{\mu\nu}$, we see that, in addition to the usual GT, one can have a GT only for ϕ_μ,

$$\phi_\mu \to \phi'_\mu = \phi_\mu + f_{,\mu} \tag{127}$$

where f is an arbitrary function, without any change in $g_{\mu\nu}$ or β, as Dirac[3] pointed out. However, from the standpoint of the Weyl theory one must regard (127) as limited only to the relation between $F_{\mu\nu}$ and ϕ_μ. The vector ϕ_μ as defined by (12) can only change by a GT that also changes $g_{\mu\nu}$ and β.

Now let us go back to the field equations (78) and (79) with $\sigma \neq 0$. This case has been discussed by Gregorash and Papini,[5] although from a standpoint different from that to be presented here.

We have seen that (79) gives (83), and if we assume charge conservation as given by (84), we get (85).

Let us take the gauge with $\beta = 1$. Then (78) and (79) give

$$G^{\mu\nu} = -8\pi(T^{\mu\nu} + M^{\mu\nu}) + \sigma(\phi^\mu\phi^\nu - \tfrac{1}{2}g^{\mu\nu}\phi_\alpha\phi^\alpha) - \Lambda g^{\mu\nu} \tag{128}$$

$$F^{\mu\nu}{}_{;\nu} = \tfrac{1}{2}\sigma\phi^\mu + 4\pi J^\mu \tag{129}$$

Consider (129) in empty space, $J^\mu = 0$. If we write $\sigma = -2\kappa^2$, we get

$$F^{\mu\nu}{}_{;\nu} + \kappa^2\phi^\mu = 0 \tag{130}$$

which is a generalization of the Proca[8] equation for a vector meson field. From it follows that

$$\phi^\mu{}_{;\mu} = 0 \tag{131}$$

If we express $F^{\mu\nu}$ in terms of ϕ_μ and note that

$$\phi^\nu{}_{;\mu\nu} = \phi^\nu{}_{;\nu\mu} - \phi_\alpha R^\alpha{}_\mu \tag{132}$$

(130) gives

$$\phi_{\mu;\alpha\underline{\alpha}} + \phi_\alpha R^\alpha{}_\mu + \kappa^2\phi_\mu = 0 \tag{133}$$

If the curvature is negligible, we can write

$$\phi^\mu{}_{;\alpha\underline{\alpha}} + \kappa^2\phi^\mu = 0 \tag{134}$$

This and (131) are the equations for the vector meson field. According to quantum mechanics, in conventional units (see Appendix)

$$\kappa = mc/\hbar = 2\pi/\lambda_c \tag{135}$$

where m is the meson mass and λ_c is its Compton wavelength.

We see that for $\sigma \neq 0$ we have, in addition to gravitation, a field which, from the quantum mechanical standpoint, consists of particles of finite rest mass. If one takes the Weyl theory seriously, one must consider this field as playing a fundamental role in nature. What is the physical significance of this field? We are now in the realm of speculation. One possibility is that we are dealing here with the electromagnetic field regarded as made up of photons of finite mass. If the mass m in (135) is very small, or λ_c very large, this meson field will be practically indistinguishable from the electromagnetic field, except for the fact that there exists the condition found earlier in (85),

$$(\beta^2\phi^\mu + \beta\beta_{,\underline{\mu}})_{;\mu} = 0$$

which gives (131) for $\beta = 1$.

Another possibility is that we have here a meson field, which interacts extremely weakly with ordinary matter. These mesons could conceivably accumulate at the centers of galaxies and galaxy clusters and could provided the "missing mass" that is needed to give a closed universe.

For $\sigma \neq 0$ the same equations of motion (90) and (104) hold as in the case $\sigma = 0$, but e is now the charge of the particle that interacts with the meson field—which may turn out to be the electromagnetic field, after all.

8. THE COSMIC GAUGE

In order to apply the field equations to cosmological models let us take $J^\mu = 0$ and $F_{\mu\nu} = 0$. From (79) it follows that

$$\sigma(\beta\phi^\mu + \beta_{,\underline{\mu}}) = 0 \tag{136}$$

so that (78) can be written

$$G^\nu_\mu = -\frac{8\pi}{\beta^2} T^\nu_\mu + \frac{2}{\beta}(\delta^\nu_\mu \beta_{;\alpha\underline{\alpha}} - \beta_{;\mu\underline{\nu}}) + \frac{1}{\beta^2}(4\beta_{,\mu}\beta_{,\underline{\nu}} - \delta^\nu_\mu \beta_{,\alpha}\beta_{,\underline{\alpha}}) - \Lambda\beta^2\delta^\nu_\mu \tag{137}$$

Assuming that the universe is homogeneous and isotropic, we can write[9]

$$ds^2 = dt^2 - \frac{R^2(t)}{R_0^2} dl^2 \tag{138}$$

with $R(t)$ describing the scale of the universe (or radius), $R_0 = R(t_0)$, where t_0 is the present value of the cosmic time t, and

$$dl^2 = (1 + kr^2/4R_0^2)^{-2}(dr^2 + r^2\, d\Omega^2) \tag{139}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \tag{140}$$

Here dl^2 is the three-dimensional time-independent line element, and k is now the spatial curvature parameter ($k=0, \pm 1$). Let us denote (t, r, θ, ϕ) by (x^0, x^1, x^2, x^3).

In accordance with our assumptions let us take $\beta=\beta(t)$ and

$$T_0^0=\rho(t), \qquad T_j^k=-p(t)\delta_j^k, \qquad T_0^k=T_k^0=0 \tag{141}$$

where ρ and p are the density and pressure, respectively, of the matter. We then get from (137), with a dot now denoting a derivative with respect to t,

$$-\frac{3\dot{R}^2}{R^2}-\frac{3k}{R^2}=-\frac{8\pi\rho}{\beta^2}+\frac{6\dot{R}\dot{\beta}}{R\beta}+\frac{3\dot{\beta}^2}{\beta^2}-\Lambda\beta^2 \tag{142}$$

$$-\frac{2\ddot{R}}{R}-\frac{\dot{R}^2}{R^2}-\frac{k}{R^2}=\frac{8\pi p}{\beta^2}+\frac{2\ddot{\beta}}{\beta}+\frac{4\dot{R}\dot{\beta}}{R\beta}-\frac{\dot{\beta}^2}{\beta^2}-\Lambda\beta^2 \tag{143}$$

The first equation is for $(\mu, \nu)=(0, 0)$, the second for $(\mu, \nu)=(1, 1)$, $(2, 2)$, $(3, 3)$. We also have to consider (90), which reduces to a single relation,

$$\dot{\rho}+\frac{3\dot{R}}{R}(\rho+p)+\frac{\dot{\beta}}{\beta}(3p-\rho)=0 \tag{144}$$

Furthermore, ρ and p are assumed to be related by an equation of state.

If one takes the gauge function $\beta=1$, one gets the usual cosmological equations of general relativity (with the cosmological term)

$$-\frac{3\dot{R}^2}{R^2}-\frac{3k}{R^2}=-8\pi\bar{\rho}-\Lambda \tag{145}$$

$$-\frac{2\ddot{R}}{R}-\frac{\dot{R}^2}{R^2}-\frac{k}{R^2}=8\pi\bar{p}-\Lambda \tag{146}$$

$$\dot{\bar{\rho}}+\frac{3\dot{R}}{R}(\bar{\rho}+\bar{p})=0 \tag{147}$$

We denote the metric in this case by $\bar{g}_{\mu\nu}$ and the density and pressure by $\bar{\rho}$ and $\bar{p}$. For a given equation of state these equations determine $R(t)$. With $\dot{R}(t_0)>0$ one can account for the Hubble red shift.[10]

Now let carry out a GT to $g_{\mu\nu}$ with

$$g_{\mu\nu}=(R_0^2/R^2)\,\bar{g}_{\mu\nu}, \qquad \beta=R/R_0 \tag{148}$$

In place of $d\bar{s}^2$ as given by (138) we now have

$$ds^2=(R_0^2/R^2)\,dt^2-dl^2 \tag{149}$$

If we introduce a new time coordinate T,

$$T = \int_0^t [R_0/R(t)]\, dt \tag{150}$$

we get

$$ds^2 = dT^2 - dl^2 \tag{151}$$

so that we now have a static universe.

The question now arises: how can one understand the existence of the Hubble effect in such a static universe, in which distances between galaxies are constant in time? For this purpose let us go back to (106). One can conclude from it that the energy levels of an atom will increase with time like β, so that, on the basis of quantum theory, the frequencies of spectral lines will increase in the same way, i.e.,

$$\nu = \nu_0 \beta \qquad (\nu_0 = \text{const}) \tag{152}$$

Let us suppose that at time T_e an atom in a distant galaxy emits light of frequency ν_e which reaches the earth at the present time T_0. It will have the same frequency ν_e on arrival in view of the static form of (151). However, an atom of the same kind on the earth will emit at time T_0 a higher frequency ν_0 according to (152) and (148). We have

$$\nu_e/\nu_0 = R(T_e)/R(T_0) \tag{153}$$

which is just the red-shift relation[10] for $R(T_0) > R(T_e)$.

With (151) holding, one can rewrite (142)–(144) accordingly by taking $R = R_0$, $\dot{R} = 0$. These then become

$$-\frac{3\dot{\beta}^2}{\beta^2} - \frac{3k}{R_0^2} = -\frac{8\pi\rho}{\beta^2} - \Lambda\beta^2 \tag{154}$$

$$-\frac{2\ddot{\beta}}{\beta} + \frac{\dot{\beta}^2}{\beta^2} - \frac{k}{R_0^2} = \frac{8\pi p}{\beta^2} - \Lambda\beta^2 \tag{155}$$

$$\dot{\rho} + \frac{\dot{\beta}}{\beta}(3p - \rho) = 0 \tag{156}$$

with a dot now denoting a derivative with respect to T. It is easily verified that, if one takes β as in (148) and one goes from T to t with the help of (150), the above equations give (145)–(147) provided one writes

$$\bar{\rho} = \rho/\beta^4, \qquad \bar{p} = p/\beta^4 \tag{157}$$

It should be remarked that, while in general the gauge function β is arbitrary, if we fix the metric, as in (151), β is determined by the field equations.

Although we are dealing with the same physical situation, we can expect that the solutions of the present equations for β expressed in terms of T will look different from those of (145)–(147) for R/R_0 in terms of t. Let us consider some simple cases.

8.1. Dust

In the case of a model of the universe filled with matter for which $p = 0$ (dust), (156) gives

$$\rho = \rho_0 \beta \qquad (\rho_0 = \text{const}) \tag{158}$$

If this is substituted into (154), we get an equation that can be easily solved if we set $\Lambda = 0$. Taking $T = 0$ as the time at which $\beta = 0$, one gets as solutions

$$k = 0: \qquad \beta = (2\pi\rho_0/3)\,T^2 \tag{159}$$

$$k = 1: \qquad \beta = (8\pi\rho_0 R_0^2/3)\sin^2(T/2R_0) \tag{160}$$

$$k = -1: \qquad \beta = (8\pi\rho_0 R_0^2/3)\sinh^2(T/2R_0) \tag{161}$$

8.2. Radiation

For a universe filled with isotropic radiation, for which

$$p = \tfrac{1}{3}\rho \tag{162}$$

(156) gives

$$\rho = \rho_0 \tag{163}$$

and, again with $\Lambda = 0$, the solutions of the equations are given by

$$k = 0: \qquad \beta = (8\pi\rho_0/3)^{1/2}\,T \tag{164}$$

$$k = 1: \qquad \beta = (8\pi\rho_0/3)^{1/2} R_0 \sin(T/R_0) \qquad (0 \leqslant T/R_0 \leqslant \pi) \tag{165}$$

$$k = -1: \qquad \beta = (8\pi\rho_0/3)^{1/2} R_0 \sinh(T/R_0) \tag{166}$$

We have here what might be called the cosmic gauge, in which the standard of length is proportional to the scale (or radius) of the universe. Hence the geometry of the universe is static; it is the gauge function β that changes with time. We can describe all phenomena with this gauge, but we may have to change some of our concepts. We see, for example, that at $T=0$ there is no "big bang," such as one gets with the Einstein gauge, $\beta=1$. During the early, radiation-dominated period the density is constant, as in (163), and at $T=0$ any particles present have zero rest masses according to (106).

We have been discussing the function β. What about the vector ϕ_μ? It is natural to take, for the cosmological models with the Einstein gauge,

$$\bar{\phi}_\mu = 0 \tag{167}$$

Going over to the cosmic gauge, we then have

$$\phi_\mu = -\beta_{,\mu}/\beta \tag{168}$$

From (12) we find for the length L of a parallely displaced vector

$$L = L_0/\beta \qquad (L_0 = \text{const}) \tag{169}$$

Let us go back to the field equations (78). We see that the coefficient of $T^{\mu\nu}$ is $8\pi/\beta^2$. In conventional units the coefficient would be written $8\pi G/c^4$, with G the Newtonian gravitational constant and c the speed of light. It follows that

$$G = G_0/\beta^2$$

where G_0 is the value of G at the present time (at which $\beta=1$).

We saw earlier that the mass of a particle m also depends on β,

$$m = m_0\beta$$

In the Newtonian approximation the gravitational potential at a distance r from the particle is therefore given by

$$\Phi = -Gm/r = -G_0 m_0/\beta r \tag{170}$$

This suggests introducing an "effective" gravitational constant, in relation to the present value of the mass m_0,

$$G_e = G_0/\beta \tag{171}$$

We see that G_e decreases with time. Its behavior, at least qualitatively, corresponds to what Dirac[3] concluded on the basis of his large-number hypothesis. However, this hypothesis will not be adopted in the present work.

Equations (170) and (171) may be significant in astronomical problems. On the other hand, it should be noted that the potential energy of masses m and M a distance r apart is given by

$$V = -GmM/r = -G_0 m_0 M_0/r \tag{172}$$

and this is independent of β.

In the discussion of the Hubble effect it was stated that, if the mass of a particle is given by (106), the same relation holds for all the energy levels of an atom. One can raise the question: is this statement in agreement with the laws determining the energy levels of an atom? In other words, can one show that, if (106) holds for all the particle masses, it also holds for all the atomic energy levels?

Consider the Schrödinger equation for a hydrogen atom consisting of a proton of mass M and an electron of mass m. The energies of the bound states are given by the Bohr relation

$$E_n = -\mu e^4/2\hbar^2 n^2 \qquad (n = 1, 2, 3, \dots) \tag{173}$$

Here μ is the reduced mass, i.e.,

$$\mu = mM/(m+M) = \mu_0 \beta \qquad (\mu_0 = \text{const}) \tag{174}$$

on the basis of (106). We see that in this case E_n indeed varies as β if we take the electron charge e and Planck's constant $2\pi\hbar$ to be constant. Similarly, one finds that the Dirac equation for an electron in the electric field of a fixed proton gives energy levels that are proportional to m and hence to β.

More generally, in the nonrelativistic case, let us suppose that we have a small system consisting of N charged particles, so that the time-independent Schrödinger equation has the form

$$H\Psi = E\Psi \tag{175}$$

where

$$H = -\sum \frac{\hbar^2}{2m_j} \nabla_j^2 + \sum_{j<k} \frac{e_j e_k}{r_{jk}} \tag{176}$$

with the usual notation. For convenience let us denote all the Cartesian coordinates of all the particles by x_i $(i = 1, 2, \dots, 3N)$. If all the masses depend on β as in (106), let us introduce a change of scale (regarding β as constant under the conditions being considered)

$$x_i = x_{0i}/\beta \tag{177}$$

It readily follows that

$$H = H_0 \beta \tag{178}$$

where H_0 is expressed in terms of x_{0i} and is independent of β. From (175) we then have

$$E = E_0 \beta \qquad (E_0 = \text{const}) \tag{179}$$

so that the energy levels behave like the masses. If, instead of electrostatic interactions, there are gravitational interactions, one gets the same result with the help of (172).

If in (175) one makes use of (106) and (178), one sees that the solution is now given by

$$\Psi = \Psi(x_{0i}) = \Psi(\beta x_i) \tag{180}$$

It follows from the last expression that a typical dimension of the physical system, say L, will change with time according to the relation

$$L = L_0/\beta \qquad (L_0 = \text{const}) \tag{181}$$

It is interesting that this has the same form as (169) for the length of a vector undergoing parallel displacement.

We see that, while in the Einstein gauge small physical systems—atoms, for example—have constant masses and sizes and the universe expands, in the cosmic gauge the universe has a constant size and, in the case of small systems, the masses increase and the sizes decrease with time.

If we assume that (181) applies also to the dimensions of an elementary particle, it follows that at $T = 0$ when there was $\beta = 0$, if elementary particles were present, they had very large sizes, presumably much larger than the distances between neighboring particles. The matter of the universe must have been in a state with very interesting properties.

9. PARTICLE FIELD IN COSMIC GAUGE

Having been considering the cosmic gauge, let us now investigate the field of a particle in this gauge. Suppose there is a particle at rest at the origin, and we consider a region around the particle which is so small that we can neglect the cosmic curvature (if there is any). Hence we take $k = 0$ in (139). Similarly let us neglect the cosmological constant Λ. Furthermore, let us assume that in this region we can neglect the other matter in the universe, so that we can take $T_{\mu\nu} = 0$ in the field equations.

Let us then write

$$ds^2 = e^{\nu}\,dt^2 - e^{\lambda}\,dr^2 - r^2\,d\Omega^2 \tag{182}$$

with ν and λ functions of t and r, and let us take the gauge function $\beta(t)$ as given. The field equations (137) for $(\mu, \nu) = (0, 0)$, $(1, 1)$, and $(1, 0)$ give

$$e^{-\lambda}\left(-\frac{\lambda_1}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = e^{-\nu}\left(\frac{\beta_0\lambda_0}{\beta} + \frac{3\beta_0^2}{\beta^2}\right) \tag{183}$$

$$e^{-\lambda}\left(\frac{\nu_1}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = e^{-\nu}\left(\frac{2\beta_{00}}{\beta} - \frac{\beta_0\nu_0}{\beta} - \frac{\beta_0^2}{\beta^2}\right) \tag{184}$$

$$-\frac{\lambda_0}{r} = \frac{\beta_0\nu_1}{\beta} \tag{185}$$

Here subscripts denoted partial derivatives. The (identical) equations for $(\mu, \nu) = (2, 2)$, $(3, 3)$ have been omitted since, if the other equations hold, they will be satisfied because of the Bianchi identities.

In view of the forms of the left sides of (183) and (184), let us assume

$$e^{\nu} = f(t)e^{-\lambda} \tag{186}$$

From (185) one then gets

$$\lambda_0/r = \beta_0\lambda_1/\beta \tag{187}$$

the solution of which is

$$\lambda = \lambda(\xi), \qquad \xi = \beta r \tag{188}$$

Making use of (186) and (188) in (183) and (184), one finds that the latter become identical if one takes f so that

$$f_0/f = 2\beta_{00}/\beta_0 - 4\beta_0/\beta \tag{189}$$

which gives

$$f = \beta_0^2/\alpha^2\beta^4 \qquad (\alpha = \text{const}) \tag{190}$$

In this case (183) and (184) take the form

$$e^{-\lambda}\left(-\frac{\lambda'}{\xi} + \frac{1}{\xi^2}\right) - \frac{1}{\xi^2} = \alpha^2 e^{\lambda}(\xi\lambda' + 3) \tag{191}$$

where a prime denotes a derivative with respect to ξ. Let us write

$$u = e^{-\lambda} \tag{192}$$

Then (191) becomes

$$\xi u' + u - 1 = \alpha^2(-\xi^3 u'/u^2 + 3\xi^2/u) \tag{193}$$

If we now take

$$u = v + (v^2 + \alpha^2\xi^2)^{1/2} \tag{194}$$

the equation simplifies to give

$$\xi v' + v - \tfrac{1}{2} = 0 \tag{195}$$

which has as its solution

$$v = \tfrac{1}{2} - m_0/\xi \qquad (m_0 = \text{const}) \tag{196}$$

Summarizing, we have in (182)

$$e^{\nu} = (\beta_0^2/\alpha^2\beta^4)\, e^{-\lambda} \tag{197}$$

where

$$e^{-\lambda} = \tfrac{1}{2} - m_0/\beta r + \Delta \tag{198}$$

with

$$\Delta = [(\tfrac{1}{2} - m_0/\beta r)^2 + \alpha^2\beta^2 r^2]^{1/2} \tag{199}$$

so that

$$e^{\lambda} = (1/\alpha^2\beta^2 r^2)(\Delta - \tfrac{1}{2} + m_0/\beta r) \tag{200}$$

The above solution is valid for any gauge function $\beta(t)$. Since we are working in the cosmic gauge, we take β as in (148).

It might be mentioned that, if we had kept the cosmological term in the field equations, we would have obtained, instead of (196),

$$v = \tfrac{1}{2} - m_0/\xi - \tfrac{1}{6}\Lambda\xi^2 \tag{201}$$

with (192) and (194) holding. However, this term is not important for our present discussion and will be omitted.

Having obtained the above solution, we can ask how the solution looks if we go from the cosmic gauge to the Einstein gauge, i.e., if

$$ds^2 \to d\bar{s}^2 = \beta^2\, ds^2, \qquad \beta \to \bar{\beta} = 1 \tag{202}$$

One then gets

$$d\bar{s}^2 = (\beta_0^2/\alpha^2\beta^2)\, e^{-\lambda}\, dt^2 - \beta^2(e^{\lambda}\, dr^2 + r^2\, d\Omega^2) \tag{203}$$

This suggests the coordinate transformation $t \to T$, with

$$dT = (\beta_0/\alpha\beta)\, dt = d\beta/\alpha\beta \qquad (\alpha > 0) \tag{204}$$

so that (with a suitable choice of the origin of T)

$$\beta = e^{\alpha T} \tag{205}$$

We then get

$$d\bar{s}^2 = e^{-\lambda}\, dT^2 - e^{2\alpha T}(e^{\lambda}\, dr^2 + r^2\, d\Omega^2) \tag{206}$$

with $e^{-\lambda}$ and e^{λ} given by (198)–(200) and (205).

This result is somewhat surprising since it is now independent of the original gauge function $\beta(t)$. It looks as if R/R_0 has been replaced by $e^{\alpha T}$. However, for large values of r we have $e^{-\lambda} \simeq \alpha r e^{\alpha T}$, so that the behavior does not correspond to any of the usual cosmological models. Evidently, the behavior is due to our having neglected the influence of the matter in the universe. (To take into account this matter in a satisfactory way would have been difficult.) However, in a small enough region around the particle, say for $\alpha r e^{\alpha T} \ll 1$, and for a small enough time interval, during which one can equate $e^{\alpha T}$ to R/R_0 and α to the Hubble constant, (206) provides a good description of the field of a particle in an expanding universe.

It should be remarked that (206) is an exact solution of the Einstein equations for space which is empty except for the particle at $r = 0$. If one lets the arbitrary constant α go to zero in order to get a static solution, one finds in the limit

$$e^{\nu} = e^{-\lambda} = 1 - 2m_0/r, \qquad r \geqslant 2m_0 \tag{207}$$

$$e^{\nu} = e^{-\lambda} = 0 \qquad r \leqslant 2m_0 \tag{208}$$

This differs from the Schwarzschild solution for $r < 2m_0$. It raises the question as to whether the usual form of the Schwarzschild solution is valid for $r < 2m_0$. This will be discussed elsewhere.[11]

10. SOME OTHER GAUGES

Although, in principle, one can introduce an arbitrary gauge, in practice one chooses a gauge for its usefulness. Thus the Einstein gauge, with $\beta = 1$, is convenient because one has the simplest formalism and because particle masses and frequencies of spectral lines are constant in time. On the other hand, the cosmic gauge given by (148), leading to the metric (151), may perhaps be useful in astronomy in that it provides a picture of a static universe with nearly fixed distances between galaxies.

One can ask whether, in the case of a cosmological model described by (138) and (139), there are other gauges of interest besides the Einstein gauge. At first glance two possibilities suggest themselves. One is to take

$$\beta = R_0/R \tag{209}$$

However, if this is substituted into (142) and (143), one finds that all the derivatives cancel out so that one gets

$$-\frac{3k}{R^2} = -\frac{8\pi\rho R^2}{R_0^2} - \frac{\Lambda R_0^2}{R^2} \tag{210}$$

$$-\frac{k}{R^2} = \frac{8\pi p R^2}{R_0^2} - \frac{\Lambda R_0^2}{R^2} \tag{211}$$

while (144) becomes

$$\dot{\rho} + (4\dot{R}/R)\rho = 0 \tag{212}$$

We see that R is an arbitrary function of t, while ρ and p satisfy the relations

$$\rho R^4 = A, \qquad pR^4 = B \qquad (A, B = \text{const}) \tag{213}$$

so that (210) and (211) become

$$-3k = -8\pi A/R_0^2 - \Lambda R_0^2 \tag{214}$$

$$-k = 8\pi B/R_0^2 - \Lambda R_0^2 \tag{215}$$

To understand this situation we note that one can carry out a GT from the present $g_{\mu\nu}$ and β given by (138) and (209) to

$$\bar{g}_{\mu\nu} = (R_0/R)^2 g_{\mu\nu}, \qquad \bar{\beta} = 1 \tag{216}$$

so that after the CT (150), $d\bar{s}^2$ is given by (151). Equations (142) and (143) then become

$$-3k/R_0^2 = -8\pi\bar{\rho} - \Lambda \tag{217}$$

$$-k/R_0^2 = 8\pi\bar{p} - \Lambda \tag{218}$$

Obviously $\bar{\rho}$ and $\bar{p}$ are constant, so that (144) is satisfied. We now have a static universe (with the Einstein gauge). For $k = 1$, $\Lambda > 0$ one gets the closed model of the universe proposed by Einstein.[11] An empty universe ($\bar{\rho} = \bar{p} = 0$) is given by $k = 0$, $\Lambda = 0$. The case $k = -1$ is impossible (for ρ and p nonnegative).

One can now take the inverse transformations to those carried out above, and one returns to (138) and (209). However, one can do this with arbitrary β, or arbitrary R. Hence it is understandable why the field equations do not have the form of differential equations in the present case. Comparing (217), (218) with (214), (215), we have

$$\bar{\rho}R_0^4 = A, \qquad \bar{p}R_0^4 = B \tag{219}$$

It should be remarked that in an expanding universe, according to (152) and (209), the frequencies of spectral lines decrease with time. It is obvious that in this model there is no Hubble effect: the red-shift due to the expansion of the universe just matches that of the spectral lines of the laboratory atoms and is therefore unobservable. One must conclude that, with (138) holding, taking β as in (209) gives disagreement with observation.

A second possibility is to take, together with the metric (138), the gauge function

$$\beta = R/R_0 \tag{220}$$

In this case the field equations (142)–(144) give

$$-\frac{12\dot{R}^2}{R^2} - \frac{3k}{R^2} = -\frac{8\pi R_0^2\rho}{R^2} - \frac{\Lambda R^2}{R_0^2} \tag{221}$$

$$-\frac{4\ddot{R}}{R} - \frac{4\dot{R}^2}{R^2} - \frac{k}{R^2} = \frac{8\pi R_0^2 p}{R^2} - \frac{\Lambda R^2}{R_0^2} \tag{222}$$

$$\dot{\rho} + \frac{\dot{R}}{R}(2\rho + 6p) = 0 \tag{223}$$

One now describes the Hubble effect as a combination of two equal effects, one due to the expansion of the universe, the other due to the change in the spectral frequencies of local atoms.

One can also carry out a GT to get the Einstein gauge $\bar{\beta} = 1$. One then finds, after a time transformation,

$$d\bar{s}^2 = dt^2 - (R/R_0)^4 \, dl^2 \tag{224}$$

so that R/R_0 here must be defined so as to correspond to $(R/R_0)^{1/2}$ in the conventional relativistic models, as in (138). We see that we could have started with (138) and the Einstein gauge in the first place.

There seems to be little gained by taking the above gauge with (220), particularly in view of the complicated description of the Hubble effect. From the standpoint of usefulness (and simplicity) the only two gauges that appear worth considering are the Einstein gauge ($\beta = 1$) and the cosmological gauge of Section 8.

However, there may be other considerations. Thus, Canuto *et al.*,[6] for a metric essentially equivalent to (138), took as gauge functions

$$\beta = t_0/t, \qquad \beta = t/t_0 \tag{225}$$

the first on the assumption of spontaneous mass creation, the second without this assumption. These were chosen in order to have the gravitational "constant" G vary like $1/t$, in accordance with Dirac's large-number hypothesis. They are not very different from the gauge functions (209) and (220) considered above.

It should be emphasized that, in the general case of a cosmological model characterized by a scale parameter (or radius) $R(t)$ and a gauge function $\beta(t)$, the description of the Hubble effect must take into account both the expansion of the universe and the change in the frequency of a spectral line as given by (152). Corresponding to (153) one should write

$$\nu_e/\nu_0 = \beta(t_e) \, R(t_e)/\beta(t_0) \, R(t_0) \tag{226}$$

In the case of galaxies not far away one gets by the usual procedure[10] for the Hubble constant H

$$H = (\dot{R}/R + \dot{\beta}/\beta) \tag{227}$$

where a dot denotes a time derivative.

11. DISCUSSION

There are two points that should be brought out. The first one is that the Weyl theory, in providing a geometrical interpretation of electromagnetism, deserves serious consideration. Dirac[3] made an important contribution to the theory by introducing the co-scalar β into the variational principle and thus into the field equations. His proposal to use two sets of units, or gauges, in order to overcome the difficulty of the nonintegrable length in the Weyl theory has some merit. However, it raises the question: how can the atomic gauge be transferred uniquely to every point in a spacetime with a Weyl geometry if an electromagnetic field is present? The suggestion made in Section 3 of working with standard vectors and modifying the law of parallel displacement may provide the answer.

The second point that needs to be emphasized is that the Weyl theory leads to co-covariant equations, so that their physical content is the same for all gauges, the choice of gauge being arbitrary. Dirac made use of the Weyl theory because, by a suitable choice of gauge, he could get the gravitational "constant" G to vary like $1/t$ (t is the age of the universe), in accordance with his large-number hypothesis. Canuto *et al.*,[6] working with this gauge, carried out elaborate astronomical and astrophysical calculations. However, there is nothing in the Weyl theory that singles out this gauge; nor do the present procedures and units of measurement show any connections with it. If future measurements confirm that G varies like $1/t$, as predicted by Dirac's large-number hypothesis, some more fundamental explanation will be required. The Weyl theory will then have to be supplemented, modified, or replaced.

APPENDIX

In the general relativity theory the relation between the component g_{00} of the metric tensor (a geometrical quantity) and the Newtonian gravitational potential (a physical quantity) is determined in the case of a weak field by comparing the equations of motion of a test particle according to general relativity and according to classical mechanics. In the Weyl theory there does not appear to be any similar way to get the numerical relation between the geometrical vector ϕ_μ and the corresponding electromagnetic potential vector in conventional units, which will be denoted by $\tilde{\phi}_\mu$.

Let us suppose that we are working with quasi-Galilean coordinates having the dimensions of length (with $x_0 = ct$) and a dimensionless metric tensor. From (12) we see that ϕ_μ has the dimension $(\text{length})^{-1}$. On the other hand, from electromagnetic theory we know that $\tilde{\phi}_\mu$ has the dimensions charge $\cdot$ $(\text{length})^{-1}$. Hence we can write

$$\tilde{\phi}_\mu = q\phi_\mu \tag{A1}$$

where q is a constant having the dimensions of charge (it could be the charge of the electron). If we denote the electromagnetic field tensor in conventional units by $\tilde{F}_{\mu\nu}$, then

$$\tilde{F}_{\mu\nu} = qF_{\mu\nu} \tag{A2}$$

Let us go back to the Lagrangian density (67). From the Einstein–Maxwell electromagnetic theory the first term should be, in conventional units,

$$(G/c^4)\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = L^2F^{\mu\nu}F^{\mu\nu} \tag{A3}$$

with

$$L^2 = Gq^2/c^4 \tag{A4}$$

so that L has the dimension of length.

If one keeps L^2 as the coefficient of the first term of (67), then (129) with $J^\mu = 0$ has the form

$$L^2F^{\mu\nu}{}_{;\nu} = \tfrac{1}{2}\sigma\phi^\mu \tag{A5}$$

Taking now $\sigma/L^2 = -2\kappa^2$, one gets (130), (133), and (134).

With (67) as it is, we have $L = 1$, so that, by (A4), $q = c^2/G^{1/2}$. In general relativity units ($c = 1$, $G = 1$) we have $q = 1$.

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