THE NATURAL GAUGE OF THE WORLD: WEYL'S SCALE FACTOR

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We now aim at a final synthesis. To be able to characterize the physical state of the world at a certain point by means of numbers we must not only refer the neighborhood of this point to a coordinate system but we must also fix on certain units of measure. This idea, when applied to geometry and the conception of distance after the step from Euclidean to Riemannian geometry had been taken, affected the final entrance into the realm of infinitesimal geometry. Removing every vestige of ideas of "action at a distance," let us assume that the world geometry is of this kind; we then find that the metrical structure of the world, besides being dependent on the quadratic form $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$*, is also dependent on the linear differential form* $\phi_\mu dx^\mu$*.*

Thus does Weyl begin the last portion of his signature book, *Space-Time-Matter*, the first edition of which appeared in 1918. Weyl's initial intention was to demonstrate that metrical space, in addition to having the metric tensor $g_{\mu\nu}$, involves a fundamental vector ϕ_μ that vanishes only in the absence of an electromagnetic field. Indeed, Weyl went so far as to identify his vector field with the electromagnetic four-potential itself, an assertion that was initially lauded by Einstein and other notable physicists of the day.

But problems quickly arose. Weyl's 1918 theory, which he also called "purely infinitesimal geometry," required that the lengths or magnitudes of all vectors be rescaled or recalibrated under physical transplantation in spacetime (Weyl believed that the Riemannian notion of invariant vector length was itself a needless carryover of the "action at a distance" concept). However, Einstein noted that a rescaling of the metric tensor $g_{\mu\nu}$ would automatically result in a rescaled line element via

$$ds^2 \longrightarrow d\hat{s}^2 = \lambda \, g_{\mu\nu} dx^\mu dx^\nu$$

where $\lambda(x)$ is a scale factor that may vary from point to point in spacetime. Since the line element ds can be made proportional to the proper time of a ticking clock, physical phenomena (such as the spacing of atomic spectral lines) would depend upon their prehistories (or paths), in obvious disagreement with observation. Nevertheless, Weyl had hit upon a powerful and intuitive notion that ultimately found application in quantum physics, where it became known as *gauge invariance*. In 1929, in response to several papers by London and others, Weyl noticed that a rescaling of the wave function $\psi \to \exp(i\lambda)\psi$, taken as a symmetry of the quantum mechanical field, led to conservation of electric charge via Noether's theorem. Weyl subsequently abandoned the idea that electrodynamics, like gravitation, was a purely geometric phenomenon, and wholeheartedly embraced gauge invariance as a quantum construct. Today, we recognize that the term "gauge invariance" (which was coined by Weyl as *Eichinvarianz*), should more appropriately be called "phase invariance."

A Little Background

In Weyl's 1918 theory, the change in the length L of an arbitrary vector under transplantation is given by

$$dL = \phi_\mu \, dx^\mu L$$

Integration then gives

$$\begin{aligned}\int \frac{dL}{L} &= \int \phi_\mu \, dx^\mu \quad \text{or} \\ L &= L_0 \, \exp\left[\int \phi_\mu \, dx^\mu\right]\end{aligned}$$

where L_0 is the original length of the vector. The exponential term is called the *Weyl scale factor*, and it alone determines how vector length varies. If we equate Weyl's gauge vector ϕ_μ with the electromagnetic four-potential A_μ, then it is easy to see from Stoke's theorem that the scale factor can vanish only if ϕ_μ is a gradient:

$$\begin{aligned}\int \phi_\mu \, dx^\mu &= \iint \text{curl } \boldsymbol{\phi} \cdot \hat{n} \, dS \\ &= \iint F_{\mu\nu} \, dx^\mu \wedge dx^\nu\end{aligned}$$

where $F_{\mu\nu} = \partial_\nu \phi_\mu - \partial_\mu \phi_\nu$ is the electromagnetic field tensor. Thus, vector length in Weyl's theory is invariant only when an electromagnetic field is absent.

The Weyl scale factor appears coincidentally in the Lagrangian for a free charged particle

$$\begin{aligned}S &= -\int \left[mc \, ds + \frac{e}{c} \phi_\mu \, dx^\mu\right] \\ &= -\int \left[mc + \frac{e}{c} \, \phi_\mu \, \frac{dx^\mu}{ds}\right] \, ds\end{aligned}$$

which, under a variation of the coordinates, δx^ν, leads immediately to the covariant form of the Lorentz force equation

$$\frac{d^2 x^\lambda}{ds^2} + \left\{\begin{matrix}\lambda \\ \mu\nu\end{matrix}\right\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = -\frac{e}{c} \, F^\lambda_{\ \mu} \frac{dx^\mu}{ds}$$

where the quantity in braces is the Christoffel symbol of the second kind:

$$\left\{\begin{matrix}\lambda \\ \mu\nu\end{matrix}\right\} = \frac{1}{2} g^{\lambda\beta} \left[\partial_\nu g_{\mu\beta} + \partial_\mu g_{\beta\nu} - \partial_\beta g_{\mu\nu}\right]$$

This motivates us to consider the Weyl scale factor in alternative definitions of the metric tensor, the line element and the coefficients of affine connection by way of a similar approach. Recall that Einstein's primary objection to Weyl's 1918 theory was that the line element ds was not gauge invariant. We may attempt to define a new metric and line element via

$$\hat{g}_{\mu\nu} = \exp\left[-a \int \phi_\mu dx^\mu\right] \, g_{\mu\nu}, \tag{1}$$

$$d\hat{s} = \exp\left[-\frac{1}{2} a \int \phi_\mu dx^\mu\right] \, ds \tag{2}$$

where a is some convenient constant. It remains to be seen how these quantities vary under a gauge transformation of the Weyl vector ϕ_μ

$$\delta\phi_\mu = \epsilon k \, \partial_\mu \pi$$

where $\epsilon \ll 0$ is a small number, k is another constant, and $\pi(x)$ is the gauge parameter, but it seems plausible that we can select the constant a so that

$$\begin{aligned} \delta \hat{g}_{\mu\nu} &= 0 \\ \delta\, d\hat{s} &= 0 \end{aligned}$$

and thus totally negate Einstein's objection.

Recall that in his 1918 theory Weyl derived a symmetric affine connection term $\Gamma^\lambda_{\mu\nu}$ that included the vector ϕ_μ, which Weyl equated with the electromagnetic four-potential:

$$\Gamma^\lambda_{\mu\nu} = -\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} + \frac{1}{2}\left[\delta^\lambda_\mu \phi_\nu + \delta^\lambda_\mu \phi_\nu - g_{\mu\nu} g^{\lambda\beta} \phi_\beta\right] \tag{3}$$

This connection necessarily results in a non-zero "non-metricity tensor" $\mathcal{D}_\beta g_{\mu\nu}$ (where $\mathcal{D}$ is the covariant derivative operator) which, using the above connection, is given by

$$\mathcal{D}_\beta g_{\mu\nu} = g_{\mu\nu} \phi_\beta \tag{4}$$

By contraction of this identity with the metric tensor, we get

$$\phi_\beta = \frac{1}{n} g^{\mu\nu} \mathcal{D}_\beta g_{\mu\nu}$$

which vanishes when space is Riemannian.

We now consider an infinitesimal gauge transformation of the metric using

$$\begin{aligned} g'_{\mu\nu} &= e^{\epsilon\pi(x)} g_{\mu\nu} = (1 + \epsilon\pi) g_{\mu\nu} \quad \text{and} \\ g^{\mu\nu\,\prime} &= e^{-\epsilon\pi(x)} g^{\mu\nu} = (1 - \epsilon\pi) g^{\mu\nu} \end{aligned}$$

so that $\delta g_{\mu\nu} = \epsilon\pi g_{\mu\nu}$ and $\delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$. Using these definition, it is easily shown that the gauge vector must transform according to

$$\delta\phi_\beta = \epsilon\, \partial_\beta \pi \tag{5}$$

a property that is identical to that of the four-potential in electrodynamics.

Now, in Riemannian geometry the connection and the equations of the geodesics for free space can be derived simply by a coordinate variation of the Lorentz action

$$\begin{aligned} S &= -mc \int ds \\ &= -mc \int g_{\mu\nu} \frac{dx^\mu}{ds} dx^\nu \end{aligned}$$

which gives us

$$\delta S = mc \int g_{\lambda\nu} \left[\frac{d^2 x^\lambda}{ds^2} + \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}\right] ds\, \delta x^\nu$$

Since the variation in the action vanishes, we recover the familiar geodesic equations for a free, vanishingly-small test particle

$$\frac{d^2 x^\lambda}{ds^2} + \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

which, in flat space, represents a straight line.

A New Method for Deriving the Weyl Connection

We next consider the possibility of deriving the Weyl connection (3) by multiplying ds by a suitable Weyl scale factor and applying same variational principle as above. Let us assume that it has the same form as given previously:

$$d\hat{s} = \exp\left[-\frac{1}{2}a\int \phi_\beta \, dx^\beta\right] ds$$

The variational problem is now given by

$$\begin{aligned} S &= -mc\int d\hat{s} \\ &= -mc\int \exp\left[-\frac{1}{2}a\int \phi_\beta \, dx^\beta\right] g_{\mu\nu}\frac{dx^\mu}{ds}dx^\nu \end{aligned}$$

It is straightforward to show that the variation now results in

$$\delta S = mc\int g_{\lambda\nu}\left[\frac{d^2x^\lambda}{ds^2} + \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + \frac{1}{4}a\, g^{\lambda\beta}\phi_\beta - \frac{1}{2}a\,\phi_\nu\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds}\right] ds\,\delta x^\nu$$

If we set $a = 2$, we recover the Weyl connection.

This is an interesting result in itself, because it demonstrates how the Weyl scale factor might be used to generate connections. But we return to the question at hand, which is: are the scaled quantities

$$\hat{g}_{\mu\nu} = \exp\left[-2\int \phi_\mu \, dx^\mu\right] g_{\mu\nu}, \tag{6}$$

$$d\hat{s} = \exp\left[-\int \phi_\mu \, dx^\mu\right] ds \tag{7}$$

invariant with respect to a gauge transformation of the Weyl vector as in (5)? If they are, then we can posit the existence of a Weylian geometry in which all scaled metric quantities ($\hat{g}_{\mu\nu}$, $d\hat{s}$, $\sqrt{-\hat{g}}$, the connection terms, etc.) are *automatically gauge invariant.* More importantly, the associated Riemann-Christoffel tensor $\hat{R}^\lambda{}_{\mu\nu\beta}$, the Ricci tensor $\hat{R}_{\mu\beta}$ and the Ricci scalar $\hat{R}$ are then all gauge invariant, making possible the use of an Einstein-Hilbert gravitational action that remains linear in the Ricci term:

$$S_G = \int \sqrt{-\hat{g}}\,\hat{R}\,d^4x$$

In the 1918 theory, Weyl had to set the Lagrangian equal to$\sqrt{-g}\,R^2$ in order to ensure gauge invariance. This Lagrangian results in field equations that are of the fourth order in the metric tensor, a problem that Einstein was also quick to point out.

In spite of the logic of our little plan, however, it doesn't quite work. If we take the gauge variation of $\hat{g}_{\mu\nu}$ in (6), we quickly find that

$$\delta\hat{g}_{\mu\nu} = -\epsilon\pi\hat{g}_{\mu\nu}$$

and not zero. Thus, if we are to preserve the possibility of having a fully gauge-invariant Weylian geometry, we have to give up the idea of deriving a connection that is consistent with this geometry (at least from a variational principle). But this simply leads to a contradiction in terms, because Weyl's geometry is really all about the connection. Furthermore, since the scaled

Christoffel symbols themselves are gauge invariant, we actually have no need for the Weyl connection in (3) at all.

Last Thoughts

The writer has argued previously that Weyl's original connection term is wrong anyway, because it demands that the length of all vectors be changed in the presence of the Weyl vector ϕ_μ. As Einstein and others noted, there are some vectors whose magnitudes are simply numbers – the Compton wavelength of an electron, the unit vector dx^μ/ds and the related four-momentum p^μ of special relativity to name three examples. The lengths of these vectors cannot change under physical transplantation, so some modification of Weyl's geometry is required to allow for them. It is easily shown that the total derivative of the unit vector equation

$$1 = g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

is simply

$$0 = \mathcal{D}_\beta g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\frac{dx^\beta}{ds}$$

so that, whatever the non-metricity tensor $\mathcal{D}_\beta(g_{\mu\nu})$ is, it is either identically zero or satisfies the peculiar cyclic symmetry condition

$$\mathcal{D}_\beta g_{\mu\nu} + \mathcal{D}_\nu g_{\beta\mu} + \mathcal{D}_\mu g_{\nu\beta} = 0,$$

an expression that seems to have first been proposed by Schrödinger in 1950. Weyl's definition of the non-metricity tensor in (2) does not satisfy this condition, and this shortcoming might have something to do with our inability to develop a fully consistent, gauge-invariant Weylian geometry that eliminates the objections Einstein raised against the theory.

There is one avenue yet open to us, and that is the fact that, in quantum mechanics, the Weyl vector ϕ_μ is a purely imaginary quantity. London showed that the change in vector length could be related to the orbital radii of an electron in the Bohr atom if one sets

$$\oint \frac{dL}{L} = 2\pi i$$

This is Cauchy's integral formula if one assumes L.to be a complex quantity. This forces the identification of Weyl's vector ϕ_μ with the electromagnetic four-potential A_μ via

$$\phi_\mu = -\frac{ie}{\hbar c}\,A_\mu\,,$$

a term that appears frequently in quantum mechanics. As yet, nothing in the original Weyl theory involves complex quantities, and it is just possible that such an approach may provide an additional degree of freedom that will ultimately make the theory consistent. At the same time, however, it would introduce further complications involving the reality of vector length variation.

These are the questions that Weyl's original gauge theory continues to pose: Does geometric gauge invariance (or conformal invariance) have any relevance in physics? And if so, what is (to quote Eddington) the Natural Gauge of the World?

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