Weyl and Schrödinger — Two Geometries

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Abstract

The non-Riemannian geometry of the German mathematical physicist Hermann Weyl, developed in 1918 shortly after Einstein's announcement of the general theory of relativity, represented the first comprehensive effort to embed electromagnetism into the geometrical formalism of general relativity — the first unified field theory. An initial admirer of Weyl's geometry, Einstein discovered a seemingly irreconcilable flaw in the theory that showed it to be unphysical. However, the sheer beauty of the theory and its fundamental notions of gauge and conformal invariance were such that researchers to this day have continued to apply the theory to physics, including cosmology and quantum mechanics. Here we outline how Weyl might have surmounted Einstein's primary objection to the theory by a simple reinterpretation of the non-metricity tensor using an approach suggested (perhaps accidentally) some thirty years later by Schrödinger, whose geometry was similar to but fundamentally different than Weyl's. We also demonstrate that either approach, which both involve the non-metricity tensor, introduces fundamental and intractable problems into general relativity that appear to render hopeless a purely classical route to the unification of gravity and electromagnetism.

Introduction

In 1918 Hermann Weyl proposed a unification of gravitation and electromagnetism based on the surmised invariance of physics with respect to a conformal (or scale) transformation of the metric tensor $g_{\mu\nu} \to e^{\pi(x)} g_{\mu\nu}$, where $\pi(x)$ is an arbitrary scalar function. A decade later Weyl's idea was recast as *gauge symmetry*, which subsequently became a cornerstone of quantum theory. More recently, the notion of conformal symmetry has been explored in numerous cosmological models, and there is increasing speculation that a conformally invariant Riemannian geometry may indeed underlie all of physics.

Weyl's theory, which introduced a non-Riemannian geometry in an effort to embed electromagnetism into general relativity as a purely geometrical construct, necessarily relied upon a Lagrangian that was invariant with respect to the local rescaling of the metric tensor $g_{\mu\nu} \to e^{\pi(x)} g_{\mu\nu}$. Weyl believed that the scale parameter $\pi(x)$ might be related to the gauge transformation property of electromagnetism $(A_\mu \to A_\mu + \partial_\mu \pi)$, and thus provide an opportunity for deriving Maxwell's equations from a geometrical foundation. The theory failed, but it has since spurred a considerable amount of interest in gravitational theories based on conformal invariance. That interest has continued to this day, with many researchers contributing to the topic, now properly called *Weyl conformal gravity*.

Meanwhile, a similar non-Riemannian geometry was suggested by Erwin Schrödinger in the late 1940s that in some ways paralleled Weyl's effort. While Schrödinger did not pursue this geometry to the extent that Weyl did, his approach nevertheless represents an arguably superior formalism, as it appeared to surmount the objections Einstein expressed regarding Weyl's earlier work. Although Schrödinger spent much of his time investigating geometries that were non-symmetric in the metric tensor and the connection term, his geometry is simpler and arguably more elegant than Weyl's.

1. Notation

Following Adler et al., ordinary partial and covariant differentiation of scalars, vectors and tensors will be denoted with a single subscripted bar and double subscripted bar, respectively, as in

$$A^{\alpha}{}_{\mu\nu||\lambda} = A^{\alpha}{}_{\mu\nu|\lambda} + A^{\beta}{}_{\mu\nu}\Gamma^{\alpha}_{\beta\lambda} - A^{\alpha}{}_{\beta\nu}\Gamma^{\beta}_{\lambda\mu} - A^{\alpha}{}_{\mu\beta}\Gamma^{\beta}_{\lambda\nu}$$

with the signs of the various terms following basic convention; unless noted otherwise, the *connections* $\Gamma^{\alpha}_{\mu\nu}$ are symmetric with respect to their lower indices.

2. Overview of Weyl's 1918 Theory

We begin with Weyl's theory of 1918, not only because it was the first comprehensive effort to derive electromagnetism from general relativity but because it contains a key element that will be of use when we turn to Schrödinger's ideas. The literature on Weyl's 1918 theory, now nearly one hundred years old, is so extensive that we will provide only the basics of his approach, along with a cursory review of the notion of *parallel transport*.

Parallel Transport, the Connection, and Vector Magnitude

One of Weyl's major contributions to early differential geometry was his development of the formalism of parallel transport, which allows two vectors at neighboring points to be compared in a covariant manner. This formalism introduced the notion of the connection $\Gamma^{\alpha}_{\mu\nu}$, which is of critical importance in differential geometry. If a vector ξ^{α} is parallel-transported from the point x to the point $x+dx$, it can be linked covariantly with its new value $\xi^{\alpha}(x+dx) = \xi^{\alpha}(x) + \xi^{\alpha}_{|\mu}$ at the new point with the quantity $\mathscr{D}\xi^{\alpha}$, where

$$\mathscr{D}\xi^{\alpha} = -\Gamma^{\alpha}_{\mu\nu}\xi^{\mu}dx^{\nu} \tag{2.1}$$

The differential operator $\mathscr{D}$ links the vector $\xi^{\alpha}(x+dx)$ with the parallel "twin" of $\xi^{\alpha}(x)$ at that point, a concept that is more fully explained in any elementary text on differential geometry (it should be distinguished from the total differential operator d, although many authors use the latter notation for both operators). The connection term $\Gamma^{\alpha}_{\mu\nu}$ is not a tensor since it transforms in a non-invariant manner under a change of coordinates, but it is otherwise completely arbitrary.

The length or magnitude L^2 of some vector A^{μ} is given by $L^2 = g_{\mu\nu}A^{\mu}A^{\nu}$, and in Riemannian geometry it is assumed to be invariant with respect to parallel transport. Using (2.1), a straightforward calculation shows that this is equivalent to

$$\mathscr{D}L^2 = 2L\mathscr{D}L = g_{\mu\nu||\alpha}A^{\mu}A^{\nu}dx^{\alpha} \tag{2.2}$$

where $g_{\mu\nu||\alpha}$ is the covariant derivative of the metric tensor. Also known as the *non-metricity tensor*, it necessarily vanishes in a Riemannian space. By considering a cyclic combination of the identity

$$g_{\mu\nu||\alpha} = g_{\mu\nu|\alpha} - g_{\mu\lambda}\Gamma^{\lambda}_{\alpha\nu} - g_{\lambda\nu}\Gamma^{\lambda}_{\alpha\mu} = 0 \tag{2.3}$$

it is easy to show that

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\}$$

Thus, the connection, like the metric tensor from which it is constructed, is symmetric in its lower indices. In a general non-Riemannian space the tensor $g_{\mu\nu||\alpha}$ does not vanish, nor is the connection necessarily required to be symmetric. The consequences of non-symmetric connections (particularly the concept known as *torsion*) have been explored by many researchers (including Einstein, Eddington and Schrödinger) and continue to this day, athough the results to date have generally not been very productive.

Weyl Geometry

For a flat space in Riemannian geometry the metric tensor is a constant, the connection vanishes and the vector parallel-transports to the new location $x+dx$ unchanged. When the space is not flat, the vector ξ^{α} can only change in direction, while the square of its magnitude $L^2 = g_{\mu\nu}\xi^{\mu}\xi^{\nu}$ remains fixed. Weyl's idea was to remove this latter constraint by allowing vector magnitude to change as well. While perhaps counterintuitive, this is arguably the simplest path to a classical non-Riemannian geometry.

In order to proceed, Weyl had to assume a form for the change in vector magnitude under parallel transport, which he believed should be structurally similar to that of the vector itself. So he simply wrote

$$\mathscr{D}L = \phi_{\alpha}Ldx^{\alpha} \tag{2.4}$$

where the as yet undefined vector field ϕ_{α} acts somewhat like a connection term. At the same time Weyl noted that the change in vector magnitude L^2 could be determined by direct calculation which, for the arbitrary vector ξ^{μ}, is given by

$$2L\mathscr{D}L = g_{\mu\nu||\alpha}\xi^{\mu}\xi^{\nu}dx^{\alpha} \tag{2.5}$$

Comparison of (2.4) and (2.5) shows that Weyl's definition for the change in vector magnitude is then equivalent to

$$g_{\mu\nu||\alpha} = 2g_{\mu\nu}\phi_\alpha \tag{2.6}$$

By expansion of $g_{\mu\nu||\alpha}$ and using cyclic permutations of the above expression it is a simple matter to show that the connection term in Weyl's geometry is

$$\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\,\nu \end{matrix} \right\} - \delta^\alpha_\mu \phi_\nu - \delta^\alpha_\nu \phi_\mu + g_{\mu\nu} g^{\alpha\beta} \phi_\beta \tag{2.7}$$

At this point Weyl made an interesting observation. Let the metric tensor $g_{\mu\nu}$, which determines vector magnitude, undergo an infinitesimal rescaling (or *regauging*) given by $g_{\mu\nu} \to e^{\epsilon\pi(x)} g_{\mu\nu}$, or $\delta g_{\mu\nu} = \epsilon\pi g_{\mu\nu}$ (similarly, $\delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$). Weyl noticed that if the vector ϕ_μ also undergoes the transformation $\delta\phi_\mu = \frac{1}{2}\epsilon\pi_{|\mu}$, then the connection $\Gamma^\alpha_{\mu\nu}$ remains unchanged. Although Weyl believed that Nature should be invariant with regard to a rescaling of the metric (also known as *conformal invariance*), more importantly he recognized that the transformation of ϕ_μ was the same as that of the electromagnetic four-potential of electrodynamics. For this reason, Weyl believed he had discovered a way to unify the forces of gravitation and electromagnetism using a purely geometrical approach.

In addition to the metric tensor, rescaling of the metric determinant quantity defined as $\sqrt{-|g_{\mu\nu}|} = \sqrt{-g}$ deserves mention at this point. It is a simple matter to show that for any variation, this quantity changes in accordance with

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu} \delta g^{\mu\nu}$$

so that, with the conformal variation $\delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$, we have

$$\delta\sqrt{-g} = \frac{1}{2}\epsilon n\pi\sqrt{-g} \tag{2.8}$$

where n is the dimension of space; in four dimensions this is simply $\delta\sqrt{-g} = 2\sqrt{-g}\epsilon\pi$. It is highly significant that the Lagrangian $\sqrt{-g}\, F_{\mu\nu}F^{\mu\nu}$ of classical electromagnetism, where $F_{\mu\nu}$ is the antisymmetric electromagnetic tensor, exhibits conformal invariance only in a four-dimensional space.

Thus, in Weyl's geometry the connection exhibits scale or conformal invariance, a type of symmetry that Weyl believed should be a basic principle not only in general relativity but in all of Nature. Although the Einstein-Hilbert Lagrangian $\sqrt{-g}\, R$, from which one derives the traditional free-space Einstein gravitational field equations, is not conformally invariant, Weyl noted that the quantity $\sqrt{-g}\, R^2$ is fully invariant, and he subsequently used this Lagrangian to derive a set of equations that appeared to provide not only an alternative version of Einstein's gravitational field equations but the entirety of Maxwell's equations as well.

To summarize, Weyl's geometry is characterized by the following:

1. There is a non-vanishing non-metricity tensor that is proportional to a new field quantity that Weyl associated with the electromagnetic four-potential, or $g_{\mu\nu||\alpha} = 2g_{\mu\nu}\phi_\alpha$
2. The geometry utilizes a symmetric, non-Riemannian connection consisting of the Christoffel term and the Weyl vector field, or $\Gamma^\alpha_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\,\nu \end{matrix} \right\} - \delta^\alpha_\mu \phi_\nu - \delta^\alpha_\nu \phi_\mu + g_{\mu\nu} g^{\alpha\beta} \phi_\beta$
3. Under an infinitesimal conformal variation of the metric tensor $g_{\mu\nu} \to (1+\epsilon\pi) g_{\mu\nu}$ (or $\delta g_{\mu\nu} = \epsilon\pi g_{\mu\nu}$, $\delta g^{\mu\nu} = -\epsilon\pi g^{\mu\nu}$), the Weyl connection remains unchanged provided the Weyl vector field varies according to $\delta\phi_\alpha = \frac{1}{2}\epsilon\pi_{|\alpha}$
4. The magnitude of any vector changes under parallel transport according to $\mathscr{D} = \phi_\alpha L dx^\alpha$. In Weyl's geometry a vector is *obliged* to change under parallel transport; there are no truly fixed-length vectors
5. The Weyl action $\int \sqrt{-g}\, R^2 d^4x$ is conformally invariant. While of fourth order, this action leads to equations of motion that are identical to those of the Einstein free-space field equations

Einstein's Objection

Although Weyl's theory was able to reproduce the classical predictions of Einstein's simpler 1915 theory (perihelion advance of the planet Mercury, gravitational redshift, etc.), Einstein — an initial admirer of Weyl's work — objected to the theory on more fundamental grounds. Under a rescaling of the metric tensor, the invariant line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ also undergoes rescaling via $ds \to \exp\frac{1}{2}\pi ds$. Einstein argued that ds can be associated with the ticking of a clock or the spacings of atomic spectral lines and concluded that if it is not absolutely invariant, many basic physical quantities (Compton wavelength, electron mass, etc.) would vary arbitrarily with time and location. Weyl tried valiantly to refute Einstein's argument, and even undertook numerous efforts to make ds a true invariant, but to no avail. Within a few years after its proposal, Weyl's theory was considered a dead end.

Overcoming Einstein's Objection

Consider the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ again, which we now write as

$$1 = g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{2.9}$$

which merely expresses the fact that the magnitude or length of the unit vector dx^μ/ds is, well, unity. Since this magnitude is a pure number, it cannot change under parallel transport. But as noted previously, in Weyl's theory the magnitude of *any* vector will change according to

$$\mathscr{D}L = \phi_\alpha L dx^\alpha \tag{2.10}$$

Indeed, in Weyl's geometry there can be are no truly "constant" vectors provided that $\phi_\mu \neq 0$, in which case we merely revert back to the Riemannian case. However, we will see that there is a way to preserve the notion of constant vectors while allowing other vector quantities to change under physical transplantation. For the unit vector dx^μ (or any vector proportional to it) we can demand that $\mathscr{D}L = 0$ if

$$g_{\mu\nu||\alpha}dx^\mu dx^\nu dx^\alpha = 0 \tag{2.11}$$

This presents two possibilities: either the non-metricity tensor $g_{\mu\nu||\alpha}$ vanishes identically (and we again have Riemannian geometry), or it satisfies the cyclic property

$$g_{\mu\nu||\alpha} + g_{\alpha\mu||\nu} + g_{\nu\alpha||\mu} = 0 \tag{2.12}$$

We note at this point that Weyl's definition of the non-metricity tensor in (2.2) cannot satisfy this condition, so if we are to maintain the notion of a non-zero $g_{\mu\nu||\alpha}$ then we must revise the geometry. In particular, it remains to be seen what the non-metricity tensor really represents, what forms it can take, and whether — as in Weyl's theory — it involves some vector quantity that we might attribute to more familiar physics.

3. Overview of Schrödinger's Connection

We now turn to Schrödinger's work of 1944-1950, a late period in the great Austrian physicist's life when he too considered non-Riemannian connections as a route to unification. He seems to have been primarily interested in purely affine theories (connections without metrics), and his investigations included ideas that would seem odd to a classical relativist today; for example, he replaced the metric determinant $\sqrt{-g}$ with its Ricci equivalent $\sqrt{|R_{\mu\nu}|}$, although similar ideas were proposed by Einstein (and even earlier by Eddington). Schrödinger also considered metrics and connections that were both symmetric and non-symmetric, again mirroring ideas that had been considered years earlier by others.

Much of this work was summarized in a series of papers Schrödinger wrote in 1947-48, but in 1950 he wrote his famous book *Space-Time Structure*, which included more conventional approaches to generalizations of Einstein's foundational 1915 theory. In that book he happened upon what he considered at the time to be the most general possible symmetric connection, although he was initially motivated by a non-symmetric formalism. What he found was a rank-three tensor $T_{\mu\nu\alpha}$ that satisfies the symmetry properties of both $g_{\mu\nu}$ and $g_{\mu\nu||\alpha}$. Although Schrödinger did not explicitly identify his T-tensor with the non-metricity tensor at the time, that identification is unavoidable, as we now show.

Schrödinger started with a symmetric metric tensor and a non-symmetric connection, but with a vanishing non-metricity tensor $g_{\mu\nu||\alpha}$. First, Schrödinger wrote the three permutations

$$g_{\mu\nu||\alpha} = g_{\mu\nu|\alpha} - g_{\mu\lambda}\Gamma^{\lambda}_{\alpha\nu} - g_{\lambda\nu}\Gamma^{\lambda}_{\alpha\mu}$$
$$g_{\alpha\mu||\nu} = g_{\alpha\mu|\nu} - g_{\alpha\lambda}\Gamma^{\lambda}_{\nu\mu} - g_{\lambda\mu}\Gamma^{\lambda}_{\nu\alpha}$$
$$g_{\nu\alpha||\mu} = g_{\nu\alpha|\mu} - g_{\nu\lambda}\Gamma^{\lambda}_{\mu\alpha} - g_{\lambda\alpha}\Gamma^{\lambda}_{\mu\nu}$$

Subtracting the first expression from the sum of the other two, and setting all the permuted $g_{\mu\nu||\alpha}$ terms to zero, we have

$$\frac{1}{2}\left(\Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\nu\mu}\right) = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + \frac{1}{2} g^{\alpha\beta} g_{\mu\lambda}\left(\Gamma^{\lambda}_{\nu\alpha} - \Gamma^{\lambda}_{\alpha\nu}\right) + \frac{1}{2} g^{\alpha\beta} g_{\nu\lambda}\left(\Gamma^{\lambda}_{\mu\alpha} - \Gamma^{\lambda}_{\alpha\mu}\right)$$

Using the notation

$$\Gamma^{\lambda}_{(\mu\nu)} = \frac{1}{2}\left(\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\nu\mu}\right), \quad \Gamma^{\lambda}_{[\mu\nu]} = \frac{1}{2}\left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right)$$

Schrödinger was able to write the above expression as

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + g^{\alpha\beta} g_{\mu\lambda}\Gamma^{\lambda}_{[\alpha\nu]} + g^{\alpha\beta} g_{\nu\lambda}\Gamma^{\lambda}_{[\beta\mu]} + \Gamma^{\alpha}_{[\mu\nu]} \tag{3.1}$$

Recognizing that the antisymmetric or skew aspect of the last term causes it to vanish when the geodesic equations are considered, Schrödinger just scrapped it, and simply wrote

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + g^{\alpha\beta}\left(g_{\mu\lambda}\Gamma^{\lambda}_{[\alpha\nu]} + g_{\nu\lambda}\Gamma^{\lambda}_{[\beta\mu]}\right) \tag{3.2}$$

Note that this expression now represents a family of purely *symmetric* connections in the indices μ, ν, which he now wrote as

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + g^{\alpha\beta} T_{\mu\nu\beta} \tag{3.3}$$

where the quantity $T_{\mu\nu\beta} = T_{\nu\mu\beta}$ consists of a combination of two antisymmetric connections whose sum is nevertheless a legitimate tensor quantity. This connection was, in Schrödinger's opinion, the most general symmetric connection possible. It is distinctly different from Weyl's' connection, and thus represents a new geometry in its own right. Parallel transport of an arbitrary vector ξ^{α} in Schrödinger's geometry is now expressible using the new differential quantity

$$\mathscr{D}\xi^{\alpha} = -\Gamma^{\alpha}_{\mu\nu}\xi^{\mu}dx^{\nu} = -\left(\left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} + g^{\alpha\beta} T_{\mu\nu\beta}\right)\xi^{\mu}dx^{\nu} \tag{3.4}$$

Schrödinger went further — unlike Weyl, he recognized that the unit vector dx^{μ}/ds must not change under parallel transport. Repeating the calculations above with the revised differential operator in (3.4) we have, after some simple algebra,

$$\mathscr{D}L^2 = 2L\mathscr{D}L = -2T_{\mu\nu\alpha}dx^{\mu}dx^{\nu}dx^{\alpha} = 0$$

from which Schrödinger surmised the condition

$$T_{\mu\nu\alpha} + T_{\alpha\mu\nu} + T_{\nu\alpha\mu} = 0 \tag{3.5}$$

Unfortunately, the connection in (3.3) presents two problems. For one, Schrödinger's elimination of the antisymmetric term $\Gamma^{\alpha}_{[\mu\nu]}$ by fiat would set all such skew terms equal to zero, making the tensor $T_{\mu\nu\alpha}$ (which is itself composed of such terms) vanish as well. And two, if we continue to assume that $g_{\mu\nu||\alpha} = 0$ for this connection, then $T_{\mu\nu\alpha}$ vanishes identically, as a quick calculation shows. To remedy this, let us expand the identity

$$g_{\mu\nu||\alpha} = g_{\mu\nu|\alpha} - g_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha} - g_{\lambda\nu}\Gamma^{\lambda}_{\mu\alpha}$$

where the connection, as Schrödinger assumed, is fully symmetric. The Christoffel term drops out, and we are left with $g_{\mu\nu||\alpha} = T_{\mu\nu\alpha}$. The conclusion is inescapable: *Schrödinger's T-tensor is the non-metricity tensor*. We can now write Schrödinger's connection as

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\,\nu \end{matrix} \right\} + g^{\alpha\beta} g_{\mu\nu||\beta} \tag{3.6}$$

4. Explicit Identification of the Non-Metricity Tensor

In his his 1918 theory, Weyl was motivated to introduce a vector field ϕ_α that he subsequently identified as the electromagnetic four-potential. In his view, the presence of an electromagnetic field causes the ordinary Christoffel connection of Riemannian geometry to acquire terms proportional to ϕ_α, resulting in a non-zero non-metricity tensor. No such field is apparent in Schrödinger's connection, but it contains a link to Weyl's theory that allows a plausible definition of the connection in terms of such a field. To see this, let us contract (2.12) with $g^{\mu\nu}$:

$$g^{\mu\nu} g_{\mu\nu||\alpha} + 2 g^{\mu\nu} g_{\mu\alpha||\nu} = 0$$

or

$$\frac{1}{2} g^{\mu\nu} g_{\mu\nu||\alpha} = -g^{\mu\nu} g_{\mu\alpha||\nu}$$

Both sides of this expression are obviously one-form vectors, and for definiteness we define

$$\phi_\alpha = \frac{1}{2} g^{\mu\nu} g_{\mu\nu||\alpha} \tag{4.1}$$

(Note that we would have arrived at essentially the same identification by contracting Weyl's non-metricity tensor.) Expanding, we have

$$\phi_\alpha = \frac{1}{2} g^{\mu\nu} \left(g_{\mu\nu|\alpha} - g_{\mu\lambda} \Gamma^{\lambda}_{\nu\alpha} - g_{\lambda\nu} \Gamma^{\lambda}_{\mu\alpha} \right)$$

which reduces to

$$\Gamma^{\mu}_{\alpha\mu} = (\ln \sqrt{-g})_{|\alpha} - \phi_\alpha$$

We thus have a definition for the contracted variant of Schrödinger's connection in terms of the covariant vector ϕ_α. But since this vector was defined in terms of the connection itself, we would seem to have a circular definition yielding an empty formalism. To avoid this we assume, as Weyl did, that there exists an *external* vector field proportional to ϕ_α whose presence causes the geometry to go from Riemannian to non-Riemannian. This is an admittedly artificial argument, but it is necessary if we are to proceed.

We now seek a definition of the non-metricity tensor in terms of the metric tensor and this new vector field. Following Weyl, the most plausible approach is to assume an expression of the form

$$g_{\mu\nu||\alpha} = A g_{\mu\nu} \phi_\alpha + B g_{\alpha\mu} \phi_\nu + C g_{\nu\alpha} \phi_\mu$$

where A, B, C are constants. From the symmetry of $g_{\mu\nu}$ we must have $B = C$, while from (2.12) we see that $A = -2B$. By considering our definition of ϕ_α in (4.1) we have $A = 2/3$, so we can finally write

$$g_{\mu\nu||\alpha} = \frac{2}{3} g_{\mu\nu} \phi_\alpha - \frac{1}{3} g_{\alpha\mu} \phi_\nu - \frac{1}{3} g_{\nu\alpha} \phi_\mu$$

as the most general definition of the non-metricity tensor. From (3.6), we can now write Schrödinger's connection as

$$\Gamma^{\alpha}_{\mu\nu} = \left\{ \begin{matrix} \alpha \\ \mu\,\nu \end{matrix} \right\} - \frac{1}{3} \delta^{\alpha}_{\mu} \phi_\nu - \frac{1}{3} \delta^{\alpha}_{\nu} \phi_\mu + \frac{2}{3} g_{\mu\nu} g^{\alpha\beta} \phi_\beta$$

which is already very similar to Weyl's connection.

5. Conformal Invariance

Although Weyl's 1918 theory failed, it was to some extent vindicated in 1929 when Weyl's idea of gauge invariance was applied to quantum physics. Weyl himself recognized that it was not the rescaling of the metric tensor that mattered in physics, but a rescaling of the wave function with a *complex phase factor* that was physically meaningful. The invariance of quantum physics with respect to such a phase factor did indeed result in a new conservation theorem, that of the conservation of electric charge. Gauge or phase invariance today represents a profoundly fundamental and important symmetry in quantum mechanics, while the importance of conformal invariance in general relativity is still largely unconfirmed. Consequently, the conformal properties of classical general relativity, which rely primarily on the conformal aspects of the connection itself, remain to be more fully explored. Here we will consider how things change when the metric tensor undergoes the infinitesimal local change of scale defined by $g_{\mu\nu} \to (1+\epsilon\pi)g_{\mu\nu}$ (or $\delta g_{\mu\nu} = \epsilon\pi g_{\mu\nu}$) (where ϵ is a small number) in Schrödinger's geometry.

Weyl was able to choose the variation of his vector one-form ϕ_α to be such that the connection $\Gamma^\alpha_{\mu\nu}$ was itself invariant to a conformal variation, but a quick glance at the Schrödinger's connection shows this to be impossible. This is because the variation of the Christoffel term, which gives

$$\delta \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} = \frac{1}{2}\epsilon\delta^\alpha_\mu \pi_{|\nu} + \frac{1}{2}\epsilon\delta^\alpha_\nu \pi_{|\mu} - \frac{1}{2}\epsilon g_{\mu\nu} g^{\alpha\beta}\pi_{|\beta},$$

is offset in Weyl's connection only if we choose $\delta\phi_\mu = \frac{1}{2}\epsilon\pi$, whereas Schrödinger's connection is incompatible with such a choice. We can, however, still straightforwardly determine how the connection changes under a change of scale. To see this, let us conduct a conformal variation of the Schrödinger connection as given in (3.6):

$$\delta\Gamma^\lambda_{\mu\nu} = \delta \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} + g_{\mu\nu||\alpha}\delta g^{\alpha\lambda} + g^{\alpha\lambda}\delta g_{\mu\nu||\alpha}$$

Using $\delta g^{\alpha\lambda} = -\epsilon\pi g^{\alpha\lambda}$, expanding $g_{\mu\nu||\alpha}$ and collecting terms, we can write this as

$$\delta\Gamma^\lambda_{\mu\nu} + g_{\mu\alpha}g^{\lambda\beta}\delta\Gamma^\lambda_{\beta\nu} + g_{\mu\alpha}g^{\lambda\beta}\delta\Gamma^\lambda_{\beta\nu} = \frac{1}{2}\epsilon g_{\mu\nu}g^{\lambda\beta}\pi_{|\beta} + \frac{1}{2}\epsilon\delta^\lambda_\mu\pi_{|\nu} + \frac{1}{2}\epsilon\delta^\lambda_\nu\pi_\mu$$

This then shows that

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\epsilon g_{\mu\nu}g^{\lambda\beta}\pi_{|\beta}$$

with the variation of the contracted form being

$$\delta\Gamma^\lambda_{\mu\lambda} = \frac{1}{2}\epsilon\pi_\mu$$

It is now an easy matter to show that

$$\delta\phi_\mu = \frac{3}{2}\epsilon\pi_{|\mu}$$

This demonstrates that a scale variation of ϕ_μ, like Weyl's vector, is a pure gradient, and any suspected relationship with the electromagnetic four-potential would seem to be preserved in the Schrödinger case. However, unlike Weyl's connection, the connection in (3.6) is not invariant with respect to rescaling. This will prove to be the formalism's undoing, as we will see in the next section.

6. The Schrödinger Equations of Motion?

As was mentioned earlier, Weyl's conformally-invariant action lagrangian is the simple quantity $\sqrt{-g}\,R^2$, which results from the fact that the Weyl connection is itself conformally invariant. This allowed Weyl to straightforwardly derive equations of motion from this Lagrangian by considering arbitrary independent variations of the metric tensor $\delta g^{\mu\nu}$ and the Weyl vector $\delta\phi_\mu$, which led him to expressions similar to Einstein's gravitational field equations and those of Maxwell's electrodynamics. Such a straightforward approach cannot be made for Schrödinger's geometry, if for no reason other than the fact that the connection $\Gamma^\lambda_{\mu\nu}$ is not scale invariant.

In 1921 Weyl used a linear combination of the quantities $\sqrt{-g}\,R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, $\sqrt{-g}\,R_{\mu\nu}R^{\mu\nu}$ and $\sqrt{-g}\,R^2$ in ordinary Riemannian geometry to show that there does indeed exist a unique action Lagrangian that is fully scale invariant. That quantity is

$$I = \sqrt{-g}\,C_{\mu\nu\alpha\beta}\,C^{\mu\nu\alpha\beta}$$

where $C_{\mu\nu\alpha\beta}$ is the familiar *Weyl conformal tensor*. In four dimensions, the Lagrangian

$$\sqrt{-g}\,C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} = \sqrt{-g}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + 2R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) \tag{6.1}$$

is fully scale invariant. In view of this, we might consider the possibility of using a similar approach to develop a scale-invariant action Lagrangian involving the Schrödinger connection, which would then yield equations of motion specific to Schrödinger's formalism. At first glance this would appear to be a tedious task, given the many components that would enter the calculation. But we will now show that such a calculation is unnecessary, since this approach is not only difficult, but in fact impossible.

Symmetry Properties of the Riemann-Christoffel Tensor in Schrödinger Geometry

We note that in Riemannian, Weyl and Schrödinger geometry, the familiar symmetry and Bianchi identities

$$R^{\alpha}{}_{\mu\nu\lambda} + R^{\alpha}{}_{\alpha\mu\nu} + R^{\alpha}{}_{\nu\alpha\mu} = 0,$$

$$R_{\alpha\mu\nu\lambda} + R_{\alpha\lambda\mu\nu} + R_{\alpha\nu\lambda\mu} = 0$$

and

$$R^{\alpha}{}_{\mu\nu\beta||\lambda} + R^{\alpha}{}_{\mu\lambda\nu||\beta} + R^{\alpha}{}_{\mu\beta\lambda||\nu} = 0 \tag{6.2}$$

are all valid for their respective connections. However, for the Weyl and Schrödinger geometries, the fully-covariant form

$$R_{\alpha\mu\nu\lambda||\beta} + R_{\alpha\mu\beta\nu||\lambda} + R_{\alpha\mu\lambda\beta||\nu} \neq 0,$$

since the non-metricity tensor is no longer zero. This is the source of the difficulty in attempting to derive equations of motion using the Schrödinger connection, because it invalidates several traditional symmetry properties of the Riemann-Christoffel tensor $R^{\alpha}{}_{\mu\nu\lambda}$, as we will now see.

The Riemann-Christoffel tensor is most commonly derived by computing the difference between the double covariant derivatives of a vector ξ_μ, as given by

$$\xi_{\mu||\alpha||\beta} - \xi_{\mu||\beta||\alpha} = -\xi_\lambda R^{\lambda}{}_{\mu\alpha\beta} = -\xi^\lambda R_{\lambda\mu\alpha\beta}$$

where

$$R^{\lambda}{}_{\mu\alpha\beta} = \Gamma^{\lambda}_{\mu\alpha|\beta} - \Gamma^{\lambda}_{\mu\beta|\alpha} + \Gamma^{\lambda}_{\beta\nu}\Gamma^{\nu}_{\mu\alpha} - \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\nu}_{\mu\beta}$$

We can do the same for any vector or tensor quantity; that of the metric tensor $g_{\mu\nu}$ is particularly interesting:

$$\begin{aligned} g_{\mu\nu||\alpha||\beta} - g_{\mu\nu||\beta||\alpha} &= -g_{\mu\lambda}R^{\lambda}{}_{\nu\alpha\beta} - g_{\lambda\nu}R^{\lambda}{}_{\mu\alpha\beta} \\ &= -\left(R_{\mu\nu\alpha\beta} + R_{\nu\mu\alpha\beta}\right) \end{aligned}$$

If the non-metricity tensor vanishes, then we have the usual Riemannian symmetry property $R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta}$ (it can also be shown that $R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}$). But in the Schrödinger geometry it does not vanish, so there is a problem. This is most evident in its inability to reproduce the traditional Bianchi identity (6.2), which can be traced to the reduction of the quantity $g^{\mu\nu}R^{\alpha}{}_{\mu\lambda\nu} = g^{\mu\nu}g^{\alpha\beta}R_{\beta\mu\lambda\nu}$. In the Schrödinger (and Weyl) geometry, interchange of the β, μ indices is no longer antisymmetric, which blocks a straightforward derivation of the usual conservation condition

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)_{||\nu} = 0$$

The problem also arises when we try to derive the same conservation condition for the action $\int\sqrt{-g}\,R\,d^4x$. The same problem exists for Weyl's geometry, but this did not prevent his proposal of the conformally invariant

Lagrangian $\sqrt{-g}\, R^2$ because his connection term was itself scale invariant. As a result, equations of motion for the Schrödinger geometry cannot be derived, since it is impossible even to propose a conformally invariant version of (6.1).

In summary, we have seen that Schrödinger's connection allows for vector quantities that are either fixed or variable under parallel transport, which provides a mathematically consistent way of avoiding Einstein's objection to Weyl's geometry. However, while Schrödinger's geometry appears to be simpler and more elegant than Weyl's, it represents a physical dead end as far as equations of motion are concerned.

7. Conclusions

We have shown that both the Weyl and Schrödinger geometries, while conceptually and mathematically intriguing (and perhaps even beautful), are problematic with regard to interpretation and meaning of any non-Riemannian formalism characterized by a non-zero non-metricity tensor. They are also problematic in that they are inconsistent with the Bianchi conservation condition (6.2). All of these drawbacks can be traced to the fact that several important symmetry properties of the Riemann-Christoffel tensor are invalidated in the presence of a non-vanishing non-metricity tensor.

Although Weyl's formalism cannot account for the existence of fixed-magnitude vector quantities under parallel transport, its built-in aspect of conformal invariance displays a certain aesthetic appeal, an appeal that has persisted for nearly a century. Furthermore, Weyl's notion of conformal invariance led directly to the fundamental concept of gauge invariance in quantum theory, which has become a cornerstone of theoretical quantum physics. The formalism also provides a means of proposing a simple scale-invariant action whose equations of motion are comparable to those of the Einstein field equations.

By comparison, the Schrödinger formalism appears to provide a mathematically legitimate way out of Einstein's objection to Weyl's theory, but at the expense of requiring a connection that cannot be made conformally invariant. This single drawback prevents formulation of a suitable action Lagrangian or associated equations of motion.

The concept of gauge invariance in general relativity, as first investigated by Weyl, and the subsequent investigations of Schrödinger, are interesting but appear to be dead ends. It is ironic that while Einstein's objections to Weyl's geometry appear to be surmountable, the existence of a non-vanishing non-metricity tensor makes such theories fundamentally unphysical. The only alternative, then, appears to be a strict adherence to Riemannian geometry, whose classical predictions to date regarding general relativity have withstood the test of time and experiment.

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