Electron Spin

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The Spin Zone

To demonstrate the Dirac notation, we will investigate some important physics involving a particle having a spin of 1/2 – the electron. The subject of electron spin is in fact much more important than any notational device, but the Dirac notation makes it particularly easy to understand. It is assumed that you already have some familiarity with the concept of particle spin, so I'm not going to discuss the basis of spin in any detail.

To begin, let's consider an investigation of great historical importance – the Stern-Gerlach experiment. In 1922, the German physicists Otto Stern and Walther Gerlach observed that a stream of silver atoms directed into an inhomogeneous magnetic field was split into two roughly equal streams. This observation was attributed to some unknown property of the single valence electron that is present in the 5s shell of an un-ionized silver atom. It was eventually realized that electrons have a quantum property that became known as *electron spin*. A kind of intrinsic angular momentum, electron spin came in just two types – up and down – and was numerically equal to $\pm 1/2\,\hbar$, where $\hbar$ is Planck's constant divided by 2π. The non-uniform magnetic field in the Stern-Gerlach apparatus was observed to be separating $+1/2\,\hbar$-spin electrons from their negative counterparts. It was also discovered that beam splitting occurred no matter how the magnet was oriented. As a result, three degrees of freedom were attributed to electron spin: S_x, S_y, and S_z, each numerically equal to $\pm 1/2\,\hbar$. Consequently, we will have the operations $\hat{S}_{\text{any}}\,|S_{\text{any}}\pm\rangle = \pm 1/2\,\hbar\,|S_{\text{any}}\pm\rangle$, where "any" stands for x, y, or z. As yet, we do not know what the operators $\hat{S}_{\text{any}}$ or their corresponding eigenfunctions look like, but we're about to find out.

This is the February 8, 1922 postcard sent by Walther Gerlach to Niels Bohr announcing the discovery of *space quantization*, which in 1927 was finally understood as a consequence of electron spin. The photos on the back of the card show the effect of the magnetic field before and after being turned on. The discovery was hastened by Otto Stern's penchant for cheap cigars! See *Physics Today*, December 2003.

Let us now investigate the Stern-Gerlach experiment a little further. Take a stream of electrons that have been generated in such as way that their spins are randomly oriented, and direct this into a z-directed Stern-Gerlach magnet. We get two streams of electrons: one has all electrons in the $|S_z+\rangle$ state, while the other stream's electrons are all in the $|S_z-\rangle$ state. Let us now direct the S_z+ stream into another Stern-Gerlach apparatus, also oriented in the z-direction. What do we observe? Since the incoming electrons are all initially in the $|S_z+\rangle$ state, the Stern-Gerlach set-up doesn't have any effect: we get just one stream coming out, and all of its electrons are still in the $|S_z+\rangle$ state. No mystery here. But now let's direct this new stream into an apparatus oriented in the y-direction. What we now observe is a splitting of the S_z+ electrons into two separate streams having the states $|S_y+\rangle$ and $|S_y-\rangle$. This is a little odd, considering that the incoming electrons had no y-spin character at all, but still no big deal. However, if we now direct either of these S_y-state electron streams into yet another z-directed Stern-Gerlach apparatus, we get two streams again in the S_z+ and S_z- states. It is as if the electrons had forgotten that the original S_z+ property had been removed prior to entering the y-oriented apparatus. How do we understand and quantify this strange behavior of electron spin?

Those of you who played with Polaroid filters as kids may see a parallel here between sequential Stern-Gerlach experiments and polarized light. If you did, then you know that if two Polaroid filters are arranged at 90 degrees to one another, no light can get through; however, by placing a third filter between them at some intermediate angle, some light does in fact pass through all three. The presence of the intermediate filter somehow "erases" the memory of the initial photons with regard to the first filter.

Although the analogy between electron spins and Polaroid filters is interesting, spin has no true counterpart in the world of classical physics, so we must rely solely on quantum mechanics to explain it. What we're going to do is define each spin state in terms of the z spin states, which for simplicity (and convention) are just called $|S_z+\rangle$ and $|S_z-\rangle$. I know this makes the z-direction a bit special, but we have to start somewhere (we could have done the same for either the x-states or the y-states). Let's begin with the S_x+ eigenstate, which we assume can be written as a linear combination of the up and down z states:

$$|S_x+\rangle = \frac{a_1}{\sqrt{2}}|S_z+\rangle + \frac{a_2}{\sqrt{2}}|S_z-\rangle$$

where a_1 and a_2 are complex coefficients to be determined. The $\sqrt{2}$ terms are there to ensure that $\langle S_x+|S_x+\rangle = 1$, as it must be. As we implied earlier, you should try to read this as saying that $\langle S_x+|S_x+\rangle$ is the probability amplitude that a $|S_x+\rangle$ state will be found later in a $|S_x+\rangle$ state. Since we're not doing anything to the $|S_x+\rangle$ state, it can't change, and the probability for this "transition" must be 100%. Alright, let's now assume that we've skimmed off a stream of S_x+ electrons from a Stern-Gerlach apparatus. If we direct the stream into another apparatus set up in the z-direction, we know that we'll get two streams representing the $|S_z+\rangle$ and $|S_z-\rangle$ states. The probability amplitude for a stream of $|S_x+\rangle$ electrons to be converted into $|S_z+\rangle$ electrons by the apparatus is

$$\langle S_z+|S_x+\rangle = \frac{1}{\sqrt{2}}$$

This result reflects the fact that the S_x+ electrons will be converted equally into S_z+ and S_z- electrons, so the transition probability is 50%. So we have

$$\langle S_z+|S_x+\rangle = \frac{1}{\sqrt{2}} = \frac{a_1}{\sqrt{2}}\langle S_z+|S_z+\rangle + \frac{a_2}{\sqrt{2}}\langle S_z+|S_z-\rangle$$

Eigenvector othogonality kills the second term on the right, so we have $a_1 = 1$. A similar argument using $\langle S_z - |S_x+\rangle$ gives $a_2 = 1$, so

$$|S_x+\rangle = \frac{1}{\sqrt{2}}|S_z+\rangle + \frac{1}{\sqrt{2}}|S_z-\rangle$$

To find $|S_x-\rangle$, we set

$$|S_x-\rangle = \frac{b_1}{\sqrt{2}}|S_z+\rangle + \frac{b_2}{\sqrt{2}}|S_z-\rangle$$

where b_1 and b_2 are again complex numbers. Using $\langle S_z+|S_x-\rangle = 1/\sqrt{2}$ again, we see that $b_1 = 1$. To get b_2, we note that $\langle S_x+|S_x-\rangle = 0$ (that is, a stream of S_x- electrons cannot be converted by a z-oriented Stern-Gerlach apparatus into S_x+ electrons). By using

$$\langle S_x+| = \frac{1}{\sqrt{2}}\langle S_x+| + \frac{1}{\sqrt{2}}\langle S_x-|$$

we easily find that $b_2 = -1$. We can therefore write the S_x eigenvectors as

$$|S_x\pm\rangle = \frac{1}{\sqrt{2}}|S_z+\rangle \pm \frac{1}{\sqrt{2}}|S_z-\rangle$$

Moving on to the y-case, we now direct a stream of S_y+ electrons into the z-oriented apparatus. We let

$$|S_y+\rangle = \frac{c_1}{\sqrt{2}}|S_z+\rangle + \frac{c_2}{\sqrt{2}}|S_z-\rangle$$

Again, multiplying $\langle S_z+|$ against this shows that $c_1 = 1$. Now, if we now follow the same approach as we did for $|S_x+\rangle$, we find that $c_2 = 1$, which makes $|S_y+\rangle$ equal to $|S_x+\rangle$, which is no good. Instead, let's look at the combination $\langle S_y+|S_y+\rangle$, which gives us $c_2^* c_2 = 1$. If $c_2 \neq 1$, the only other choice we have is $c_2 = \pm i$. Let's pick $c_2 = i$, so that

$$|S_y+\rangle = \frac{1}{\sqrt{2}}|S_z+\rangle + \frac{i}{\sqrt{2}}|S_z-\rangle$$

A completely analogous argument for $|S_y-\rangle$ shows we can write the S_y states as

$$|S_y\pm\rangle = \frac{1}{\sqrt{2}}|S_z+\rangle \pm \frac{i}{\sqrt{2}}|S_z-\rangle$$

As mentioned, we could have defined the spin eigenvectors in terms of S_x or S_y, but it is conventional to use the S_z eigenstates. The reason for this has to do with the fact that the S_z spin operator (will we will now derive) has a very simple form in spherical coordinates.

If you don't know already, state vectors and eigenvectors can also be represented as column matrices, while operators can be represented by square Hermitian matrices (also differential operators). This affords us the luxury of working either in ket and bra notation or with the (perhaps) more familiar matrix forms. As an example of this, we'll now convert the above results for spin-1/2 systems into the matrix representation.

Since we picked the S_z eigenvectors as the basis for the S_x and S_y kets, we'll pick the simplest column vectors possible for $|S_z\pm\rangle$:

$$|S_z+\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \quad |S_z-\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

There's only two components, reflecting the fact that spin-1/2 systems have only two possible spin states: up and down, corresponding to $\pm\hbar/2$. Similarly, we have the bras

$$\langle S_z+| = \begin{bmatrix}1 & 0\end{bmatrix}, \quad \langle S_z-| = \begin{bmatrix}0 & 1\end{bmatrix}$$

You can check for yourself that the inner products of these vectors are either zero or one, depending on whether they're orthogonal (for example, $\langle S_z-|S_z+\rangle = 0$). We can also determine the outer products, and they are

$$\begin{aligned}
|S_z+\rangle\langle S_z+| &= \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \\
|S_z-\rangle\langle S_z-| &= \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \\
|S_z\pm\rangle\langle S_z\mp| &= \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}
\end{aligned}$$

It is now a simple matter to compute the operator $\hat{S}_z$ that operates on the S_z eigenvectors. Recalling that the closure relation for the eigenvectors of some observable A is $\sum_k |a_k\rangle\langle a_k| = 1$, we can derive the useful relation

$$\begin{aligned}
\hat{A} &= \sum_k \hat{A}\,|a_k\rangle\langle a_k| \\
&= \sum_k a_k\,|a_k\rangle\langle a_k|
\end{aligned}$$

where the a_k are the eigenvalues of the operator $\hat{A}$. The eigenvalues associated with $\hat{S}_z$ are $\pm\hbar/2$, so this gives us

$$\hat{S}_z = \frac{1}{2}\hbar\,|S_z+\rangle\langle S_z+| - \frac{1}{2}\hbar\,|S_z-\rangle\langle S_z-|$$

Using the matrix forms for the outer products, we can also write this as

$$\hat{S}_z = \frac{1}{2}\hbar \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

You should now be able to verify for yourself that the operators for $\hat{S}_x$ and $\hat{S}_y$ are

$$\begin{aligned} \hat{S}_x &= \frac{1}{2}\hbar\,|S_z+\rangle\langle S_z-| - \frac{1}{2}\hbar\,|S_z-\rangle\langle S_z+| = \frac{1}{2}\hbar \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \hat{S}_y &= \frac{1}{2}i\hbar\,|S_z-\rangle\langle S_z+| - \frac{1}{2}i\hbar\,|S_z+\rangle\langle S_z-| = \frac{1}{2}\hbar \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \end{aligned}$$

One last note before we wrap this up. Consider the following somewhat peculiar combinations of the $\hat{S}_x$ and $\hat{S}_y$ operators, which are defined as

$$\begin{aligned} \hat{S}_+ &= \hat{S}_x + i\hat{S}_y = \frac{1}{2}\hbar \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \hat{S}_- &= \hat{S}_x - i\hat{S}_y = \frac{1}{2}\hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

If we multiply any spin-down ket $|-\rangle$ by $\hat{S}_+$, the operator turns it into a spin-up ket:

$$\hat{S}_+\,|-\rangle = \frac{1}{2}\hbar \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\hbar \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

while the operator $\hat{S}_-$ turns a spin-up ket into its spin-down counterpart. In addition, the quantities $\hat{S}_+\,|+\rangle$ and $\hat{S}_-\,|-\rangle$ are both zero, in keeping with the observation that electron spins higher than $\hbar/2$ and lower than $-\hbar/2$ don't exist. The operators $\hat{S}_+$ and $\hat{S}_-$ are respectively called the *raising* and *lowering* operators (sometimes called "ladder" operators), and they are of tremendous utility in the quantum mechanical description of all kinds of angular momentum (not just spin), while another version of these operators plays an equally important role in the raising and lowering of energy states (as in the harmonic oscillator). The raising and lowering operators in effect create states that have higher or lower levels of angular momentum or energy. In quantum field theory, similar operators are used to create and annihilate particles and fields. The discovery of ladder operators is credited to Dirac – no big surprise, perhaps!

More Spin

The three 2×2 matrices S_x, S_y and S_z are known as the *Pauli spin matrices*. The great Austrian physicist Wolfgang Pauli first used them to "force fit" electron spin into the Schrödinger equation for the hydrogen atom. Then in 1928, an even greater physicist (our friend Mr. Dirac) found a set of four 4×4 matrices, closely related to the ones above, that he used in his celebrated relativistic electron equation. The Dirac equation has electron spin already built into the formalism, so it doesn't have to be jammed in artificially. Because the Dirac matrices are 4×4, they allow for a description of not only electrons (with spin up and spin down), but also for spin-up and spin-down *positrons*, which were the first forms of antimatter to be discovered.

Dirac's discovery of the relativistic electron equation ranks as one of the most profound achievements of the human mind. Though Dirac's religious inclinations were somewhat ambiguous (I believe he was a member of the Anglican Church, though he was also considered to be an agnostic), in my opinion there's little doubt that God spoke to Dirac's mind one day back in 1928, and we're the better for it.