On a Remarkable Property of the Quantum-Orbits of a Single Electron

by Erwin Schrödinger in Zurich

In Weyl's geometry [reference omitted] there appears in addition to the well-known quadratic differential form, which determines the metric at individual points, a linear differential form

$$\phi_o dx^o + \phi_1 dx^1 + \phi_2 dx^2 + \phi_3 dx^3 = \phi_i dx^i,$$

which determines the metrical connection between different points. Its geometrical meaning is that the scale of a "length" l (the square of the absolute value of a vector) does not remain invariant with respect to "congruent transfer" of the length to a neighbouring point, but undergoes the change

$$dl = -l\phi_i dx^i. \tag{1}$$

Weyl discovered that, so long as one requires that the transfer of a vector be accompanied by the congruent transfer of its length, the two of them together (the metric of the individual points + the metrical relationship) define an affine connection (i.e. a concept of the parallel transport of a vector). On congruent transfer of a length along a finite section of a world-line—e.g. by parallel transfer of a vector along such a section—the scale of the length is multipled by the factor

$$e^{-\int \phi_i dx^i}, \tag{2}$$

where the line-integral is to be taken, of course, along the world-line, and depends non-trivially on the path so long as the quantities

$$f_{ik} = \frac{\partial \phi_i}{\partial x_k} - \frac{\partial \phi_k}{\partial x_i} \tag{3}$$

do not vanish. —Physically the components of the connection are identified with the gravitational field and the f_{ik} with the electromagnetic field. If these are the correct relations—and the coordinates in a space-time region are so chosen—that, at least in some approximation, x_o denotes the time (in sec) and

$x_1 x_2 x_3$ the Cartesian coordinates (in cm), then the ϕ_i are, up to a universal proportionality-factor, the electromagnetic potentials in the usual sense:

$$V, \quad -\frac{1}{c}\mathcal{A}_x, \quad -\frac{1}{c}\mathcal{A}_y, \quad -\frac{1}{c}\mathcal{A}_z. \tag{4}$$

If we write this factor as $\gamma^{-1}e$ where e is the elementary quantum of electric charge in CGS units, that is to say,

$$\phi_o = \gamma^{-1}eV, \quad \phi_1 = -\gamma^{-1}\frac{e}{c}\mathcal{A}_x, \quad \phi_2 = -\gamma^{-1}\frac{e}{c}\mathcal{A}_y, \quad \phi_3 = -\gamma^{-1}\frac{e}{c}\mathcal{A}_z,$$

then, since ϕ_o has the dimension $\sec^{-1}$ and eV the dimension of "energy", γ has the dimension of action ($g\ cm^2\ sec^{-1}$).—The scale-factor (2) becomes

$$e^{-\frac{e}{\gamma}\int (Vdt - \mathcal{A}_x dx - \mathcal{A}_y dy - \mathcal{A}_z dz)}. \tag{5}$$

The property announced in the title and which to me seems remarkable is that the "true" quantum-conditions i.e. those that are sufficient to determine the energy and thus the spectrum, are just sufficient to make the exponent of the path-factor (5) an integer multiple of $\gamma^{-1}h$ (which is a pure number according to the above) for all approximate periods of the system. As there are some ifs and buts to be attached to this statement I shall first establish it for the individual cases for which it is valid in the simple form just stated. Only then shall I discuss the the possible meaning of the statement—concerning which—let me hasten to admit—I have not made much progress.

[Schrödinger then proceeds to give five examples for which the statement is valid, namely, (1) unperturbed Kepler orbits, (2) Zeeman effect, (3) Stark effect, (4) mixed Zeeman and Stark effects with parallel axes, and (5) relativistic mass variation. But as the examples are rather long and the London analysis discussed above gives a general summary of the situation, we shall omit the details and proceed to the discussion, which is the part of Schrödinger's paper that is most relevant from the gauge-theory point of view.]

Discussion of the Results

To summarize, we have the following situation. Were the electron in its orbit to bring along with it a "length" which remained unchanged by the transfer, then, starting from any arbitrary point on the orbit, each time that the electron returned to its approximate initial position and initial state the scale of this length would appear to be multiplied by an approximately integer power of

$$e^{\frac{h}{\gamma}} \tag{22}$$

It is difficult to believe that this result is merely an accidental mathematical consequence of the quantum conditions, and has no deeper physical meaning. The somewhat imprecise form of the approximations in which it is encountered, changes nothing in this respect; we know, in any case, that the quantum orbits are not sharply defined for two reasons: First, because of the electromagnetic radiation, which, although it certainly does not exist in its classical form, must certainly correspond to something quantum-mechanical of the same order of magnitude, since otherwise the frequencies could not be correctly deduced from the correspondence principle. Second, a lack of sharpness in the quantum orbits is caused also by the fact that in most cases the motion is periodic only in a certain approximation. (E.g. in the case of the Zeeman effect the terms quadratic in the field-strengths must be neglected in principle; and, if one takes relativistic corrections into account, the Stark effect no longer belongs to the class of strictly separable problems).

Whether the electron really brings a "length" along with it in its motion is more than questionable. It is very possible that, in the course of its motion, it continually "reestablishes" itself in the sense of Weyl. It may be that the meaning of our result is to be sought in the fact that not every re-establishment tempo is permitted to the electron, but rather that the re-establishment must take place subject to a certain dependence on the quasi-periodic orbital cyles.

One is tempted to guess what value the universal constant γ must have. There are two well-established constants with the dimensions of action, namely h and $\frac{e^2}{c}$ (though for my own part I am convinced that they are not independent). Were $\gamma \simeq \frac{e^2}{c}$ the universal factor (22) would be a very large number, of the order of e^{1000}. The other possibility $\gamma \simeq h$ suggests that the pure imaginary value

$$\gamma = \frac{h}{2\pi\sqrt{-1}}$$

might be a possibility, in which case the universal factor would be equal to unity and the scale of any accompanying distance would reproduce itself after each quasi-period.—I do not dare to judge whether this would make sense in the context of Weyl geometry.

The fact that e, h, c are not the only known universal constants should also be taken into account. If one includes the (usual) gravitational constant k and any universal mass, e.g. the electron mass, one has

$$\frac{e^2}{km^2} = \text{pure number} \simeq 10^{+40}$$

Hence

$$\frac{he^2}{km^2}$$

is a "universal quantum of action" of the order of magnitude 10^{+13} ergsec.—We should recall, however, that in this matter dimensional-considerations alone are not conclusive.