Renormalization of Higher Derivative Quantum Gravity

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Renormalization of higher-derivative quantum gravity*

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Gravitational actions which include terms quadratic in the curvature tensor are renormalizable. The necessary Slavnov identities are derived from Becchi-Rouet-Stora (BRS) transformations of the gravitational and Faddeev-Popov ghost fields. In general, non-gauge-invariant divergences do arise, but they may be absorbed by nonlinear renormalizations of the gravitational and ghost fields (and of the BRS transformations). Fortunately, these artifactual divergences may be eliminated by letting the coefficient of the harmonic gauge-fixing term tend to infinity, thus considerably simplifying the renormalization procedure. Coupling to other renormalizable fields may then be handled in a straightforward manner.

I. INTRODUCTION

It has been suggested by various authors[1] that the action for quantum gravity should contain, in addition to the Einstein action, certain nonminimal functionals of the metric tensor which involve more than two derivatives. These suggestions have recently been highlighted by the nonrenormalizability of general relativity.[2] Although higher-derivative terms in the action would have a negligible influence in the low-frequency domain of classical experiments, at high frequencies they would dominate the behavior of the theory, leading to a stabilization of the divergence structure and consequently to power-counting renormalizability.

The principal candidates for such higher-derivative additions to the action are contracted quadratic products of the curvature tensor. These contain four derivatives, and lead to a graviton propagator which behaves like k^{-4} for large momenta, provided one takes care to supply an appropriate gauge-fixing term. Power counting then shows that all the divergences involving gravitons have degree of divergence four or less.

In principle, one could also include terms with even higher numbers of derivatives. One would only have to be careful to include those terms which contribute to the propagator, and not just to the vertices. For example, $\kappa^2 \int (R_{\mu\nu;\lambda} R^{\mu\nu;\lambda} + R^3)$ $\times \sqrt{-g}$ would be an admissible addition but $\kappa^2 \int R^3 \sqrt{-g}$ alone would not, since it does not contain any terms quadratic in the gravitational field. However, such terms with more than four derivatives would not be renormalized, since the maximum degree of divergence would remain at four. Also, although the addition of terms with more than four derivatives would make the theory finite after a certain order in the loop expansion, it would not remove the divergences at the one-loop order. We shall therefore restrict ourselves to

considering nonminimal terms involving just four derivatives, the simplest extensions of the gravitational action sufficient to obtain renormalizability.

To demonstrate renormalizability in a gauge theory is an exercise in elucidating the consequences of gauge invariance, which is always haunting the calculations even though it has been temporarily broken by the covariant quantization procedure.[3] The consequences of gauge invariance are expressed by the Slavnov identities, which relate the various Feynman diagrams of the theory. In deriving these identities, we shall make use of a residual supergauge symmetry of the quantum effective action which is analogous to the residual symmetry of Yang-Mills theories discovered by Becchi, Rouet, and Stora[4] (hereafter referred to as BRS).

The Slavnov identity for the generating functional of proper vertices leads to a renormalization equation which governs the structure of the divergent parts of the proper vertices. The solution of this equation shows that some of the divergences may be eliminated by renormalizations of the coefficients of the gauge-invariant terms in the action, while others are non-gauge-invariant in structure and must be eliminated by nonlinear renormalizations of the gravitational and Faddeev-Popov ghost fields and of the BRS transformations themselves.

We shall find that the procedure followed to introduce the gauge-fixing term into the effective action has a considerable influence on the non-gauge-invariant divergences of the theory. However, a special kind of functional identity shows that the physically important renormalizations, those necessary to make the S matrix finite, do not depend upon such artifacts of the quantization procedure. What is more, the non-gauge-invariant divergences may actually be eliminated by letting the coefficient of the harmonic gauge-fix-

ing term tend to infinity. This considerably simplifies the renormalization procedure, which can then be extended to include coupling of the gravitational field to other renormalizable fields. Coupling to a massive scalar field shall be discussed as an example.

In our notation, we use the signature $(-+++)$. The curvature tensor is defined by $R^\lambda_{\mu\alpha\nu} = \partial_\nu \Gamma^\lambda_{\mu\alpha} + \cdots$, and the Ricci tensor by $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$.

II. HIGHER-DERIVATIVE THEORIES OF GRAVITATION

Adding quadratic products of the curvature tensor to the gravitational action leads to field equations in which some terms involve four derivatives. While it is not the purpose of this paper to investigate the novel consequences of these classical field equations, a brief summary of some of the salient features is in order to give a grounding to the following discussion of renormalization. Details will be left to a separate publication.[17]

The generic form of the action may be written

$$I_{\text{sym}} = -\int d^4x\sqrt{-g}\,(\alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2 + \kappa^{-2}\gamma R)\,,$$

(2.1)

where $\gamma = 2$ and $\kappa^2 = 32\pi G$. There is no need to include $\int d^4x\sqrt{-g}\,R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ in the action because of the Gauss-Bonnet topological invariance in four dimensions:

$$\int d^4x\sqrt{-g}\,(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)$$

vanishes for space-times topologically equivalent to flat space.

The most convenient definition of the gravitational field variable for our work is given in terms of the contravariant metric density:

$$\kappa h^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu} - \eta^{\mu\nu}\,.$$

(2.2)

This definition, together with our choice of the harmonic gauge, will considerably simplify some of the later discussion (Sec. VIII).

The parameters α and β in the action (2.1) may be limited to satisfy the various experimental constraints. For example, the Newtonian limit of the static field is

$$h^{00} \sim \frac{1}{r} - \frac{4}{3}\frac{e^{-m_2 r}}{r} + \frac{1}{3}\frac{e^{-m_0 r}}{r}\,,$$

(2.3)

where $m_2 = (\frac{1}{2}\alpha\kappa^2)^{-1/2}$ and $m_0 = [(3\beta - \alpha)\kappa^2]^{-1/2}$. In regions where such a weak-field limit is appropriate, this may be made to approach the Newtonian $1/r$ as closely as one wishes by ensuring that m_2 and m_0 are large enough. This is not true, however, of models which omit the Einstein term in the action. The action $\int d^4x\sqrt{-g}\,(\alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2)$

leads to field equations in which all terms contain four derivatives. The potential due to a point mass is then linear, since $\nabla^2\nabla^2\phi = \delta^3(r) \to \phi \sim r$.

Analysis of the linearized radiation shows that there are eight dynamical degrees of freedom in the field. Two of these excitations correspond to the familiar massless spin-2 graviton. Five more correspond to a massive spin-2 particle with mass m_2. The eighth corresponds to a massive scalar particle with mass m_0. Although the linearized field energy of the massless spin-2 and massive scalar excitations is positive definite, the linearized energy of the massive spin-2 excitations is negative definite. This feature is characteristic of higher-derivative models, and poses the major obstacle to their physical interpretation.

In the quantum theory, there is an alternative problem which may be substituted for the negative energy. It is possible to recast the theory so that the massive spin-2 eigenstates of the free-field Hamiltonian have positive-definite energy, but also negative norm in the state vector space. These negative-norm states cannot be excluded from the physical sector of the vector space without destroying the unitarity of the S matrix, as was shown by Pais and Uhlenbeck in 1950.[5]

The requirement that the graviton propagator behave like k^{-4} for large momenta makes it necessary to choose the indefinite-metric vector space over the negative-energy states. This amounts to a choice of the signs of the $i\epsilon$ terms in the propagator, as is shown in the Appendix. Except for this choice, the problem of unitarity does not directly affect the ultraviolet divergences, which are our main concern here. It should be stressed, however, that there can be no sensible physical interpretation of these higher-derivative models until the unitarity problem is resolved. We shall return to this point briefly at the end of the paper.

III. QUANTUM THEORY[6]

The presence of massive quantum states of negative norm which cancel some of the divergences due to the massless states is analogous to the Pauli-Villars regularization of other field theories. For quantum gravity, however, the resulting improvement in the ultraviolet behavior of the theory is sufficient only to make it renormalizable, but not finite. Thus, a further regularization scheme is needed. In the following, we shall use dimensional regularization.

The gauge choice which we adopt in defining the quantum theory is the familiar harmonic gauge,

$$\partial_\nu h^{\mu\nu} = 0\,.$$

(3.1)

Green's functions are then given by a generating functional

$$Z(T_{\mu\nu}) = N \int \left[\prod_{\mu \leq \nu} dh^{\mu\nu} \right] [dC^{\sigma}][d\overline{C}_{\tau}] \delta^4(F^{\tau}) \exp \left[i \left(I_{\text{sym}} + \int d^4x \overline{C}_{\tau} \vec{F}{}^{\tau}_{\mu\nu} D^{\mu\nu}_{\alpha} C^{\alpha} + \kappa \int d^4x T_{\mu\nu} h^{\mu\nu} \right) \right] ,$$

(3.2)

where $F^{\tau} = \vec{F}{}^{\tau}_{\mu\nu} h^{\mu\nu}$ and $\vec{F}{}^{\tau}_{\mu\nu} = \delta^{\tau}_{\mu} \overset{\leftarrow}{\partial}_{\nu}$ (the arrow indicates the direction in which the derivative acts). N is an irrelevant normalization constant. C^{σ} is the Faddeev-Popov ghost field, and $\overline{C}_{\tau}$ is the antighost field; both C^{σ} and $\overline{C}_{\tau}$ are anticommuting quantities. $D^{\mu\nu}_{\alpha}$ is the operator which generates gauge transformations in $h^{\mu\nu}$, given an arbitrary spacetime-dependent vector $\xi^{\alpha}(x)$ (corresponding to $x^{\mu'} = x^{\mu} + \kappa \xi^{\mu}$): $\delta h^{\mu\nu} = D^{\mu\nu}_{\alpha} \xi^{\alpha}$, where

$$D^{\mu\nu}_{\alpha} \xi^{\alpha} = \partial^{\mu} \xi^{\nu} + \partial^{\nu} \xi^{\mu} - \eta^{\mu\nu} \partial_{\alpha} \xi^{\alpha}$$

$$+ \kappa(\partial_{\alpha} \xi^{\mu} h^{\alpha\nu} + \partial_{\alpha} \xi^{\nu} h^{\alpha\mu} - \xi^{\alpha} \partial_{\alpha} h^{\mu\nu} - \partial_{\alpha} \xi^{\alpha} h^{\mu\nu}) .$$

(3.3)

In the functional integral (3.2), we have written the metric for the gravitational field as

$$\left[\prod_{\mu \leq \nu} dh^{\mu\nu} \right] ,$$

without any local factors of $g = \det g_{\mu\nu}$. Such factors do not contribute to the Feynman rules because their effect is to introduce terms proportional to $\delta^4(0) \int d^4x \ln(-g)$ into the effective action,[7] and $\delta^4(0)$ is set equal to zero in dimensional regularization.

In calculating the generating functional (3.2) via the loop expansion, one may represent the δ function which fixes the gauge as the limit of a Gaussian, discarding an infinite normalization constant:

$$\delta^4(F^{\tau}) \sim \lim_{\Delta \to 0} \exp \left[i \left(\tfrac{1}{2} \Delta^{-1} \int d^4x \, F_{\tau} F^{\tau} \right) \right] .$$

(3.4)

In this expression, the index τ has been lowered using the flat-space metric tensor $\eta_{\mu\nu}$. For the remainder of this paper, we shall adopt the standard approach to the covariant quantization of gravity, in which only Lorentz tensors occur, and all raising and lowering of indices is done with respect to flat space. The graviton propagator may be calculated from $I_{\text{sym}} + \tfrac{1}{2} \Delta^{-1} \int d^4x F_{\tau} F^{\tau}$ in the usual fashion, letting $\Delta \to 0$ after inverting.

In another method of calculation, introduced by 't Hooft,[8] the gauge condition is smeared out with a weighting functional. First, the gauge is changed to read $F^{\tau} = e^{\tau}(x)$, where $e^{\tau}(x)$ is an arbitrary four-vector function. Then, it is anticipated that the renormalized S matrix will be gauge invariant, hence independent of $e^{\tau}(x)$. Consequently, the generating functional of Green's functions may be multiplied by some weighting functional $\omega(e^{\tau})$

and functionally integrated over $[de^{\tau}]$ without changing the renormalized S matrix. If the chosen weighting functional is

$$\omega_2(e^{\tau}) = \exp \left[i \left(\tfrac{1}{2} \Delta^{-1} \int d^4x e_{\tau} e^{\tau} \right) \right] ,$$

(3.5)

the graviton propagator is obtained as before from $I_{\text{sym}} + \tfrac{1}{2} \Delta^{-1} \int d^4x F_{\tau} F^{\tau}$, but with Δ now remaining as a finite parameter.

Whenever one performs manipulations of this sort with functional integrals, one must make sure that one has not been too cavalier about questions of convergence. For the lack of an intrinsic definition of the functional integral, one is restricted to perturbation theory in checking whether any new infinities have been introduced. The rationale for using the weighted integral over $[de^{\tau}]$ is the gauge invariance of the renormalized S matrix. This procedure neglects the fact that the Green's functions are not gauge invariant, and by using the weighted functional integral, one possibly may have introduced infinities into the Green's functions which cancel out only when the renormalized S matrix is constructed.

The need for caution in choosing the gauge-fixing term is underscored by the fact that with the commonly used weighting functional (3.5), we have already introduced infinities into the Green's functions. The expression $\tfrac{1}{2} \Delta^{-1} \int d^4x F_{\tau} F^{\tau}$ contains only two derivatives. Consequently, there are parts of the graviton propagator which behave like k^{-2} for large momenta. Specifically, the k^{-2} terms consist of everything but those parts of the propagator which are transverse in all indices. These terms give rise to unpleasant infinities already at the one-loop order. For example, the graviton self-energy diagram shown in Fig. 1 has a divergent part with the general structure $(\partial^4 h)^2$. Such divergences do cancel when they are connected to tree diagrams whose outermost lines are on the mass shell, as they must if the S matrix is to be made finite without introducing counterterms for them. However, they great-
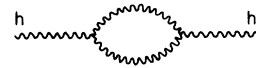

FIG. 1. The one-loop graviton self-energy diagram.

ly complicate the renormalization of Green's functions.

We may attempt to extricate ourselves from the situation described in the last paragraph by picking a different weighting functional.[9] Keeping in mind that we want no part of the graviton propagator to fall off slower than k^{-4} for large momenta, we now choose the weighting functional

$$\omega_4(e^\tau) = \exp\left[i\left(-\tfrac{1}{2}\kappa^2\Delta^{-1}\int d^4x\, e_\tau\Box^2 e^\tau\right)\right]. \quad (3.6)$$

The corresponding gauge-fixing term in the effective action is

$$-\tfrac{1}{2}\kappa^2\Delta^{-1}\int d^4x F_\tau\Box^2 F^\tau. \quad (3.7)$$

The graviton propagator resulting from the gauge-fixing term (3.7) is derived in the Appendix. For most values of the parameters α and β in I_{sym} it satisfies the requirement that all its leading parts fall off like k^{-4} for large momenta. There are, however, specific choices of these parameters which must be avoided. If $\alpha = 0$, the massive spin-2 excitations disappear, and inspection of the graviton propagator shows that some terms then behave like k^{-2}. Likewise, if $3\beta - \alpha = 0$, the massive scalar excitation disappears, and there are again terms in the propagator which behave like k^{-2}.

Unfortunately, even if we avoid the special cases $\alpha = 0$ and $3\beta - \alpha = 0$, and if we use the propagator derived from (3.7), we still do not obtain a clean renormalization of the Green's functions. To examine the question further, we must now turn to the implications of gauge invariance.

IV. BRS TRANSFORMATIONS FOR GRAVITY

The basic tools which we shall employ in the following are a set of supergauge transformations of the gravitational and ghost-field variables that are analogous to those introduced for Yang-Mills theories by Becchi, Rouet, and Stora.[4] These transformations express a residual symmetry of the effective action which remains after the original gauge invariance has been broken by the addition of the gauge-fixing term and the ghost action term.

Before we write down the BRS transformations for gravity, let us first establish the commutation relation for gravitational gauge transformations, which reveals the group structure of the theory. Take the gauge transformation (3.3) of $h^{\mu\nu}$, generated by ξ^μ, and perform a second gauge transformation, generated by η^μ, on the $h^{\mu\nu}$ fields appearing there. Then antisymmetrize in ξ^μ and η^μ.

The result is

$$\frac{\delta D_\alpha^{\mu\nu}}{\delta h^{\rho\sigma}} D_\beta^{\rho\sigma}(\xi^\alpha\eta^\beta - \eta^\alpha\xi^\beta) = \kappa D_\lambda^{\mu\nu}(\partial_\alpha\xi^\lambda\eta^\alpha - \partial_\alpha\eta^\lambda\xi^\alpha).$$

$$(4.1)$$

In this equation, we use an extended summation convention: Repeated indices denote both summation over the discrete values of the indices and integration over the spacetime arguments of the functions or operators indexed. This abbreviated notation is necessary to simplify the writing of the often complicated equations to follow. When there is a possible ambiguity, we shall write out equations in fuller detail.

The BRS transformations for gravity[10] appropriate for the gauge-fixing term (3.6) are

$$\delta_{BRS} h^{\mu\nu} = \kappa D_\alpha^{\mu\nu} C^\alpha\delta\lambda, \quad (4.2a)$$

$$\delta_{BRS} C^\alpha = -\kappa^2\partial_\beta C^\alpha C^\beta\delta\lambda, \quad (4.2b)$$

$$\delta_{BRS}\overline{C}_\tau = -\kappa^3\Delta^{-1}\Box^2 F_\tau\delta\lambda, \quad (4.2c)$$

where $\delta\lambda$ is an infinitesimal anticommuting constant parameter.

The importance of these transformations resides in the quantities which they leave invariant. First, note that the transformation (4.2a) is just a gauge transformation of $h^{\mu\nu}$ generated by $\kappa C^\alpha\delta\lambda$, so gauge-invariant functionals of $h^{\mu\nu}$ alone, like I_{sym}, are BRS invariant too. Another BRS invariance shows that the transformation of C^σ is nilpotent,

$$\delta_{BRS}(\partial_\beta C^\sigma C^\beta) = 0. \quad (4.3)$$

The transformation of $h^{\mu\nu}$ is nilpotent also,

$$\delta_{BRS}(D_\alpha^{\mu\nu} C^\alpha) = 0. \quad (4.4)$$

This last equation follows from the commutation relation (4.1) and the anticommuting nature of C^σ and $\delta\lambda$.

As a result of Eq. (4.4), the only part of the ghost action which varies under the BRS transformations is the antighost $\overline{C}_\tau$. Accordingly, the transformation (4.2c) has been chosen to make the variation of the ghost action just cancel the variation of the gauge-fixing term. Therefore, the entire effective action is BRS invariant:

$$\delta_{BRS}(I_{sym} - \tfrac{1}{2}\kappa^2\Delta^{-1}F_\tau\Box^2 F^\tau + \overline{C}_\tau F_{\mu\nu}^\tau D_\alpha^{\mu\nu} C^\alpha) = 0.$$

$$(4.5)$$

Equations (4.3), (4.4), and (4.5) now enable us to write the Slavnov identities in an economical way.

V. SLAVNOV IDENTITIES

In order to carry out the renormalization program, we will need to have Slavnov identities for

the proper vertices. Our development continues to follow the lines of recent work on Yang-Mills theories.[11,12] We proceed in steps, first discussing the Slavnov identities for Green's functions.

A. Green's functions

To simplify writing the Slavnov identities, we consider an expanded generating functional of Green's functions,

$$Z[T_{\mu\nu},\bar{\beta}_\sigma,\beta^\tau,K_{\mu\nu},L_\sigma]=N\int\left[\prod_{\mu\leq\nu}dh^{\mu\nu}\right][dC^\sigma][d\bar{C}_\tau]\exp[i\,(\bar{\Sigma}\,(h^{\mu\nu},C^\sigma,\bar{C}_\tau,K_{\mu\nu},L_\sigma)+\bar{\beta}_\sigma C^\sigma+\bar{C}_\tau\beta^\tau+\kappa T_{\mu\nu}h^{\mu\nu})]\,.$$

$$(5.1)$$

Anticommuting sources have been included for the ghost and antighost fields, and the effective action $\bar{\Sigma}$ has been enlarged by the inclusion of BRS invariant couplings of the ghosts and gravitons to some external fields $K_{\mu\nu}$ (anticommuting) and L_σ (commuting),

$$\bar{\Sigma}=I_{\text{sym}}-\tfrac{1}{2}\kappa^2\Delta^{-1}F_\tau\Box^2F^\tau+\bar{C}_\tau\vec{\tilde{F}}^\tau_{\mu\nu}D^{\mu\nu}_\alpha C^\alpha$$
$$+\kappa K_{\mu\nu}D^{\mu\nu}_\alpha C^\alpha+\kappa^2 L_\sigma\partial_\beta C^\sigma C^\beta\,.$$

$$(5.2)$$

$\bar{\Sigma}$ is BRS invariant by virtue of Eqs. (4.3), (4.4), and (4.5). We may use the new couplings to write this invariance as

$$\frac{\delta\bar{\Sigma}}{\delta K_{\mu\nu}}\frac{\delta\bar{\Sigma}}{\delta h^{\mu\nu}}+\frac{\delta\bar{\Sigma}}{\delta L_\sigma}\frac{\delta\bar{\Sigma}}{\delta C^\sigma}+\kappa^3\Delta^{-1}\Box^2F_\tau\frac{\delta\bar{\Sigma}}{\delta\bar{C}_\tau}=0\,.$$

$$(5.3)$$

In this equation, and throughout this paper, we use left variational derivatives with respect to anticommuting quantities: $\delta f(C^\sigma)=\delta C^\tau\,\delta f/\delta C^\tau$. Equation (5.3) may be simplified by rewriting it in terms of a reduced effective action,

$$\Sigma=\bar{\Sigma}+\tfrac{1}{2}\kappa^2\Delta^{-1}F_\tau\Box^2F^\tau\,.$$

$$(5.4)$$

Substitution of (5.4) into (5.3) yields

$$\frac{\delta\Sigma}{\delta K_{\mu\nu}}\frac{\delta\Sigma}{\delta h^{\mu\nu}}+\frac{\delta\Sigma}{\delta L_\sigma}\frac{\delta\Sigma}{\delta C^\sigma}=0\,,$$

$$(5.5)$$

where we have used the relation

$$\kappa^{-1}\vec{\tilde{F}}^\tau_{\mu\nu}\frac{\delta\Sigma}{\delta K_{\mu\nu}}-\frac{\delta\Sigma}{\delta\bar{C}_\tau}=0\,.$$

$$(5.6)$$

When we examine the Jacobian which arises upon performing the BRS transformations on the integration variables of a functional integral, we find that the metric

$$\left[\prod_{\mu\leq\nu}dh^{\mu\nu}\right][dC^\sigma][d\bar{C}_\tau]$$

is BRS invariant. For infinitesimal transformations, the Jacobian is 1, because of the trace relations

$$\frac{\delta^2\bar{\Sigma}}{\delta K_{(\mu\nu)}\delta h^{(\mu\nu)}}=0\,,$$

$$(5.7a)$$

$$\frac{\delta^2\bar{\Sigma}}{\delta C^\sigma\delta L_\sigma}=0\,,$$

$$(5.7b)$$

both of which follow from $\int d^4x\,\partial_\alpha C^\alpha=0$. The parentheses surrounding the indices in (5.7a) indicate that the summation is to be carried out only for $\mu\leq\nu$.

The Slavnov identity for the generating functional of Green's functions is obtained by performing the BRS transformations (4.2) on the integration variables in the generating functional (5.1). This transformation does not change the value of the

generating functional, so we obtain

$$N\int\left[\prod_{\mu\leq\nu}dh^{\mu\nu}\right][dC^\sigma][d\bar{C}_\tau](\kappa^2 T_{\mu\nu}D^{\mu\nu}_\alpha C^\alpha-\kappa^2\bar{\beta}_\sigma\partial_\beta C^\sigma C^\beta+\kappa^3\Delta^{-1}\beta^\tau\Box^2\vec{\tilde{F}}_{\tau\mu\nu}h^{\mu\nu})\exp[i(\bar{\Sigma}+\kappa T_{\mu\nu}h^{\mu\nu}+\bar{\beta}_\sigma C^\sigma+\bar{C}_\tau\beta^\tau)]=0\,.$$

$$(5.8)$$

Another identity which we shall need is the ghost equation of motion. To derive this equation, we shift the antighost integration variable $\bar{C}_\tau$ to $\bar{C}_\tau+\delta\bar{C}_\tau$, again with no resulting change in the value of the generating functional:

$$N\int\left[\prod_{\mu\leq\nu}dh^{\mu\nu}\right][dC^\sigma][d\bar{C}_\tau]\left(\frac{\delta\bar{\Sigma}}{\delta\bar{C}_\tau}+\beta^\tau\right)\exp[i(\bar{\Sigma}+\kappa T_{\mu\nu}h^{\mu\nu}+\bar{\beta}_\sigma C^\sigma+\bar{C}_\tau\beta^\tau)]=0\,.$$

$$(5.9)$$

We can now define the generating functional of connected Green's functions as the logarithm of the functional (5.1),

$$W[T_{\mu\nu},\bar{\beta}_\sigma,\beta^\tau,K_{\mu\nu},L_\sigma]$$
$$=-i\ln Z[T_{\mu\nu},\bar{\beta}_\sigma,\beta^\tau,K_{\mu\nu},L_\sigma]\,,\quad(5.10)$$

and make use of the couplings to the external fields $K_{\mu\nu}$ and L_σ to rewrite (5.8) in terms of W:

$$\kappa T_{\mu\nu}\frac{\delta W}{\delta K_{\mu\nu}} - \bar{\beta}_\sigma \frac{\delta W}{\delta L_\sigma} + \kappa^2 \Delta^{-1}\beta^\tau \Box^2 \vec{F}_{\tau\mu\nu}\frac{\delta W}{\delta T_{\mu\nu}} = 0 .$$

(5.11)

Similarly, we can rewrite the ghost equation of motion:

$$\kappa^{-1}\vec{F}_{\mu\nu}^\tau \frac{\delta W}{\delta K_{\mu\nu}} + \beta^\tau = 0 .$$

(5.12)

B. Proper vertices

A Legendre transformation takes us from the generating functional of connected Green's functions (5.10) to the generating functional of proper vertices. First, we define the expectation values of the gravitational, ghost, and antighost fields in the presence of the sources $T_{\mu\nu}$, $\bar{\beta}_\sigma$, and β^τ and the external fields $K_{\mu\nu}$ and L_σ:

$$h^{\mu\nu}(x) = \frac{\delta W}{\kappa \delta T_{\mu\nu}(x)} ,$$

(5.13a)

$$C^\sigma(x) = \frac{\delta W}{\delta \bar{\beta}_\sigma(x)} ,$$

(5.13b)

$$\bar{C}_\tau(x) = -\frac{\delta W}{\delta \beta^\tau(x)} .$$

(5.13c)

We have chosen to denote the expectation values of the fields by the same symbols which were used for the fields in the effective action (5.2).

The Legendre transformation can now be performed, giving us the generating functional of proper vertices as a functional of the new variables (5.13) and the external fields $K_{\mu\nu}$ and L_σ,

$$\tilde{\Gamma}[h^{\mu\nu}, C^\sigma, \bar{C}_\tau, K_{\mu\nu}, L_\sigma]$$
$$= W[T_{\mu\nu}, \bar{\beta}_\sigma, \beta^\tau, K_{\mu\nu}, L_\sigma] - \kappa T_{\mu\nu}h^{\mu\nu} - \bar{\beta}_\sigma C^\sigma - \bar{C}_\tau \beta^\tau .$$

(5.14)

In this equation, the quantities $T_{\mu\nu}$, $\bar{\beta}_\sigma$, and β^τ are given implicitly in terms of $h^{\mu\nu}$, C^σ, $\bar{C}_\tau$, $K_{\mu\nu}$, and L_σ by Eq. (5.13).

The relations dual to (5.13) are

$$\kappa T_{\mu\nu}(x) = -\frac{\delta \tilde{\Gamma}}{\delta h^{\mu\nu}(x)} ,$$

(5.15a)

$$\bar{\beta}_\sigma(x) = \frac{\delta \tilde{\Gamma}}{\delta C^\sigma(x)} ,$$

(5.15b)

$$\beta_\tau(x) = -\frac{\delta \tilde{\Gamma}}{\delta \bar{C}_\tau(x)} .$$

(5.15c)

Since the external fields $K_{\mu\nu}$ and L_σ do not participate in the Legendre transformation (5.14), for them we have the relations

$$\frac{\delta \tilde{\Gamma}}{\delta K_{\mu\nu}(x)} = \frac{\delta W}{\delta K_{\mu\nu}(x)} ,$$

(5.16a)

$$\frac{\delta \tilde{\Gamma}}{\delta L_\sigma(x)} = \frac{\delta W}{\delta L_\sigma(x)} .$$

(5.16b)

Finally, the Slavnov identity for the generating functional of proper vertices is obtained by transcribing (5.11) using the relations (5.13), (5.15), and (5.16):

$$\frac{\delta \tilde{\Gamma}}{\delta K_{\mu\nu}}\frac{\delta \tilde{\Gamma}}{\delta h^{\mu\nu}} + \frac{\delta \tilde{\Gamma}}{\delta L_\sigma}\frac{\delta \tilde{\Gamma}}{\delta C^\sigma} + \kappa^3 \Delta^{-1}\Box^2 \vec{F}_{\tau\mu\nu}h^{\mu\nu}\frac{\delta \tilde{\Gamma}}{\delta \bar{C}_\tau} = 0 .$$

(5.17)

We also have the ghost equation of motion,

$$\kappa^{-1}\vec{F}_{\mu\nu}^\tau \frac{\delta \tilde{\Gamma}}{\delta K_{\mu\nu}} - \frac{\delta \tilde{\Gamma}}{\delta \bar{C}_\tau} = 0 .$$

(5.18)

Since Eq. (5.17) has exactly the same form as (5.3), we follow the example set by (5.4) and define a reduced generating functional of proper vertices,

$$\Gamma = \tilde{\Gamma} + \tfrac{1}{2}\kappa^2 \Delta^{-1}(\vec{F}_{\tau\mu\nu}h^{\mu\nu})\Box^2(\vec{F}_{\rho\sigma}^\tau h^{\rho\sigma}) .$$

(5.19)

Substituting this into (5.17) and (5.18), the Slavnov identity becomes

$$\frac{\delta \Gamma}{\delta K_{\mu\nu}}\frac{\delta \Gamma}{\delta h^{\mu\nu}} + \frac{\delta \Gamma}{\delta L_\sigma}\frac{\delta \Gamma}{\delta C^\sigma} = 0 ,$$

(5.20)

and the ghost equation of motion becomes

$$\kappa^{-1}\vec{F}_{\mu\nu}^\tau \frac{\delta \Gamma}{\delta K_{\mu\nu}} - \frac{\delta \Gamma}{\delta \bar{C}_\tau} = 0 .$$

(5.21)

Equations (5.20) and (5.21) are of exactly the same form as (5.5) and (5.6). This is as it should be, since at the zero-loop order,

$$\Gamma^{(0)} = \Sigma .$$

(5.22)

VI. STRUCTURE OF THE DIVERGENCES

The Slavnov identity (5.20) is quadratic in the functional Γ. This nonlinearity is reflected in the fact that the renormalization of the effective action generally also involves the renormalization of the BRS transformations which must leave the effective action invariant.

There are two different approaches to this problem in the literature on the renormalization of Yang-Mills theories. The first of these[4] begins by showing formally that the effective action may be renormalized in such a way that the analog of our equation (5.3) holds exactly. This equation then determines the structure of the renormalized effective action. The second approach[13,14] uses the Slavnov identity for the generating functional of proper vertices to derive a linear equation for the divergent parts of the proper vertices. This

equation is then solved to display the structure of the divergences. From this structure, it can be seen how to renormalize the effective action so that it remains invariant under a renormalized set of BRS transformations.

We shall follow the second of these two approaches because it keeps to the forefront the structure of the divergences, and is thus more appropriate for establishing renormalizability.

A. Renormalization equation

Suppose that we have successfully renormalized the reduced effective action up to $n-1$ loop order; that is, suppose we have constructed a quantum extension of Σ which satisfies Eqs. (5.5) and (5.6) exactly, and which leads to finite proper vertices when calculated up to order $n-1$. We will denote this renormalized quantity by $\Sigma^{(n-1)}$. In general, it contains terms of many different orders in the loop expansion, including orders greater than $n-1$. The $n-1$ loop part of the reduced generating functional of proper vertices will be denoted by $\Gamma^{(n-1)}$.

When we proceed to calculate $\Gamma^{(n)}$, we find that it contains divergences. Some of these come from n-loop Feynman integrals. Since all the subintegrals of an n-loop Feynman integral contain less than n loops, they are finite by assumption. Therefore, the divergences which arise from n-loop Feynman integrals come only from the overall divergences of the integrals, so the corresponding parts of $\Gamma^{(n)}$ are local in structure. In the dimensional regularization procedure, these divergences are of order $\epsilon^{-1} = (d-4)^{-1}$, where d is the dimensionality of spacetime in the Feynman integrals.

There may also be divergent parts of $\Gamma^{(n)}$ which do not arise from loop integrals, and which contain higher-order poles in the regulating parameter ϵ. Such divergences comes from n-loop order parts of $\Sigma^{(n-1)}$ which are necessary to ensure that (5.5) is satisfied. Consequently, they too have a local structure.

We may separate the divergent and finite parts of $\Gamma^{(n)}$:

$$\Gamma^{(n)} = \Gamma^{(n)}_{\text{div}} + \Gamma^{(n)}_{\text{finite}} . \tag{6.1}$$

If we insert this breakup into Eq. (5.20), and keep only the terms of the equation which are of n-loop order, we get

$$\frac{\delta\Gamma^{(n)}_{\text{div}}}{\delta K_{\mu\nu}}\frac{\delta\Gamma^{(0)}}{\delta h^{\mu\nu}} + \frac{\delta\Gamma^{(0)}}{\delta K_{\mu\nu}}\frac{\delta\Gamma^{(n)}_{\text{div}}}{\delta h^{\mu\nu}} + \frac{\delta\Gamma^{(n)}_{\text{div}}}{\delta L_\sigma}\frac{\delta\Gamma^{(0)}}{\delta C^\sigma} + \frac{\delta\Gamma^{(0)}}{\delta L_\sigma}\frac{\delta\Gamma^{(n)}_{\text{div}}}{\delta C^\sigma}$$

$$= -\sum_{i=0}^{n}\left(\frac{\delta\Gamma^{(n-i)}_{\text{finite}}}{\delta K_{\mu\nu}}\frac{\delta\Gamma^{(i)}_{\text{finite}}}{\delta h^{\mu\nu}} + \frac{\delta\Gamma^{(n-i)}_{\text{finite}}}{\delta L_\sigma}\frac{\delta\Gamma^{(i)}_{\text{finite}}}{\delta C^\sigma}\right).$$

$$\tag{6.2}$$

Since each term on the right-hand side of (6.2) remains finite as $\epsilon \to 0$, while each term on the left-hand side contains a factor with at least a simple pole in ϵ, each side of the equation must vanish separately. Remembering (5.22), we can write the following equation, called the renormalization equation:

$$\mathcal{G}\Gamma^{(n)}_{\text{div}} = 0 , \tag{6.3}$$

where

$$\mathcal{G} = \frac{\delta\Sigma}{\delta h^{\mu\nu}}\frac{\delta}{\delta K_{\mu\nu}} + \frac{\delta\Sigma}{\delta C^\sigma}\frac{\delta}{\delta L_\sigma}$$

$$+ \frac{\delta\Sigma}{\delta K_{\mu\nu}}\frac{\delta}{\delta h^{\mu\nu}} + \frac{\delta\Sigma}{\delta L_\sigma}\frac{\delta}{\delta C^\sigma} . \tag{6.4}$$

In similar fashion, collecting the n-loop order divergences in the ghost equation of motion (5.21) gives us

$$\kappa^{-1}\vec{F}^\tau_{\mu\nu}\frac{\delta\Gamma^{(n)}_{\text{div}}}{\delta K_{\mu\nu}} - \frac{\delta\Gamma^{(n)}_{\text{div}}}{\delta\overline{C}_\tau} = 0 . \tag{6.5}$$

B. Local solutions

Our task now is to construct local solutions to Eqs. (6.3) and (6.5). This may be done if we note that the operator defined in (6.4) is nilpotent:

$$\mathcal{G}^2 = 0 . \tag{6.6}$$

To show this, it is convenient to write $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_0$, where

$$\mathcal{G}_1 = \frac{\delta\Sigma}{\delta h^{\mu\nu}}\frac{\delta}{\delta K_{\mu\nu}} + \frac{\delta\Sigma}{\delta C^\sigma}\frac{\delta}{\delta L_\sigma}$$

and

$$\mathcal{G}_0 = \frac{\delta\Sigma}{\delta K_{\mu\nu}}\frac{\delta}{\delta h^{\mu\nu}} + \frac{\delta\Sigma}{\delta L_\sigma}\frac{\delta}{\delta C^\sigma} .$$

The operator $\mathcal{G}_0$ just generates the zero-loop BRS transformations (4.2a),(4.2b), so by (4.3) and (4.4) we know that it is nilpotent. Explicit calculation then shows that the remaining terms cancel:

$$\mathcal{G}_1^2 + \{\mathcal{G}_0, \mathcal{G}_1\} = 0 .$$

Equation (6.6) leads us to consider local solutions to Eq. (6.3) of the form

$$\Gamma^{(n)}_{\text{div}} = \mathcal{S}(h^{\mu\nu}) + \mathcal{G}[X(h^{\mu\nu}, C^\sigma, \overline{C}^\tau, K_{\mu\nu}, L_\sigma)] , \tag{6.7}$$

where $\mathcal{S}$ is an arbitrary gauge-invariant local functional of $h^{\mu\nu}$ and its derivatives, and X is an arbitrary local functional of $h^{\mu\nu}$, C^σ, $\overline{C}^\tau$, $K_{\mu\nu}$, and L_σ and their derivatives. Kluberg-Stern and Zuber[13] have made the conjecture that the analog of our Eq. (6.7) is in fact the most general local solution to the Yang-Mills renormalization equation. This conjecture has been proven for Yang-

Mills theories by Joglekar and Lee,[14] using a rather involved argument. We shall not attempt to duplicate that argument here, and shall be content to leave (6.7) as a conjecture of the general local solution to Eq. (6.3). It will be sufficient for our purposes to show the difficulties which solutions of the form (6.7) get us into.

In order to satisfy the ghost equation of motion (6.5) we require that

$$\Gamma_{\text{div}}^{(n)} = \Gamma_{\text{div}}^{(n)}(h^{\mu\nu}, C^{\sigma}, K_{\mu\nu} - \kappa^{-1}\bar{C}_{\tau}\bar{\mathbb{F}}_{\mu\nu}^{\tau}, L_{\sigma}) . \qquad (6.8)$$

C. Ghost number and power counting

A glance at the effective action (5.2) shows that we may define the following conserved quantity, called ghost number:

$$N_g[h^{\mu\nu}] = 0, \quad N_g[C^{\sigma}] = +1, \quad N_g[\bar{C}_{\tau}] = -1,$$

$$N_g[K_{\mu\nu}] = -1, \quad N_g[L_{\sigma}] = -2 . \qquad (6.9)$$

It follows from this definition that

$$N_g[\Sigma] = N_g[\Gamma] = 0 . \qquad (6.10)$$

Since

$$N_g[\mathbb{9}] = +1 , \qquad (6.11)$$

we require of the functional X that

$$N_g[X] = -1 . \qquad (6.12)$$

To complete our analysis of the structure of $\Gamma_{\text{div}}^{(n)}$, we must supplement the symmetry equations (6.7), (6.8), and (6.12) with the constraints on the divergences which arise from power counting. Accordingly, we introduce the following notations: n_E = number of graviton vertices with two derivatives, n_G = number of antighost-graviton-ghost vertices, n_K = number of K-graviton-ghost vertices, n_L = number of L-ghost-ghost vertices, I_G = number of internal-ghost propagators, E_C = number of external ghosts, $E_{\bar{C}}$ = number of external antighosts.

Since graviton propagators behave like k^{-4}, and ghost propagators like k^{-2}, we are led by standard power counting to the degree of divergence of an arbitrary diagram,

$$D = 4 - 2n_E + 2I_G - 2n_G - 3n_L - 3n_K - E_{\bar{C}} . \qquad (6.13)$$

The last term in (6.13) arises because each external antighost line carries with it a factor of external momentum. We can make use of the topo-
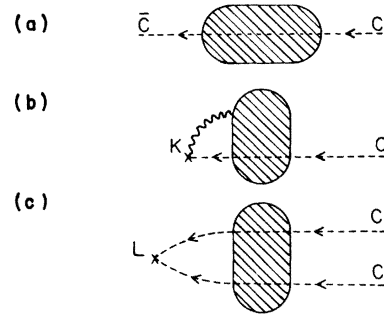

FIG. 2. The three types of divergent diagram which involve external ghost lines. Arbitrarily many gravitons may emerge from each of the central regions. (a) Ghost action type. (b) K type. (c) L type.

logical relation

$$2I_G - 2n_G = 2n_L + n_K - E_C - E_{\bar{C}} \qquad (6.14)$$

to write the degree of divergence as

$$D = 4 - 2n_E - n_L - 2n_K - E_C - 2E_{\bar{C}} . \qquad (6.15)$$

Together with conservation of ghost number, Eq. (6.15) enables us to catalog three different types of divergent structures involving ghosts. These are illustrated in Fig. 2. Each of the three types has degree of divergence $D = 1 - 2n_E$. Consequently, all the divergences which involve ghosts have $n_E = 0$. Since the degree of divergence is then 1, the associated divergent structures in $\Gamma_{\text{div}}^{(n)}$ have an extra derivative appearing on one of the fields. Diagrams whose external lines are all gravitons have degree of divergence $D = 4 - 2n_E$.

Combining (6.15) with (6.12), (6.8), and (6.7), we can finally write the most general expression for $\Gamma_{\text{div}}^{(n)}$ which satisfies all the constraints of symmetries and power counting:

$$\Gamma_{\text{div}}^{(n)} = \mathbb{S}(h^{\mu\nu}) + \mathbb{9}[(K_{\mu\nu} - \kappa^{-1}\bar{C}\bar{\mathbb{F}}_{\mu\nu}^{\tau})P^{\mu\nu}(h^{\alpha\beta})$$

$$+ L_{\sigma}Q_{\tau}^{\sigma}(h^{\alpha\beta})C^{\tau}] , \qquad (6.16)$$

where $P^{\mu\nu}(h^{\alpha\beta})$ and $Q_{\tau}^{\sigma}(h^{\alpha\beta})$ are arbitrary Lorentz-covariant functions of the gravitational field $h^{\mu\nu}$, but not of its derivatives, at a single spacetime point. $\mathbb{S}(h^{\mu\nu})$ is a local gauge-invariant functional of $h^{\mu\nu}$ containing terms with four, two, and zero derivatives.

Expanding (6.16), we obtain an array of possible divergent structures,

$$\Gamma_{\text{div}}^{(n)} = \mathbb{S} + \frac{\delta I_{\text{sym}}}{\delta h^{\mu\nu}}P^{\mu\nu} + (\kappa K_{\rho\sigma} - \bar{C}_{\tau}\bar{\mathbb{F}}_{\rho\sigma}^{\tau})\left(\frac{\delta D_{\alpha}^{\rho\sigma}}{\delta h^{\mu\nu}}C^{\alpha}\right)P^{\mu\nu} - (\kappa K_{\rho\sigma} - \bar{C}_{\tau}\bar{\mathbb{F}}_{\rho\sigma}^{\tau})\frac{\delta P^{\rho\sigma}}{\delta h^{\mu\nu}}D_{\alpha}^{\mu\nu}C^{\alpha}$$

$$- (\kappa K_{\mu\nu} - \bar{C}_{\tau}\bar{\mathbb{F}}_{\mu\nu}^{\tau})D_{\sigma}^{\mu\nu}(Q_{\epsilon}^{\sigma}C^{\epsilon}) - \kappa^2 L_{\sigma}\partial_{\beta}(Q_{\tau}^{\sigma}C^{\tau})C^{\beta}$$

$$- \kappa^2 L_{\sigma}\partial_{\beta}C^{\sigma}Q_{\tau}^{\beta}C^{\tau} - \kappa L_{\sigma}\frac{\delta Q_{\tau}^{\sigma}}{\delta h^{\mu\nu}}C^{\tau}D_{\alpha}^{\mu\nu}C^{\alpha} + \kappa^2 L_{\sigma}Q_{\tau}^{\sigma}\partial_{\beta}C^{\tau}C^{\beta} . \qquad (6.17)$$

The breakup between the gauge-invariant divergences S and the rest of (6.17) is determined only up to a term of the form

$$\int d^4x (\eta^{\mu\nu} + \kappa h^{\mu\nu}) \frac{\delta I_{\text{sym}}}{\kappa \delta h^{\mu\nu}}, \qquad (6.18)$$

which can be generated by adding to $P^{\mu\nu}$ a term proportional to $\eta^{\mu\nu} + \kappa h^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$. Explicit calculations of sample one-loop diagrams reveal that divergences such as those allowed by (6.17) do occur. Aside from the fact of their not vanishing, the results of these calculations are not particularly enlightening, and will not be presented here.

VII. RENORMALIZATION

The profusion of divergences allowed by (6.17) appears to make the task of renormalizing the effective action rather complicated. Although the many divergent structures do pose a considerable nuisance for practical calculations, the situation is still reminiscent in principle of the renormalization of Yang-Mills theories. There, the non-gauge-invariant divergences may be eliminated by a number of field renormalizations. We shall find the same to be true here, but because the gravitational field $h^{\mu\nu}$ carries no weight in the power counting, there is nothing to prevent the field renormalizations from being nonlinear, or from mixing the gravitational and ghost fields.

A. Field and transformation renormalizations

Many of the divergences in (6.17) are canceled if we replace the reduced effective action $\Sigma^{\langle n-1 \rangle}$ by

$$\Sigma^{\langle n-1 \rangle}[h^{\mu\nu} - P^{\mu\nu}(h^{\alpha\beta}), C^\sigma + Q_\tau^\sigma(h^{\alpha\beta})C^\tau, \overline{C}_\tau, K_{\mu\nu}, L_\sigma].$$

$$(7.1)$$

Leaving aside the gauge-invariant divergences, (7.1) is not yet suitable as a renormalization of $\Sigma^{\langle n-1 \rangle}$ because it does not satisfy Eq. (5.5), and because the non-gauge-invariant divergences are not all canceled until we add the additional counterterms

$$[\kappa K_{\rho\sigma} - (\overline{C}\overline{F}_{\rho\sigma}^\tau)] \frac{\delta P^{\rho\sigma}}{\delta h^{\mu\nu}} D_\alpha^{\mu\nu} C^\alpha \qquad (7.2a)$$

and

$$\kappa L_\sigma \frac{\delta Q_\tau^\sigma}{\delta h^{\mu\nu}} C^\tau D_\alpha^{\mu\nu} C^\alpha - \kappa^2 L_\sigma Q_\tau^\sigma \partial_\beta C^\tau C^\beta. \qquad (7.2b)$$

Fortunately, these two problems solve each other to a certain extent. Addition of the terms (7.2) to (7.1) results in an expression which satisfies Eq. (5.5) up to n-loop order, since the n-loop-order parts of this expression satisfy (6.3).

The additional counterterms (7.2) provide cor-

rections to the forms of the BRS transformations and the ghost action. The necessity for these corrections may be appreciated if we remember that the BRS transformations (4.2) were derived for a specific choice of the gravitational and ghost fields. In (7.1), we have made substitutions in $\Sigma^{\langle n-1 \rangle}$ corresponding to nonlinear transformations and mixing of these fields, and should change accordingly the forms of the ghost action and of the K and L interaction terms.

To explain these changes in more detail, let us first concentrate on the renormalizations which are necessitated by the divergences dependent upon $P^{\mu\nu}$. The gauge transformation (3.3) does not leave $I_{\text{sym}}[h^{\mu\nu} - P^{\mu\nu}(h^{\alpha\beta})]$ invariant, because the functional dependence on $h^{\mu\nu}$ has been changed. However, if we define

$$h_{(u)}^{\mu\nu}(h^{\alpha\beta}) = h^{\mu\nu} - P^{\mu\nu}(h^{\alpha\beta}), \qquad (7.3)$$

we find that the gauge transformation (3.3) does correctly transform $h_{(u)}^{\mu\nu}$.

The inverse of the field transformation (7.3) is given by

$$h^{\mu\nu} = h_{(u)}^{\mu\nu} + P*^{\mu\nu}(h_{(u)}^{\alpha\beta}). \qquad (7.4)$$

Although $P^{\mu\nu}$ contains only terms of n-loop order, since all the lower-order divergences are assumed to have been canceled by previous renormalizations, its nonlinear structure implies that the inverse transformation (7.4) does contain terms of order higher than n loop. These higher-order terms in $P*^{\mu\nu}$ are also of higher order in the dimensional regularization poles in ϵ. Such terms must be taken care of in subsequent renormalizations, but since at this stage we are only trying to make the proper vertices finite to n-loop order, they do not concern us now. Limited to the n-loop-order terms only, we have

$$[P*^{\mu\nu}]_{n\text{ loop}} = P^{\mu\nu}. \qquad (7.5)$$

It follows from Eq. (7.4) that the correctly renormalized gauge transformation of $h^{\mu\nu}$ is given by the operator

$$D_\sigma^{\langle n \rangle\mu\nu}[h^{\alpha\beta}] = D_\sigma^{\mu\nu}[h_{(u)}^{\alpha\beta}] + \frac{\delta P*^{\mu\nu}[h_{(u)}^{\alpha\beta}]}{\delta h_{(u)}^{\rho\tau}} D_\sigma^{\rho\tau}[h_{(u)}^{\alpha\beta}]. \qquad (7.6)$$

This renormalized gauge transformation operator can now be used in constructing the renormalized ghost action and K interaction terms. Equations (7.5) and (7.6) show that when $D_\sigma^{\langle n \rangle\mu\nu}[h^{\alpha\beta}]$ is used in these constructions, the resulting n-loop-order changes in the reduced effective action are, aside from the replacement of $h^{\mu\nu}$ by $h_{(u)}^{\mu\nu}(h^{\alpha\beta})$, the addition of precisely the expression given in (7.2a).

Similarly, the change in the functional dependence on C^σ leads to a change in the BRS trans-

formation of C^σ. We define

$$C^\sigma_{(u)}(C^\rho, h^{\alpha\beta}) = C^\sigma + Q^\sigma_T(h^{\alpha\beta})C^\tau . \qquad (7.7)$$

The inverse transformation is

$$C^\sigma = C^\sigma_{(u)} - Q^{*\sigma}_T(h^{\alpha\beta}_{(u)})C^\tau_{(u)} , \qquad (7.8)$$

and here also we have, at n-loop order,

$$[Q^{*\sigma}_T]_{n\ \text{loop}} = Q^\sigma_T . \qquad (7.9)$$

The renormalized gauge transformation (7.6) still satisfies the commutation relation (4.1). Consequently, Eq. (7.8) shows that the correctly renormalized L interaction term is

$$\kappa^2 L_\sigma U^{<n>\sigma}_{\rho\tau}[h^{\alpha\beta}]C^\rho C^\tau = \kappa^2 L_\sigma \partial_\tau C^\sigma_{(u)}C^\tau_{(u)} + \kappa L_\sigma \frac{\delta Q^{*\sigma}_T}{\delta h^{\mu\nu}_{(u)}}(h^{\alpha\beta}_{(u)})C^\tau_{(u)}D^{\mu\nu}_\lambda[h^{\alpha\beta}_{(u)}]C^\lambda_{(u)} - \kappa^2 L_\sigma Q^{*\sigma}_T(h^{\alpha\beta}_{(u)})\partial_\lambda C^\tau_{(u)}C^\lambda_{(u)} .$$

$$(7.10)$$

At the n-loop order, the changes in the reduced effective action brought about by (7.10) are, aside from the replacement of C^σ by $C^\sigma_{(u)}(C^\rho, h^{\alpha\beta})$, the addition of the expressions given in (7.2b).

B. Gauge-invariant renormalizations

We have not yet taken into account the gauge-invariant divergences $S(h^{\alpha\beta})$ which occur in (6.17). These must be canceled by gauge-invariant counterterms in the reduced effective action.

As we noted in Sec. VI, the gauge-invariant divergences involve either four, two, or zero derivatives. There are only four such invariants for us to consider:

$$S_I = \alpha^{(n)} \int R_{\mu\nu}R^{\mu\nu}\sqrt{-g} - \beta^{(n)} \int R^2\sqrt{-g}$$

$$+ \gamma^{(n)}\kappa^{-2} \int R\sqrt{-g} \qquad (7.11a)$$

and

$$S_\lambda = -\lambda^{(n)}\kappa^{-4} \int \sqrt{-g} , \qquad (7.11b)$$

where $\alpha^{(n)}$, $\beta^{(n)}$, $\lambda^{(n)}$, and $\lambda^{(n)}$ are some divergent coefficients.

The divergences (7.11a) may be canceled by renormalizations of the appropriate coefficients in I_{sym}. The divergent structure (7.11b) is more troublesome. Its presence may be verified by an explicit calculation of the one-loop tadpole diagram shown in Fig. 3. Such tadpoles do not occur in general relativity, or in a model whose action consists solely of quadratic products of the curvature tensor, because all the poles in the propagators for such models are massless, and dimensional regularization sets the integral $\int d^4k$ equal to zero. The graviton propagator derived from the action (2.1) contains massive poles, however, so the tadpole no longer vanishes. The parameters α and β occur in the tadpole divergence in the form $b_1\alpha^{-2} + b_2(3\beta - \alpha)^{-2}$, where b_1 and b_2 have the same sign, so there are no special

values of α and β which give cancellation.

If we had started with an action which included a cosmological term $\lambda\kappa^{-4}\int d^4x\sqrt{-g}$, we could renormalize the parameter λ to eliminate divergences like (7.11b). We have not done so because flat space would then no longer be a solution of the classical field equations. Instead of getting involved in the resulting complications of defining initial and final states, etc., we simply chose to make the renormalized value of λ equal zero. Our renormalized action does then contain a cosmological term, but it is present only to cancel out the divergences like (7.11b).

To summarize this section, we give the prescription for constructing the renormalized reduced effective action $\Sigma^{<n>}$. The gauge-invariant divergences in $\Gamma^{(n)}_{\text{div}}$ are eliminated by adding $\alpha^{(n)}$, $\beta^{(n)}$, $\gamma^{(n)}$, and $\lambda^{(n)}$ to the corresponding coefficients in $\Sigma^{<n-1>}$. The non-gauge-invariant divergences are then eliminated by substituting $h^{\mu\nu} - P^{\mu\nu}(h^{\alpha\beta})$ for $h^{\mu\nu}$ as the argument of the (gauge-invariant) terms depending on $h^{\mu\nu}$ alone, and by replacing the terms involving ghosts with

$$\Sigma^{<n>}_{\text{ghosts}} = (\kappa K_{\mu\nu} - \bar{C}_\tau \vec{F}^\tau_{\mu\nu})D^{<n>\mu\nu}_\sigma[h^{\alpha\beta}][C^\sigma + Q^\sigma_T(h^{\alpha\beta})C^\tau]$$

$$+ \kappa^2 L_\sigma U^{<n>\sigma}_{\rho\tau}[h^{\alpha\beta}]C^\rho C^\tau . \qquad (7.12)$$

$\Sigma^{<n>}$ satisfies Eq. (5.5) by construction.

The full effective action $\bar{\Sigma}^{<n>}$ is obtained by adding back the gauge-fixing term:

$$\bar{\Sigma}^{<n>} = \Sigma^{<n>} - \tfrac{1}{2}\kappa^2\Delta^{-1}(\vec{F}^\tau_{\mu\nu}h^{\mu\nu})\Box^2(\vec{F}_{\tau\rho\sigma}h^{\rho\sigma}) .$$

$$(7.13)$$

There are no divergences in (6.17) which must be removed by renormalizing the gauge-fixing term.
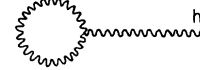

FIG. 3. The one-loop graviton tadpole diagram.

This term has been taken care of automatically in our discussion through the definition (5.4) of the reduced effective action. In particular, it should be noted that the substitution of $h^{\mu\nu} - P^{\mu\nu}(h^{\alpha\beta})$ for $h^{\mu\nu}$ is *not* carried out in the gauge-fixing term. Also, the sources $T_{\mu\nu}$ and $\bar{\beta}_\sigma$ in the generating functional of Green's functions remain coupled just to $h^{\mu\nu}$ and C^σ. It is on account of these unrenormalized terms that we have absorbed the field renormalizations into changes in the functional form of Σ, instead of making more direct use of the "bare" quantities $h^{\mu\nu}_{(u)}$ and $C^\sigma_{(u)}$. Our derivation of the Slavnov identities has relied upon the fact that the gauge-fixing term (3.7) is derived from a gauge condition (3.1) which is linear in the gravitational field $h^{\mu\nu}$. If we were to rewrite the theory in terms of the field $h^{\mu\nu}_{(u)}$, the corresponding gauge condition would be nonlinear. Such gauge conditions can be handled by the BRS methods,[15] but we shall not pursue the details here.

VIII. CLEANER METHODS

The renormalization procedure described in the last section is sufficiently complicated to make practical calculations unappealing. We now turn to other choices of the gauge-fixing term which greatly simplify matters by eliminating the need for the field and transformation renormalizations.

A. Unweighted gauge condition

Explicit calculations of samples of the non-gauge-invariant divergences allowed by (6.17) reveal that they depend upon the gauge-fixing parameter Δ which was introduced into the effective action by the weighting functional (3.6). This suggests that if we take the limit $\Delta \to 0$, all the field and transformation renormalizations may disappear. This limit as $\Delta \to 0$ returns us to the unweighted gauge condition

$$\partial_\nu h^{\mu\nu} = 0 , \tag{8.1}$$

with the same Feynman rules as those obtained using the simple Gaussian representation (3.4) of the gauge-fixing δ function.

The graviton propagator in the limit $\Delta \to 0$ may be calculated as suggested above, setting $\Delta = 0$ in the propagator calculated for finite Δ (cf. Appendix), or by substituting the gauge condition (8.1) into the linearized classical field equations and then inverting. The resulting propagator is constructed entirely from projectors which are transverse in all their indices:

$$D^{\Delta=0}_{\mu\nu\,\rho\sigma}(k)$$
$$= \frac{1}{(2\pi)^4 i}\left(\frac{2P^{(2)}_{\mu\nu\rho\sigma}(k)}{k^2(\alpha\kappa^2 k^2 + \gamma)} - \frac{2P^{(0-s)}_{\mu\nu\rho\sigma}(k)}{k^2[(3\beta - \alpha)\kappa^2 k^2 + \frac{1}{2}\gamma]} \right) . \tag{8.2}$$

The definitions of the projectors $P^{(2)}$ and $P^{(0-s)}$ are given in the Appendix.

The antighost-graviton-ghost interaction is

$$V_{\bar{C}hC} = \partial_\sigma \bar{C}_\alpha \partial_\rho h^{\rho\sigma} C^\alpha + \partial_\alpha \bar{C}_\sigma \partial_\rho h^{\rho\sigma} C^\alpha$$
$$+ \partial_{\rho\sigma}{}^2 \bar{C}_\alpha h^{\rho\sigma} C^\alpha . \tag{8.3}$$

The first two terms in this expression contain the gauge condition (8.1), and consequently do not connect to the graviton propagator (8.2). Similarly, integration by parts in the remaining term can be used to move the derivatives onto the ghost field C^α. When these derivatives fall on $h^{\rho\sigma}$, they form the gauge condition (8.1) again, so we have effectively

$$V_{\bar{C}hC} \approx \partial_{\rho\sigma}{}^2 \bar{C}_\alpha h^{\rho\sigma} C^\alpha \approx \bar{C}_\alpha h^{\rho\sigma} \partial_{\rho\sigma}{}^2 C^\alpha . \tag{8.4}$$

The symbol $\approx$ is used to indicate that terms containing $\partial_\rho h^{\rho\sigma}$ or $\partial_{\rho\sigma}{}^2 h^{\rho\sigma}$ have been dropped, since they do not connect to the graviton propagator.

The power-counting rule given in Sec. VI must be modified as a results of (8.4). In one-particle-irreducible (1PI) diagrams, there is a separate vertex $V_{\bar{C}hC}$ for each external ghost and antighost line. Consequently, each of these lines carries with it two factors of external momentum. The resulting degree of divergence of an arbitrary 1PI diagram is

$$D^{1PI}_{(\Delta=0)} = 4 - 2n_E - n_L - 2n_K - 3E_C - 3E_{\bar{C}} . \tag{8.5}$$

This result would hold even if we had not chosen (2.2) as our definition of the gravitational field variable. However, the simple relation (8.4) is dependent upon that choice, which accords with the harmonic gauge condition (8.1). Otherwise there would be a complicated cancellation between vertices.

From the power-counting rule (8.5), we see that each of the three types of diagrams shown in Fig. 2 is now convergent: The ghost action type has $D^{1PI}_{(\Delta=0)} = -2 - 2n_E$, the K type has $D^{1PI}_{(\Delta=0)} = -1 - 2n_E$ and the L type has $D^{1PI}_{(\Delta=0)} = -3 - 2n_E$. Therefore, there are no parts of $\Gamma^{(n)}_{\text{div}\,(\Delta=0)}$ which depend upon ghosts:

$$\frac{\delta\Gamma^{(n)}_{\text{div}}(\Delta=0)}{\delta C^\sigma} = 0 , \tag{8.6a}$$

$$\frac{\delta\Gamma^{(n)}_{\text{div}}(\Delta=0)}{\delta K_{\mu\nu}} = 0 , \tag{8.6b}$$

$$\frac{\delta\Gamma^{(n)}_{\text{div}}(\Delta=0)}{\delta L_\sigma} = 0 . \tag{8.6c}$$

Insertion of Eqs. (8.6) into the renormalization Eq. (6.3) yields

$$\frac{\delta \Sigma}{\delta K_{\mu\nu}} \frac{\delta \Gamma_{\rm div}^{(n)}(\Delta = 0)}{\delta h^{\mu\nu}} = 0 \, . \tag{8.7}$$

Together with (8.6a), this implies that $\Gamma_{\rm div}^{(n)}(\Delta = 0)$ is gauge invariant. All the divergences may therefore be eliminated by renormalizations of the parameters α, β, and γ in $I_{\rm sym}$ and by the addition of a cosmological counterterm. The field variables and the BRS transformations do not need to be renormalized.

The contrast between the complicated renormalization procedure which one must use when the quantum theory is defined with the gauge-fixing term (3.7) and the much simpler procedure for the unweighted gauge condition is reminiscent of the situation in the axial gauge in Yang-Mills theory. There, the ghosts decouple entirely from the Yang-Mills fields if one uses the unweighted axial gauge condition. However, if one smears the axial gauge with a weighting functional, the resulting propagator does connect to the ghosts, and then there arise non-gauge-invariant divergences. These Yang-Mills divergences are similar to those we would have obtained in the gravitational theory had we kept the two-derivative gauge-fixing term derived from (3.5). In both cases, the part of the propagator which depends upon the gauge-fixing parameter has a bad asymptotic behavior for large momenta, leading to non-gauge-invariant divergences of progressively higher order as the calculation proceeds in perturbation theory.

Taking the limit $\Delta \to 0$ is necessary for the axial gauge quantization of Yang-Mills theory to avoid these artifactual divergences. However, this limit is less useful in other gauges: Although one obtains an improvement in the power counting just as we have found for gravitation, the improvement is not sufficient to eliminate all the non-gauge-invariant divergences, and one must still renormalize the Yang-Mills gauge transformation. Thus, although taking the limit $\Delta \to 0$ is perfectly acceptable in Yang-Mills theory, it is generally of no particular advantage, and has not been much used in the literature.

B. Third-derivative gauge

Since we are dealing with theories in which the classical field equations involve fourth derivatives, the Cauchy data which must be initially specified to determine the classical evolution of the field include the values of the field and up to its third derivatives on some spacelike hypersurface. Accordingly, we should also be prepared to use gauge conditions which involve up to third derivatives. A gauge condition of this type which has the same structure as the harmonic gauge condition (8.1) is

$$\kappa^2 \Box^2 \partial_\nu h^{\mu\nu} = 0 \, . \tag{8.8}$$

If we weight the gauge condition (8.8) with the Gaussian functional (3.5), we get the gauge-fixing term

$$\tfrac{1}{2} \kappa^4 \Delta^{-1} (\Box^2 \partial_\nu h^{\mu\nu})(\Box^2 \partial_\lambda h_\mu^\lambda) \, . \tag{8.9}$$

Another way to arrive at (8.9) is to start from the usual harmonic gauge condition (8.1) and to weight it with the functional

$$\omega_6(e_\tau) = \exp\left[i \left(\tfrac{1}{2} \kappa^4 \Delta^{-1} \int (\Box^2 e_\tau)(\Box^2 e^\tau) \right) \right] \, . \tag{8.10}$$

When we obtain (8.9) this second way, it is clear that the ghost action which we must use is exactly the same that we had before in the generating functional (3.2). This also follows from the first method of arriving at (8.9), because we may always redefine the antighost field: $\Box^2 \overline{C}_\tau \to \overline{C}_\tau$.

The gauge-fixing term (8.9) requires us to change the BRS transformation of the antighost field $\overline{C}_\tau$. The new transformation is

$$\delta_{(6)\,\rm BRS} \overline{C}_\tau = \kappa^5 \Delta^{-1} \Box^2 \Box^2 F_\tau \, \delta\lambda \, . \tag{8.11}$$

The Slavnov identities for the generating functionals of Green's functions and of proper vertices must be changed too, but the identity for the reduced generating functional of proper vertices,

$$\Gamma_{(6)} = \tilde{\Gamma}_{(6)} - \tfrac{1}{2} \kappa^4 \Delta^{-1} (\Box^2 F_{\tau\mu\nu} h^{\mu\nu})(\Box^2 F_{\rho\sigma}^\tau h^{\rho\sigma}) \, , \tag{8.12}$$

remains the same as (5.20). Consequently, the renormalization equation is the same as (6.3).

The Feynman rules which we obtain using (8.9) differ from those obtained using (3.7) only in the replacement of the factors of $\Delta \kappa^{-2} k^{-4}$ in the graviton propagator by $\Delta \kappa^{-4} k^{-6}$. This change brings about a reduction in the degree of divergence of those parts of diagrams which depend on the parameter Δ. The degree of divergence is reduced by 2 for each factor of Δ, so that once again all three types of diagram involving ghosts shown in Fig. 2 are convergent. The renormalization equation then implies that all the divergences in $\Gamma_{(6)\,\rm div}^{(n)}$ are gauge invariant.

IX. GAUGE-FIXING PARAMETER

We have found that the non-gauge-invariant divergences of the theory are dependent on the choice of weighting functional used to derive the gauge-fixing term. We may ask whether the

gauge-invariant divergences (7.11) also depend on the choice of weighting functional through a dependence of the coefficients $\alpha^{(n)}$, $\beta^{(n)}$, $\gamma^{(n)}$, and $\lambda^{(n)}$ on the gauge-fixing parameter Δ. We do not expect this to be so, because unlike the non-gauge-invariant divergences, the divergences (7.11) must be eliminated if we are to make the S matrix finite. Therefore, with the possible exception of a term proportional to the classical field equation, the divergences (7.11) should not depend on artifacts of our quantization procedure. On the other hand, power counting does not exclude a dependence on Δ.

To resolve this question, we now write another kind of identity[12,13] for the reduced generating functional Γ, an identity which expresses the dependence of the proper vertices on the parameter Δ. We will derive this identity using the gauge-fixing term (3.7), but since the final equation will be written in terms of the reduced gen-

erating functional Γ, it will also be valid for the theory quantized with the gauge fixing term (8.9).

We begin by adding to the effective action (5.2) another interaction term which couples the antighost and gravitational fields:

$$\bar{\Sigma}(h^{\mu\nu}, \ldots, Y, \Delta) = \bar{\Sigma}(h^{\mu\nu}, \ldots, 0, \Delta)$$
$$+ \kappa^{-1} Y \int \vec{F}^{\sigma}_{\mu\nu} h^{\mu\nu} \overline{C}_{\sigma}, \qquad (9.1)$$

where Y is a constant anticommuting parameter with ghost number $+1$. Next, we define the reduced effective action,

$$\Sigma = \bar{\Sigma} - \kappa^{-1} Y F^{\sigma} \overline{C}_{\sigma} + \tfrac{1}{2}\kappa^2 \Delta^{-1} F_{\tau} \square^2 F^{\tau}. \qquad (9.2)$$

The new term in (9.1) is not BRS invariant:

$$\delta_{\text{BRS}}(\kappa^{-1} Y F^{\sigma} \overline{C}_{\sigma}) = [-Y\kappa^2\Delta^{-1} F^{\sigma} \square^2 F_{\sigma}$$
$$+ Y\overline{C}_{\sigma} \vec{F}^{\sigma}_{\mu\nu} D^{\mu\nu}_{\alpha} C^{\alpha}]\delta\lambda. \qquad (9.3)$$

Taking (9.3) into account, the Slavnov identity for Green's functions becomes

$$N \int \left[\prod_{\mu \le \nu} dh^{\mu\nu} \right] [dC^{\sigma}][d\overline{C}_{\tau}] \left(\kappa T_{\mu\nu} \frac{\delta}{\delta K_{\mu\nu}} - \bar{\beta}_{\sigma} \frac{\delta}{\delta L_{\sigma}} + \kappa^2\Delta^{-1}\beta^{\tau} \square^2 \vec{F}_{\tau\mu\nu} \frac{\delta}{\delta T_{\mu\nu}} \right.$$
$$\left. - 2Y\Delta \frac{d}{d\Delta} + iY\overline{C}_{\sigma} \vec{F}^{\sigma}_{\mu\nu} D^{\mu\nu}_{\alpha} C^{\alpha} \right) \exp[i(\bar{\Sigma} + \kappa T_{\mu\nu} h^{\mu\nu} + \bar{\beta}_{\sigma} C^{\sigma} + \overline{C}_{\tau} \beta^{\tau})] = 0. \qquad (9.4)$$

The new ghost equation of motion is

$$N \int \left[\prod_{\mu \le \nu} dh^{\mu\nu} \right] [dC^{\sigma}][d\overline{C}_{\tau}](\vec{F}^{\tau}_{\mu\nu} D^{\mu\nu}_{\alpha} C^{\alpha} - \kappa^{-1} Y \vec{F}^{\tau}_{\mu\nu} h^{\mu\nu} + \beta^{\tau}) \exp[i(\cdots)] = 0. \qquad (9.5)$$

Combining this with (9.4), we may write the Slavnov identity for connected Green's functions,

$$\kappa T_{\mu\nu} \frac{\delta W}{\delta K_{\mu\nu}} - \bar{\beta}_{\sigma} \frac{\delta W}{\delta L_{\sigma}} + \kappa^2\Delta^{-1}\beta^{\tau} \square^2 \vec{F}_{\tau\mu\nu} \frac{\delta W}{\delta T_{\mu\nu}} - 2Y\Delta \frac{dW}{d\Delta} - Y\beta^{\tau} \frac{\delta W}{\delta\beta^{\tau}} = 0. \qquad (9.6)$$

In order to arrive at (9.6), we have used the fact that $Y^2 = 0$ and we have dropped a term proportional to $\delta^4(0)$.

The generating functional of proper vertices is still defined as in Eq. (5.14). Noting that $d\bar{\Gamma}/d\Delta = dW/d\Delta$, we may transcribe (9.6) to obtain the Slavnov identity for the generating functional of proper vertices:

$$\frac{\delta\bar{\Gamma}}{\delta h^{\mu\nu}} \frac{\delta\bar{\Gamma}}{\delta K_{\mu\nu}} + \frac{\delta\bar{\Gamma}}{\delta L_{\sigma}} \frac{\delta\bar{\Gamma}}{\delta C^{\sigma}} + \kappa^3\Delta^{-1} \square^2 \vec{F}_{\tau\mu\nu} h^{\mu\nu} \frac{\delta\bar{\Gamma}}{\delta \overline{C}_{\tau}}$$
$$+ 2Y\Delta \frac{d\bar{\Gamma}}{d\Delta} - Y\overline{C}_{\tau} \frac{\delta\bar{\Gamma}}{\delta \overline{C}_{\tau}} = 0. \qquad (9.7)$$

Following the example of (9.2), we define the reduced generating functional of proper vertices,

$$\Gamma = \bar{\Gamma} - \kappa^{-1} Y \vec{F}^{\sigma}_{\mu\nu} h^{\mu\nu} \overline{C}_{\sigma}$$
$$+ \tfrac{1}{2}\kappa^2\Delta^{-1}(\vec{F}_{\tau\mu\nu} h^{\mu\nu})\square^2(\vec{F}^{\tau}_{\rho\sigma} h^{\rho\sigma}). \qquad (9.8)$$

When we rewrite the Slavnov identity (9.7) in terms of Γ, we obtain the simpler equation

$$\frac{\delta\Gamma}{\delta h^{\mu\nu}} \frac{\delta\Gamma}{\delta K_{\mu\nu}} + \frac{\delta\Gamma}{\delta L_{\sigma}} \frac{\delta\Gamma}{\delta C^{\sigma}} + 2Y\Delta \frac{d\Gamma}{d\Delta} = 0. \qquad (9.9)$$

We have also the ghost equation of motion,

$$\kappa^{-1} \vec{F}^{\tau}_{\mu\nu} \frac{\delta\Gamma}{\delta K_{\mu\nu}} - \frac{\delta\Gamma}{\delta \overline{C}_{\tau}} = 0. \qquad (9.10)$$

Equation (9.9) leads to a modified renormalization equation for the n-loop divergences $\Gamma^{(n)}_{\text{div}}(h^{\mu\nu}, \ldots, Y, \Delta)$:

$$\left(\mathcal{G} + 2Y\Delta \frac{d}{d\Delta} \right) \Gamma^{(n)}_{\text{div}}(h^{\mu\nu}, \ldots, Y, \Delta) = 0, \qquad (9.11)$$

where the operator $\mathcal{G}$ is the same as in (6.4). Taking a derivative with respect to Y in (9.11), and then setting Y equal to zero, we obtain finally

$$\Delta \frac{d}{d\Delta} \Gamma^{(n)}_{\text{div}}(h^{\mu\nu}, \dots, 0, \Delta)$$

$$= \mathbf{g} \left(\frac{d}{dY} \Gamma^{(n)}_{\text{div}}(h^{\mu\nu}, \dots, Y, \Delta) \right)_{Y=0}. \quad (9.12)$$

Comparison of (9.12) with (6.16) shows that the gauge-invariant divergences $\mathbf{g}(h^{\mu\nu})$ are independent of the gauge-fixing parameter Δ, up to terms of the form

$$\int d^4x (\eta^{\mu\nu} + \kappa h^{\mu\nu}) \frac{\delta I_{\text{sym}}}{\kappa \delta h^{\mu\nu}};$$

these latter terms may be absorbed by a field renormalization [cf. Eq. (6.18)].

If we use the gauge-fixing term (8.9) in the quantization procedure, we can avoid making use of the conjecture (6.7), since then conservation of ghost number and power counting imply that

$$\left(\frac{d}{dY} \Gamma^{(n)}_{(\mathbf{g}) \text{div}}(h^{\mu\nu}, \dots, Y, \Delta) \right)_{Y=0} = 0. \quad (9.13)$$

Equation (9.12) then yields directly

$$\frac{d\Gamma^{(n)}_{(\mathbf{g}) \text{div}}}{d\Delta} = 0. \quad (9.14)$$

X. COUPLING TO SCALAR MATTER

Now that we know how to carry out the renormalization procedure for a purely gravitational model, it is straightforward to include coupling to other renormalizable fields. As an example, we discuss a massive scalar field in interaction with the gravitational field alone, adding to the action (2.1) the additional term

$$I_\phi = \int d^4x (-\tfrac{1}{2} \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} - \tfrac{1}{2} m^2 \phi^2) \sqrt{-g}. \quad (10.1)$$

The BRS transformations must now include a transport term for the scalar field,

$$\delta_{\text{BRS}} \phi = -\kappa^2 \partial_\mu \phi C^\mu \delta\lambda. \quad (10.2)$$

This transformation is nilpotent:

$$\delta_{\text{BRS}} (\partial_\mu \phi C^\mu) = 0. \quad (10.3)$$

In order to write the Slavnov identities, we make use of (10.3) by adding a term coupling the scalar and ghost fields to a new anticommuting external field $B(x)$:

$$\Sigma_\phi = I_{\text{sym}} + I_\phi + (\kappa K_{\mu\nu} - \overline{C}_\tau \overline{F}^\tau_{\mu\nu}) D^{\mu\nu}_\alpha C^\alpha$$

$$+ \kappa^2 L_\sigma \partial_\beta C^\sigma C^\beta - \kappa^2 B \partial_\mu \phi C^\mu. \quad (10.4)$$

In the generating functional of Green's functions, the scalar field is coupled to a source $J(x)$; the Legendre transformation then trades this dependence on $J(x)$ for a dependence on $\phi(x) = \delta W/\delta J(x)$ in the generating functional of proper vertices. The Slavnov identity for the reduced generating functional of proper vertices reads

$$\frac{\delta \Gamma_\phi}{\delta B} \frac{\delta \Gamma_\phi}{\delta \phi} + \frac{\delta \Gamma_\phi}{\delta K_{\mu\nu}} \frac{\delta \Gamma_\phi}{\delta h^{\mu\nu}} + \frac{\delta \Gamma_\phi}{\delta L_\sigma} \frac{\delta \Gamma_\phi}{\delta C^\sigma} = 0. \quad (10.5)$$

As before, this identity leads to the renormalization equation for $\Gamma^{(n)}_{\phi \text{div}}$.

Power counting, using the unweighted gauge condition, gives the degree of divergence of an arbitrary 1PI diagram,

$$D^{1\text{PI}}_\phi(\Delta = 0) = 4 - 2n_E - n_L - 2n_K - n_B$$

$$- 3E_C - 3E_{\overline{C}} - 2E_S, \quad (10.6)$$

where n_B is the number of B-scalar-ghost vertices and E_S is the number of external scalar lines. The external scalar lines are counted twice in (10.6) because of the linkage of scalar fields and derivatives in the interaction between scalars and gravitons (the mass term is super-renormalizable and is not included in the power counting). This linkage is similar to the linkage of ghosts and derivatives which we have already encountered.

The power-counting rule (10.6), together with the conservation of ghost number, shows that all 1PI diagrams with external ghost lines are convergent, so that

$$\frac{\delta \Sigma_\phi}{\delta B} \frac{\delta \Gamma^{(n)}_{\phi \text{div}}}{\delta \phi} + \frac{\delta \Sigma_\phi}{\delta K_{\mu\nu}} \frac{\delta \Gamma^{(n)}_{\phi \text{div}}}{\delta h^{\mu\nu}} = 0. \quad (10.7)$$

Consequently, $\Gamma^{(n)}_{\phi \text{div}}$ is gauge invariant. The only gauge-invariant structures consistent with (10.6) are

$$\Gamma^{(n)}_{\phi \text{div}} = \alpha^{(n)} \int R_{\mu\nu} R^{\mu\nu} \sqrt{-g} - \beta^{(n)} \int R^2 \sqrt{-g} + \gamma^{(n)} \kappa^{-2} \int R \sqrt{-g} - \lambda^{(n)} \kappa^{-4} \int \sqrt{-g}$$

$$+ \tfrac{1}{2} f^{(n)} \int \partial_\mu \phi \partial_\nu \phi g^{\mu\nu} \sqrt{-g} + \tfrac{1}{2} (\delta m^2)^{(n)} \int \phi^2 \sqrt{-g}. \quad (10.8)$$

These divergences may be eliminated by renormalizations of the appropriate coefficients in I_{sym} and I_ϕ, and by the addition of a cosmological counterterm. It should be noted that the absence

of a term like $\int R \phi^2 \sqrt{-g}$ in (10.8) is due to the linkage of scalars and derivatives. If this linkage were broken by the inclusion in (10.1) of a scalar self-interaction $\int \phi^4 \sqrt{-g}$, then it would be nec-

essary to include as well the nonminimal gravitational-scalar interaction.

The scalar field example shows that once renormalizability has been established for a purely gravitational model, the inclusion of couplings to other renormalizable fields poses no further problems (except possibly the necessity for a nonminimal gravitational-scalar interaction). In particular, the Faddeev-Popov ghost machinery remains unrenormalized just as it did in the purely gravitational case. The allowed divergences may be summarized by assigning a power-counting weight to each field, and then requiring that divergent structures be gauge invariant and of power-counting weight four or less. It is necessary to take into account any linkages of fields and derivatives in the interactions by augmenting the weight of a field by the number of derivatives linked to it. The weight of the gravitational field is zero, and before linkages with derivatives are taken into account, the weights of other fields are simply given by their canonical dimensions.

XI. A NOTE ON GENERAL RELATIVITY

The gauge-invariant divergences which make general relativity nonrenormalizable are by now familiar.[2] We can use power counting and the renormalization equation (6.3) to predict the kind of non-gauge-invariant divergences that are also likely to occur.

The renormalization equation (6.3) is valid for all values of α and β, including zero, so it holds unchanged for general relativity. The power-counting rule is now quite different, however, since the graviton propagator behaves like k^{-2} for large momenta. This leads to the degree of divergence of an 1PI diagram,

$$D_{(\Delta=0)}^{\text{Einstein, 1PI}} = 4 + 2 I_h + 2 I_G - 2 n_E - 2 n_G$$

$$- 3 n_L - 3 n_K - 2 E_C - 2 E_{\bar{C}}, \qquad (11.1)$$

where I_h is the number of internal graviton propagators. To eliminate I_h and I_G, we use the Euler relation for the number of loops in a diagram,

$$l - 1 = I_h + I_G - n_E - n_L - n_G - n_K. \qquad (11.2)$$

Thus,

$$D_{(\Delta=0)}^{\text{Einstein, 1PI}} = 2 + 2l - n_L - n_K - 2 E_C - 2 E_{\bar{C}}. \qquad (11.3)$$

At first glance, it would appear that a divergence of the form $\partial^2 \bar{C} \partial^2 C$ could develop at the one-loop order. However, according to (6.7), this divergence would have to come from $\mathcal{G}[(K_{\mu\nu} - \kappa^{-1} \bar{C}_\tau \bar{F}_{\mu\nu}^\tau) P^{\mu\nu}]$, and power counting now would require that $P^{\mu\nu}$ contain two derivatives, so there would also have to be divergences of the

form $\partial \bar{C} \partial h \partial^2 C$. These do not occur, since two derivatives must appear on each ghost and antighost field. Thus, $P_{1\text{loop}}^{\mu\nu} = 0$, so all the divergences at the one-loop order are gauge invariant (provided one uses the unweighted gauge condition).

In higher orders, non-gauge-invariant divergences should be as plentiful as the gauge-invariant ones presumably are, in any covariant gauge and regardless of whether one weights the gauge condition or not. Like the gauge-invariant divergences, they will also involve progressively higher derivatives of the gravitational field.

XII. CONCLUSION

The renormalizable models which we have considered in this paper should be regarded as constructs for a study of the ultraviolet problem of quantum gravity. The difficulties with unitarity appear to preclude their direct acceptability as physical theories. They do have some promise as phenomenological models, however, for their unphysical behavior may be restricted to arbitrarily large energy scales by an appropriate limitation on the renormalized masses m_2 and m_0. Actually, it is only the massive spin-two excitations of the field which give the trouble with unitarity and thus require a very large mass. The limit on the mass m_0 is determined only by the observational constraints on the static field.

It remains to be seen just how much the analogy to Pauli-Villars regularization remains valid in the full quantum theory for these models. The higher-derivative additions to the action do not make them finite, but do control the divergences enough to make them renormalizable. The Pauli-Villars analogy would seem to indicate that letting the mass m_2 go to infinity will just bring back some infinities which have been postponed by the higher derivatives. On the other hand, the quantum interactions may change the structure of the theory in such a way that letting the renormalized mass m_2 tend to infinity makes sense even though the same limit with the unrenormalized mass is unacceptable. Alternatively, perhaps there is a way to obtain a unitary theory without going to the infinite-mass limit. Further investigation of these questions appears to require the use of nonperturbative techniques.

APPENDIX: THE GRAVITON PROPAGATOR

The inversion of the gravitational kinetic matrix which is necessary to calculate the graviton propagator involves a substantial amount of Lorentz algebra on symmetric rank-two tensors. To organize the calculation, it is convenient to use a set of orthogonal projectors in momentum space. We choose a set of projectors which emphasises transversality,[16] since this is important in Sec. VIII.

Our projectors are constructed using the transverse and longitudinal projectors for vector quantities,

$$\theta_{\mu\nu} = \eta_{\mu\nu} - k_\mu k_\nu / k^2 , \tag{A1a}$$

$$\omega_{\mu\nu} = k_\mu k_\nu / k^2 . \tag{A1b}$$

The four projectors for symmetric rank-two tensors are then

$$P^{(2)}_{\mu\nu\,\rho\sigma} = \tfrac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - \tfrac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} , \tag{A2a}$$

$$P^{(1)}_{\mu\nu\,\rho\sigma} = \tfrac{1}{2}(\theta_{\mu\rho}\omega_{\nu\sigma} + \theta_{\mu\sigma}\omega_{\nu\rho} + \theta_{\nu\rho}\omega_{\mu\sigma} + \theta_{\nu\sigma}\omega_{\mu\rho}) , \tag{A2b}$$

$$P^{(0-s)}_{\mu\nu\,\rho\sigma} = \tfrac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} , \tag{A2c}$$

$$P^{(0-w)}_{\mu\nu\,\rho\sigma} = \omega_{\mu\nu}\omega_{\rho\sigma} . \tag{A2d}$$

For a massive tensor field in the rest frame, the projectors (A2a)–(A2d) select out the spin-two, spin-one, and two spin-zero parts of the field. However, the projectors (A2) do not span the operator space of the gravitational field equations. In order to have a complete basis, we must also include the two spin-zero transfer operators,

$$P^{(0-sw)}_{\mu\nu\,\rho\sigma} = 3^{-1/2}\theta_{\mu\nu}\omega_{\rho\sigma} , \tag{A3a}$$

$$P^{(0-ws)}_{\mu\nu\,\rho\sigma} = 3^{-1/2}\omega_{\mu\nu}\theta_{\rho\sigma} . \tag{A3b}$$

The orthogonality relations of the projectors (A2) and the transfer operators (A3) are

$$P^{i\,-a}P^{j\,-b} = \delta^{ij}\delta^{ab}P^{i\,-b} , \tag{A4a}$$

$$P^{i\,-ab}P^{j\,-cd} = \delta^{ij}\delta^{bc}P^{j\,-a} , \tag{A4b}$$

$$P^{i\,-a}P^{j\,-bc} = \delta^{ij}\delta^{ab}P^{j\,-ac} , \tag{A4c}$$

$$P^{i\,-ab}P^{j\,-c} = \delta^{ij}\delta^{bc}P^{j\,-ac} , \tag{A4d}$$

where i and j run from 0 to 2, and a and b take on the values w and s.

In order to calculate the graviton propagator, we must first write out the part of the effective action (5.2) which is purely quadratic in the gravitational field $h^{\mu\nu}$. Going over to momentum space and using (A2) and (A3), we have

$$\frac{1}{4}\int d^4k h^{\mu\nu}(-k)(-(\alpha\kappa^2 k^2 + \gamma)k^2 P^{(2)}_{\mu\nu\,\rho\sigma}(k) + \Delta^{-1}\kappa^2 k^4 P^{(1)}_{\mu\nu\rho\sigma}(k) + \{3k^2[(3\beta - \alpha)\kappa^2 k^2 + \tfrac{1}{2}\gamma] + 2\Delta^{-1}\kappa^2 k^4\} P^{(0-w)}_{\mu\nu\rho\sigma}(k)$$

$$+ k^2[(3\beta - \alpha)\kappa^2 k^2 + \tfrac{1}{2}\gamma]\{P^{(0-s)}_{\mu\nu\rho\sigma}(k) + \sqrt{3}\,[P^{(0-ws)}_{\mu\nu\rho\sigma}(k) + P^{(0-sw)}_{\mu\nu\rho\sigma}(k)]\})h^{\rho\sigma}(k) . \tag{A5}$$

The combination of parameters $(3\beta - \alpha)$ which occurs throughout this expression is an echo of the conformally invariant action $\int d^4x \sqrt{-g}\,(R_{\mu\nu}R^{\mu\nu} - \tfrac{1}{3}R^2) = \tfrac{1}{2}\int d^4x \sqrt{-g}\,C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}$, where $C_{\mu\nu\alpha\beta}$ is the Weyl tensor. The orthogonality relations (A4) may now be used in inverting the kinetic matrix shown in (A5) to obtain the graviton propagator:

$$D_{\mu\nu\rho\sigma}(k) = \frac{1}{(2\pi)^4 i}\left(\frac{2P^{(2)}_{\mu\nu\rho\sigma}(k)}{k^2(\alpha\kappa^2 k^2 + \gamma)} - \frac{2P^{(0-s)}_{\mu\nu\rho\sigma}(k)}{k^2([3\beta - \alpha]\kappa^2 k^2 + \tfrac{1}{2}\gamma)} - \frac{2\Delta P^{(1)}_{\mu\nu\rho\sigma}(k)}{\kappa^2 k^4}\right.$$

$$\left. - \frac{\Delta(3P^{(0-s)}_{\mu\nu\rho\sigma}(k) - \sqrt{3}\,[P^{(0-sw)}_{\mu\nu\rho\sigma}(k) + P^{(0-ws)}_{\mu\nu\rho\sigma}(k)] + P^{(0-w)}_{\mu\nu\rho\sigma}(k))}{\kappa^2 k^4}\right) . \tag{A6}$$

To determine the propagator (A6) completely, we must specify how the k_0 integration contour is to skirt the poles in calculating Feynman integrals. We do this in the customary way by including $i\epsilon$ terms in the denominators of the individual poles, which must first be obtained by separating (A6) into partial fractions. Ignoring for the moment the terms proportional to Δ, we find

$$\frac{1}{(2\pi)^4 i}\left(\frac{2[P^{(2)}_{\mu\nu\rho\sigma}(k) - 2P^{(0-s)}_{\mu\nu\rho\sigma}(k)]}{\gamma k^2} - \frac{2P^{(2)}_{\mu\nu\rho\sigma}(k)}{\gamma(k^2 + \gamma[\alpha\kappa^2]^{-1})} + \frac{4P^{(0-s)}_{\mu\nu\rho\sigma}(k)}{\gamma(k^2 + \gamma[2(3\beta - \alpha)\kappa^2]^{-1})}\right) . \tag{A7}$$

Normally, one requires that quantum states have positive-definite norm and energy. Such states give rise to poles in the propagator with positive residues. Since both the massless pole and the pole at $k^2 = -\gamma[2(3\beta - \alpha)\kappa^2]^{-1}$ in (A7) do have positive residues, we shift them in the standard fash-

ion, replacing the denominators respectively by

$$(k^2 - i\epsilon) \tag{A8}$$

and by

$$\{k^2 + \gamma[2(3\beta - \alpha)\kappa^2]^{-1} - i\epsilon\} . \tag{A9}$$

On the other hand, the negative residue of the massive spin-two pole at $k^2 = -\gamma[\alpha\kappa^2]^{-1}$ faces us with a choice between two unfortunate alternatives: to give up either the positive definiteness of the norm or of the energy of the corresponding quantum states. Both choices give the required negative residue, but they differ in the way the pole must be shifted.

If the massive spin-two states are taken to have negative norm, the situation is analogous to a Pauli-Villars regularized theory. We recall that in the usual derivation of the propagator, one starts from $\langle 0| T[h_{\mu\nu}(x) h_{\rho\sigma}(x')] |0\rangle$, transforms to momentum space, and sums over a complete set of momentum eigenstates inserted between the two field operators. The only difference in the present case is that the negative-norm states must be accompanied by a vector space metric factor of (-1) in the sum over states. This gives rise to a negative residue for the massive spin-two pole, but does not affect the location of the pole, whose denominator is consequently given by

$$(k^2 + \gamma[\alpha\kappa^2]^{-1} - i\epsilon).\tag{A10a}$$

As the Pauli-Villars analogy leads us to expect, the choice (A10a), together with (A8) and (A9), gives a high-energy behavior of the total propagator which is like k^{-4}. To see this, one may, for example, perform a Wick rotation into Euclidean space and then drop the $i\epsilon$ terms. This is allowed because (A10a), (A8), and (A9) all shift the poles in the same way.

If the massive spin-two states are taken to have negative energy, the pole in the propagator acquires a negative residue for a different reason. In this case, there are no vector space metric factors in the sum over states, but the expansion of the field operators into creation and annihilation operators involves normalization factors $(2|k_0|)^{-1/2} = (-2k_0)^{-1/2}$. These contribute an overall minus sign to the massive spin-two part of the propagator. In addition, the sign of the energy flow for a given time ordering is opposite to that for a positive-energy field, so the denominator of the pole is now given by

$$(k^2 + \gamma[\alpha\kappa^2]^{-1} + i\epsilon).\tag{A10b}$$

The difference between the poles given by (A10a) and (A10b) is a term proportional to $\delta(k^2 + \gamma[\alpha\kappa^2]^{-1})$. While the choice of (A10a) leads to the desired k^{-4} behavior, this additional term effectively spoils the high-energy behavior of (A10b). Thus, our power-counting requirements lead us to adopt an indefinite-metric state vector space, following the analogy to Pauli-Villars regularization.

The pure k^{-4} terms in (A6), proportional to Δ, may be handled by confluence, replacing them by $\zeta^{-1}[(k^2 - i\epsilon)^{-1} - (k^2 + \zeta - i\epsilon)^{-1}]$, and then letting $\zeta \to 0$ at the end of the calculation.

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