NORMAL COÖRDINATES FOR THE GEOMETRY OF PATHS

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1. The normal coördinates introduced by Riemann have been of the greatest utility in a variety of researches in Riemann geometry and, are likely to be important in the theory of relativity. An analogous coördinate system is fundamental in what Professor Eisenhart and I have called the Geometry of Paths (Vol. 8, p. 19 of these PROCEEDINGS) i.e., in the theory of the differential equations.

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \qquad (1.1)$$

in which

$$\Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha} \qquad (1.2)$$

the Γ's being functions of the variables, $x^1, x^2, \ldots x^n$, and the paths being the curves which satisfy (1.1). The purpose of the present note is to define the new normal coördinates, to study a set of tensors connected with them, and to obtain a set of identities. Some of the formulae are believed to be new even for those manifolds in which the geometry of paths reduces to the Riemann geometry.

2. From the differential equations (1.1) we obtain by differentiation a sequence of differential equations.

$$\frac{d^3x^i}{ds^3} + \Gamma^i_{\alpha\beta\gamma}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0 \qquad (2.1)$$

$$\frac{d^4x^i}{ds^3} + \Gamma^i_{\alpha\beta\gamma\delta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\frac{dx^\delta}{ds} = 0 \qquad (2.2)$$

and so on, in which

$$\Gamma^i_{\alpha\beta\gamma} = \frac{\partial\Gamma^i_{\alpha\beta}}{\partial x^\gamma} - \Gamma^i_{j\beta}\,\Gamma^j_{\alpha\gamma} - \Gamma^i_{\alpha j}\,\Gamma^j_{\beta\gamma} \qquad (2.3)$$

and in general

$$\Gamma^i_{\alpha\beta\gamma\ldots\xi\eta} = \frac{\partial\Gamma^i_{\alpha\beta\gamma\ldots\xi}}{\partial x^\eta} - \Gamma^i_{j\beta\gamma\ldots\xi}\,\Gamma^j_{\alpha\eta} - \Gamma^i_{\alpha j\gamma\ldots\xi}\,\Gamma^j_{\beta\eta}$$
$$\ldots - \Gamma^i_{\alpha\beta\gamma\ldots j}\,\Gamma^j_{\xi\eta}. \qquad (2.4)$$

3. The differential equation (1.1) has a unique set of solutions determined by the initial conditions, $x^i = p^i$ and $dx^i/ds = \xi^i$ when $s = 0$. These solutions may be written in the form:

$$x^i = p^i + \xi^i s - \frac{1}{2!} (\Gamma^i_{\alpha\beta})_P \, \xi^\alpha \, \xi^\beta s^2 - \frac{1}{3!} (\Gamma^i_{\alpha\beta\gamma})_P \, \xi^\alpha \xi^\beta \xi^\gamma s^3 - \dots \qquad (3.1)$$

in which the subscripts mean that the parentheses are evaluated for $x^i = p^i$, i.e., at the point P.

Let us now substitute $y^i = \xi^i s$ and solve (3.1) for y^i, obtaining

$$y^i = \psi^i(x^1, x^2 \dots, x^n) \qquad (3.2)$$

This determines a transformation of the x's into new coördinates $y^1, y^2 \dots, y^n$. The y's are normal coördinates. They are determined uniquely by the x's, the point P, and the differential equation (1.1). They have the characteristic property that every curve

$$y^i = \xi^i s \qquad (3.3)$$

is a path, i.e., a solution of the equation,

$$\frac{d^2 y^i}{ds^2} + C^i_{\alpha\beta} \frac{dy^\alpha}{ds} \frac{dy^\beta}{ds} = 0 \qquad (3.4)$$

into which (1.1) is transformed by the substitution (3.1). Moreover every path through P is given by (3.3).

4. Substituting (3.3) in (3.4) we obtain

$$C^i_{\alpha\beta} \, \xi^\alpha \, \xi^\beta = 0 \qquad (4.1)$$

in which the functions $C^i_{\alpha\beta}$ are evaluated for the values of y such that $y^i = \xi^i s$. This is more simply written in the form

$$C^i_{\alpha\beta} \, y^\alpha \, y^\beta = 0 \qquad (4.2)$$

which is an identity in y^i. Differentiating this with regard to s, we obtain

$$\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} y^\alpha y^\beta + 2 C^i_{\alpha\gamma} y^\alpha \right) \frac{dy^\gamma}{ds} = 0. \qquad (4.21)$$

Since $\dfrac{dy^\gamma}{ds}$ is arbitrary this gives

$$\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} y^\alpha y^\beta + 2 C^i_{\alpha\gamma} y^\alpha = 0. \qquad (4.22)$$

Multiplying this by y^γ, summing, and using (4.2) we obtain

$$\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} y^\alpha y^\beta y^\gamma = 0. \qquad (4.3)$$

Differentiating once more and repeating the process just described, we obtain

$$\frac{\partial^2 C^i_{\alpha\beta}}{\partial y^\gamma \partial y^\delta} y^\alpha y^\beta y^\gamma y^\delta = 0, \tag{4.4}$$

and in general,

$$\frac{\partial^m C^i_{\alpha\beta}}{\partial y^\gamma \partial y^\delta \ldots \partial y^\xi} y^\alpha y^\beta \ldots y^\xi = 0. \tag{4.m}$$

5. Since the directions ξ^i at the origin are entirely arbitrary it follows from (4.1) that

$$(C^i_{\alpha\beta})_0 = 0. \tag{5.2}$$

If we substitute $y^i = \xi^i s$ in (4.3), divide by s^3, and evaluate for $y^i = 0$ we have

$$\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma}\right)_0 \xi^\alpha \xi^\beta \xi^\gamma = 0.$$

Rewriting this with α,β,γ permuted cyclically and adding, we obtain

$$\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial C^i_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial C^i_{\gamma\alpha}}{\partial y^\beta}\right)_0 \xi^\alpha \xi^\beta \xi^\gamma = 0.$$

Since this form is symmetric and the ξ's are arbitrary,

$$\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial C^i_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial C^i_{\gamma\alpha}}{\partial y^\beta}\right)_0 = 0. \tag{5.3}$$

By a similar argument we obtain from (4.4) the relation:

$$\left(\frac{\partial^2 C^i_{\alpha\beta}}{\partial y^\gamma \partial y^\delta} + \frac{\partial^2 C^i_{\alpha\gamma}}{\partial y^\beta \partial y^\delta} + \frac{\partial^2 C^i_{\alpha\delta}}{\partial y^\beta \partial y^\gamma} + \frac{\partial^2 C^i_{\beta\gamma}}{\partial y^\alpha \partial y^\delta} + \frac{\partial^2 C^i_{\beta\delta}}{\partial y^\alpha \partial y^\gamma} + \frac{\partial^2 C^i_{\gamma\delta}}{\partial y^\alpha \partial y^\beta}\right)_0 = 0. \tag{5.4}$$

By repeating this process we find that *the sum of the* $(m + 2)\ (m + 1)/2$ *derivatives of the* m^{th} *order of the functions, C, in which any set of* $m + 2$ *integers,* $\alpha,\beta,\ldots,\xi$, $(\leqq m)$ *appear as the subscripts of the C's and the superscripts of the y's, is zero at the origin of normal coördinates.*

6. We now define a sequence of systems of functions of $(x^1,x^2,\ldots,x^n)$, $A^i_{\alpha\beta\ldots\xi}$, by the condition that the value of $A^i_{\alpha\beta\gamma\ldots\xi}$ at any point $(x^1,x^2,\ldots,x^n)$ of the manifold is the value of $\left(\frac{\partial^m C^i_{\alpha\beta}}{\partial y^\gamma \ldots \partial y^\xi}\right)_0$ determined in the system of normal coördinates having $(x^1,x^2,\ldots,x^n)$ as origin.

From (1.2) it follows that

$$A^i_{\alpha\beta\gamma\ldots\xi} = A^i_{\beta\alpha\gamma\ldots\xi}. \tag{6.1}$$

From the fact that all the subscripts of $A^i_{\beta\alpha\gamma\ldots\xi}$ after the first two correspond to differentiation of $C^i_{\alpha\beta}$ it follows that

$$A^i_{\alpha\beta\gamma\ldots\xi} = A^i_{\beta\alpha\delta\ldots\eta}, \tag{6.2}$$

where $\delta\ldots\eta$ stands for any permutation of the subscripts, $\gamma\ldots\xi$. From (5.3) there follows

$$A^i_{\alpha\beta\gamma} + A^i_{\beta\gamma\alpha} + A^i_{\gamma\alpha\beta} = 0. \tag{6.3}$$

This is a special case of an identity,

$$A^i_{\alpha\beta\gamma\ldots\xi} + A^i_{\alpha\gamma\beta\ldots\xi} + \ldots = 0 \tag{6.4}$$

in which there are $m\,(m - 1)/2$ terms, each pair of the m subscripts $\alpha\beta\gamma\ldots\xi$ being the first pair in one and only one term. This identity follows directly from the last theorem of §5.

7. *The system of functions $A^i_{\alpha\beta\gamma\ldots\xi}$ with m subscripts ($m \geqq 3$) is a tensor of order $m + 1$.* This theorem can be inferred directly from the invariantive character of the normal coördinates. But we prefer to prove it here by showing how to express the functions A explicitly in terms of the curvature tensor and its covariant derivatives which have already been proved to be tensors in the paper by Professor Eisenhart and the writer.

8. The curvature tensor is defined by the equation,

$$B^i_{\alpha\beta\gamma} = \left(\frac{\partial\Gamma^i_{\alpha\beta}}{\partial x^\gamma} - \Gamma^i_{j\beta}\,\Gamma^j_{\alpha\gamma}\right) - \left(\frac{\partial\Gamma^i_{\alpha\gamma}}{\partial x^\beta} - \Gamma^i_{j\gamma}\,\Gamma^j_{\alpha\beta} =\right)\Gamma^i_{\alpha\beta\gamma} - \Gamma^i_{\alpha\gamma\beta}. \tag{8.1}$$

(This is the negative of what we denoted by $B^i_{\alpha\beta\gamma}$ in the former paper.) If it is computed in a normal coördinate system and evaluated at the origin, P, of these coördinates, it must satisfy the equations,

$$(B^i_{\alpha\beta\gamma})_P = \left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} - \frac{\partial C^i_{\alpha\gamma}}{\partial y^\beta}\right)_0 \tag{8.2}$$

These equations are equivalent to

$$(2B^i_{\alpha\beta\gamma} + B^i_{\gamma\alpha\beta})_P = 3\left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma}\right)_0 - \left(\frac{\partial C^i_{\alpha\beta}}{\partial y^\gamma} + \frac{\partial C^i_{\beta\gamma}}{\partial y^\alpha} + \frac{\partial C^i_{\gamma\alpha}}{\partial y^\beta}\right)_0. \tag{8.3}$$

By means of (5.3) and the definition of $A^i_{\alpha\beta\gamma}$ this leads to

$$A^i_{\alpha\beta\gamma} = \frac{1}{3}\,(2B^i_{\alpha\beta\gamma} + B^i_{\gamma\alpha\beta}). \tag{8.4}$$

By means of the well-known identity,

$$B^i_{\alpha\beta\gamma} + B^i_{\gamma\alpha\beta} + B^i_{\beta\gamma\alpha} = 0, \tag{8.5}$$

this reduces to

$$A^i_{\alpha\beta\gamma} = \frac{1}{3}\,(B^i_{\alpha\beta\gamma} + B^i_{\beta\alpha\gamma}). \tag{8.6}$$

The identity (8.5) is itself a direct consequence of (8.2).

9. In order to extend the formulas of §8 to our sequence of sets of functions $A^i_{\alpha\beta\gamma\ldots\xi}$ we observe that by repeated covariant differentiation of (8.1) we obtain

$$B^i_{\alpha\beta\gamma\delta\ldots\xi} = \frac{\partial^m \Gamma^i_{\alpha\beta}}{\partial x^\gamma \partial x^\delta \ldots \partial x^\xi} - \frac{\partial^m \Gamma^i_{\alpha\gamma}}{\partial x^\beta \partial x^\delta \ldots \partial x^\xi} + \ldots \tag{9.1}$$

where the term on the left of the equality sign represents an $(m - 3)$rd covariant derivative of $B^i_{\alpha\beta\gamma}$, and the three dots at the right represent terms involving derivatives of the Γ's of order less than m and covariant derivatives of $B^i_{\alpha\beta\gamma}$ of order less than $m - 3$. By writing (9.1) in normal coördinates so that the Γ's in the right member become C's and then evaluating at the origin by means of the definition of the A's, we obtain

$$B^i_{\alpha\beta\gamma\delta\ldots\xi} = A^i_{\alpha\beta\gamma\delta\ldots\xi} - A^i_{\alpha\gamma\beta\delta\ldots\xi} + \ldots \tag{9.2}$$

in which the three dots represent terms involving A's and B's with fewer than $m - 1$ subscripts.

The equations (9.2) by themselves do not determine the A's uniquely in terms of the B's. When taken in connection with the equations in § 6, however, they can be solved.

10. In order to find this solution we observe that among the A's with a given set of n subscripts any one is determined by its first two subscripts, and these two subscripts are interchangeable (§ 6). We consider the following $m(m - 1)/2$ permutations of the subscripts $(\beta\alpha\gamma\ldots\xi,)$ $(\gamma\beta\alpha\ldots\xi)$, $(\gamma\alpha\delta\ldots\xi)$, $(\delta\gamma\beta\ldots\xi)$, $(\delta\beta\alpha\ldots\xi)$, $(\delta\alpha\epsilon\ldots\xi)$, $(\epsilon\delta\gamma\ldots\xi)$, $(\epsilon\gamma\beta\ldots\xi)\ldots$. The A's with any two successive permutations of this set as subscripts are capable of entering in an equation of the form (9.2). There is thus determined a set of $M - 1$ equations like (9.2), where $M = \frac{1}{2} m(m - 1)$. Let the first of these equations be multiplied by $(M - 1)$, the second by $(M - 2)\ldots$ and the last by 1. On adding the resulting equations we obtain

$$\begin{aligned}(M-1)\, B^i_{\beta\alpha\gamma\ldots\xi} + (M-2)\, B^i_{\gamma\beta\alpha\ldots\xi} + (M-3)\, B^i_{\gamma\alpha\delta\ldots\xi} + \ldots \\ = MA^i_{\beta\alpha\gamma\ldots\xi} - (A^i_{\beta\alpha\gamma\ldots\xi} + A^i_{\gamma\beta\alpha\ldots\xi} + \ldots) + \ldots\end{aligned} \tag{10.1}$$

The parenthesis of the right member is zero by (6.4). Hence (10.1) reduces to

$$A^i_{\alpha\beta\gamma\ldots\xi} = \frac{1}{M}\left\{(M-1)\, B^i_{\beta\alpha\gamma\ldots\xi} + (M-2)\, B^i_{\gamma\beta\alpha\ldots\xi} + \ldots + \ldots\right\} \tag{10.2}$$

in which the coefficients of the terms in the parenthesis are the integers from 1 to $M - 1$, the permutations of the subscripts of the B's are those indicated in the paragraph above, and the final three dots represent a polynomial in A's and B's with fewer than $(m - 1)$ subscripts. By using (10.2) as a recursion formula this last polynomial is converted into a polynomial in the B's with fewer than $(m - 1)$ subscripts.

This completes the proof that each A is expressible as a polynomial in the B's and determines explicitly the coefficients of the linear expression in the B's with m subscripts which forms a part of this polynomial.

Each of the terms represented by the three dots in (9.2) is evidently a sum of products of A's and B's (such, for example, as $A^i_{\alpha\beta j} B^j_{\gamma\delta\epsilon}$) of such a form that it must represent a tensor if the A's and B's which enter into it represent tensors. The same remark follows directly for the terms represented by the three dots in (10.2). Hence by the use of (10.2) as a recursion formula it follows that the A's are all tensors.

11. The rule for determining the permutations of the subscripts of the $M - 1$ functions B which appear linearly in (10.2) can be regarded as a rule for tracing out the points and lines of a configuration analogous to the Desargues Configuration (cf. Veblen and Young, *Projective Geometry*, Vol. I, Chap. 2). To see this it is only necessary to observe that the M permutations of the subscripts of $A^i_{\alpha\beta\gamma\ldots\xi}$ which give functions which are not identical according to (6.1) and (6.2) are in $(1 - 1)$ correspondence with the points of the configuration obtained by taking a plane section of a complete m-point in a projective 3-space.

12. For the case $m = 3$ the formula (10.2) reduces to (8.4). For the case $m = 4$ the terms represented by the three dots in (9.2) are all zero and hence they are all zero in (10.2). Hence the latter formula reduces to

$$A^i_{\alpha\beta\gamma\delta} = \frac{1}{6}\,(5B^i_{\beta\alpha\gamma\delta} + 4\,B^i_{\gamma\beta\alpha\delta} + 3\,B^i_{\gamma\alpha\delta\beta} + 2\,B^i_{\delta\gamma\beta\alpha} + B^i_{\delta\beta\alpha\gamma}) \qquad (11.1)$$

Another formula which follows at once from (9.1) or (11.1) is

$$\begin{aligned} A^i_{\alpha\beta\gamma\delta} = \frac{1}{24}\,[5(B^i_{\alpha\beta\gamma\delta} + B^i_{\beta\alpha\gamma\delta} + B^i_{\alpha\beta\delta\gamma} + B^i_{\beta\alpha\delta\gamma}) \\ - (B^i_{\gamma\delta\alpha\beta} + B^i_{\delta\gamma\alpha\beta} + B^i_{\gamma\delta\beta\alpha} + B^i_{\delta\gamma\beta\alpha})] \end{aligned} \qquad (11.2)$$

From (9.1) there also follows at once the identity of Bianchi,

$$B^i_{\alpha\beta\gamma\delta} + B^i_{\alpha\delta\beta\gamma} + B^i_{\alpha\gamma\delta\beta} = 0, \qquad (11.3)$$

as well as

$$B^i_{\beta\alpha\gamma\delta} + B^i_{\gamma\beta\delta\alpha} + B^i_{\delta\gamma\alpha\beta} + B^i_{\alpha\delta\beta\gamma} = 0 \qquad (11.4)$$

which is one of a sequence of important identities.