

WEYL SPINORS AND DIRAC'S ELECTRON EQUATION

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I've been planning for some time now to provide a simplified write-up of Weyl's seminal 1929 paper on gauge invariance. I'm still planning to do it, but since Weyl's paper covers so much ground I thought I would first address a discovery that he made kind of in passing that (as far as I know) has nothing to do with gauge-invariant gravitation. It involves the mathematical objects known as spinors. Although Weyl did not invent spinors, I believe he was the first to explore them in the context of Dirac's relativistic electron equation. However, without a doubt Weyl was the first person to investigate the consequences of zero mass in the Dirac equation and the implications this has on parity conservation. In doing so, Weyl unwittingly anticipated the existence of a particle that does not respect the preservation of parity, an unheard-of idea back in 1929 when parity conservation was a sacred cow. Following that, I will use this opportunity to derive the Dirac equation itself and talk a little about its role in particle spin.

Those of you who have studied Dirac's relativistic electron equation may know that the 4-component Dirac spinor is actually composed of two 2-component spinors that Weyl introduced to physics back in 1929. The Weyl spinors have unusual parity properties, and because of this Pauli was initially very critical of Weyl's analysis because it postulated massless fermions (neutrinos) that violated the then-cherished notion of parity conservation. In this write-up, we explore the concept of a spinor, which is what Nature uses to describe fermions. We then proceed to the Dirac equation and discuss Weyl's contribution to what is surely one of the most profound discoveries of modern physics. Because the spinor formalism is closely tied to the Lorentz Group of spacetime rotations, we'll first review this topic in three and four dimensions. The Weyl spinors will then fall out automatically from this analysis.

Incidentally, you may be aware that there are two ways to derive Dirac's electron equation. The easy way consists of following Dirac's original approach, in which he basically took the square root of the relativistic mass-energy equation $E^2 = m^2c^4 + c^2p^2$. Unfortunately, this approach allows the student to pretty much avoid understanding what a spinor really is (when I first learned about the Dirac equation, I persisted in looking upon the Dirac spinor as an ordinary four-component scalar that happened to have an odd way of transforming under Lorentz transformations). Sooner or later you're going to have to learn about spinors, and the second approach I'll describe in the following pages is probably the simplest you're likely to find. In a very real way, spinors are more fundamental than the scalars, vectors and tensors that were, after all, pretty easy to learn when you were first exposed to them. Since most of the observable matter in the universe is composed of fermions (electrons, protons, etc.), it's a good idea to acquire a basic understanding of spinors.

The following pages are kind of dry and boring and so, to liven things up at least a bit, consider the following excerpt from an interview that Dirac gave in America to a rather obnoxious newspaperman way back in April 1929:

"And now I want to ask you something more: They tell me that you and Einstein are the only two real sure-enough high-brows and the only ones who can really understand each other. I won't ask you if this is straight stuff for I know you are too modest to admit it. But I want to know this — Do you ever run across a fellow that even you can't understand?"

"Yes," says he.

"This will make a great reading for the boys down at the office," says I. "Do you mind releasing to me who he is?"

"Weyl," says he.

The interview came to a sudden end just then, for the doctor pulled out his watch and I dodged and jumped for the door. But he let loose a smile as we parted and I knew that all the time he had been talking to me he was solving some problem that no one else could touch. But if that fellow Professor Weyl ever lectures in this town again I sure am going to take a try at understanding him! A fellow ought to test his intelligence once in a while.

1. Strolling Over a Bridge

On the evening of October 16, 1843, the great Irish mathematician William Rowan Hamilton was walking with his wife (with whom he shared a notoriously unhappy marriage) on the path along the Royal Canal in Dublin. Hamilton had been thinking of about the mathematics of complex numbers, and how it might be extended from the plane of one real and one imaginary component ($x + iy$) to one real and three imaginary components (his reasoning for this has been well documented, but that's another story). When the couple reached the Brougham Bridge, the solution suddenly jumped into Hamilton's mind. Taking out his pocket knife, and possibly to the trepidation of his wife, he scratched out the following expression on one of the stones of the bridge:

$$I^2 = J^2 = K^2 = IJK = -1$$

Here, Hamilton's I, J, K are all imaginary quantities that form the basis of what became known as quaternions, quantities that pre-date the modern form of today's vectors. In Hamilton's time quaternions enjoyed a vogue of sorts, but nowadays they are considered just part of the quaint lore of mathematics. Nevertheless, they provide a means of understanding the mathematical objects known as spinors, which are of considerable importance in quantum theory. In modern notation, Hamilton's imaginary quantities are nothing more than the Pauli spin matrices multiplied by the imaginary number i :

$$\begin{aligned} I &= i\sigma_x \\ J &= i\sigma_y \\ K &= i\sigma_z \end{aligned}$$

where

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (1.1)$$

whose properties you are surely already aware of.



Brougham Bridge, Dublin

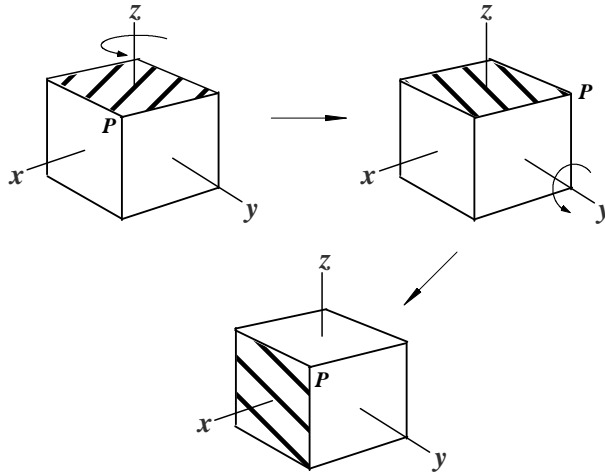
Hamilton's quaternions provide a compact way of understanding the geometrical basis of rotations in three and four dimensions, and at the same time they provide a means of introducing one of the most bizarre

aspects of the spinor formalism (which you're about to discover). Hamilton also used quaternions to show how multiple rotations could be expressed as a single rotation. Let's take a cube, centered at the origin of an xyz coordinate system, and observe one of the corners, P (see figure below). Let us now rotate the cube 90° counterclockwise about the x axis as shown. Having done this, let us now rotate the cube again 90° counterclockwise about the y axis. This returns P to its original position in the coordinate system, although the cube itself has been turned around somewhat. Now ask yourself how we might have achieved this result in a *single* rotation about some particular axis. A little thought will tell you that by rotating the cube counterclockwise about the diagonal axis joining the original point and the cube's center, we can achieve the same result with one rotation. Hamilton came up with a neat formula for this, although you'll have to do some digging in the math literature to find the damn thing (I found the equation in Misner-Thorne-Wheeler's *Gravitation*). It is

$$R(\theta) = \cos \frac{\theta}{2} - i [\sigma_x \cos \varphi_x + \sigma_y \cos \varphi_y + \sigma_z \cos \varphi_z] \sin \frac{\theta}{2} \quad (1.2)$$

Here, θ is the amount of rotation about some axis, and the φ_n are the angles between that axis and those of the unit vectors in the coordinate system. For the example of our rotated cube, we have for the first rotation

$$\begin{aligned} R_1(90) &= \cos \frac{90}{2} - i\sigma_z \sin \frac{90}{2} \\ &= \frac{1}{\sqrt{2}}(1 - i\sigma_z) \end{aligned}$$



(This results because $\varphi_x = \varphi_y = 90^\circ$ and $\varphi_z = 0$.) Similarly, the second rotation gives

$$\begin{aligned} R_2(90) &= \cos \frac{90}{2} - i\sigma_y \sin \frac{90}{2} \\ &= \frac{1}{\sqrt{2}}(1 - i\sigma_y) \end{aligned}$$

The resultant combination of these two rotations can be expressed as $R_f = R_2 \cdot R_1$, or

$$\begin{aligned} R_f &= \frac{1}{2}(1 - i\sigma_y)(1 - i\sigma_z) \\ &= \frac{1}{2}(1 - i\sigma_y - i\sigma_z - i\sigma_x) \end{aligned} \quad (1.3)$$

where I have used the identity $\sigma_y\sigma_z = i\sigma_x$ in the last step. Thus

$$R_f = \cos \frac{\theta_f}{2} - i [\sigma_x \cos \varphi_{xf} + \sigma_y \cos \varphi_{yf} + \sigma_z \cos \varphi_{zf}] \sin \frac{\theta_f}{2} \quad (1.4)$$

Equating like terms in (1.3) and (1.4), we see that

$$\begin{aligned}\theta_f &= 120^\circ \\ \varphi_{\text{all } f} &= \arccos\left(\frac{1}{\sqrt{3}}\right) = 54.7^\circ\end{aligned}$$

Hamilton's rotation equation is pretty neat all by itself, but the point I'm trying to make here is that there seems to be a connection between *half angles* and the Pauli matrices. As we will see, there is indeed a deep connection.

2. The Lorentz Group of Rotations

In order to understand the spinor concept, it is helpful to have a command of the algebra of ordinary rotations in three dimensions. If we rotate the coordinates counterclockwise about the z axis, the coordinates in the rotated system are given by

$$\begin{aligned}\bar{x} &= x \cos \theta + y \sin \theta \\ \bar{y} &= -x \sin \theta + y \cos \theta \\ \bar{z} &= z\end{aligned}\tag{2.1}$$

or $\bar{X} = R_z(\theta)X$, where

$$R_z(\theta) = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

You should also be familiar with the forms for rotations about the x and y axes as well:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix}, \quad R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}\tag{2.2}$$

It is easy to express these matrices as exponential operators. Let's expand the infinitesimal rotation matrix $R_z(d\theta)$ to first order about $\theta = 0$:

$$R_z(d\theta) = R_z(0) + \left. \frac{\partial R_z}{\partial \theta} \right|_{\theta=0} d\theta + \dots$$

where $R_z(0) = I$. We define the rotation generator for the z axis to be

$$\begin{aligned}J_z &= -i \left. \frac{\partial R_z}{\partial \theta} \right|_{\theta=0} \\ &= -i \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

For rotations about the x and y axes, we have the similar definitions

$$J_x = -i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad J_y = -i \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

These matrices do not commute, so the rotation order is important. However, we have the familiar commutation relation

$$[J_i, J_k] = J_i J_k - J_k J_i = i \epsilon_{ikl} J_l\tag{2.3}$$

where ϵ_{ikl} is the completely-antisymmetric Levi-Civita tensor. For a finite rotation around the z axis, we can build up the rotation angle θ in the usual way by using a succession of n infinitesimal angles $d\theta$; i.e., $\theta = nd\theta$, so that

$$\begin{aligned}R_z(\theta) &= \lim_{n \rightarrow \infty} \left[1 + i J_z \frac{\theta}{n} \right]^n \\ &= \exp i J_z \theta\end{aligned}$$

For rotations about any combined axis, we have

$$R_{\mathbf{n}}(\theta) = \exp i\mathbf{J} \cdot \mathbf{n}\theta \quad (2.4)$$

where \mathbf{n} is a unit vector in the direction of \mathbf{J} . All of this should be familiar from your classical mechanics courses. The extension to quantum mechanical systems is straightforward.

In view of the commutation relation (2.3), it is easy to see that any combination of successive rotations will ultimately reduce to either J_x , J_y or J_z . Therefore the J s form a closed and complete symmetry group which is called the *special orthogonal rotation group for three dimensions*, or $SO(3)$. It's called special and orthogonal because each of the three rotation matrices R has a unit determinant ($\det R = 1$) and is orthogonal (i.e., the transpose is equal to the inverse, $R^T = R^{-1}$).

Mathematicians observed many years ago that the $SO(3)$ group is closely associated with another group called $SU(2)$. This group involves the Pauli spin matrices, which also form a closed and complete group. The starting point for $SU(2)$, which means *special unitary group in two dimensions*, involves taking a linear combination of the Pauli matrices, which we'll write as

$$\begin{aligned} \boldsymbol{\sigma} \cdot \mathbf{x} &= \sigma_0 x^0 + \sigma_x x + \sigma_y y + \sigma_z z \\ &= \begin{bmatrix} x^0 + z & x - iy \\ x + iy & x^0 - z \end{bmatrix} \end{aligned} \quad (2.5)$$

where $\sigma_0 \equiv I$ (the four quantities σ_μ make up the Hamilton quaternion). (Actually, by adding the $\sigma_0 x^0$ term it's looks like we're really dealing with $SO(4)$ symmetry, but we're not.) The above matrix is Hermitian and has the determinant $(x^0)^2 - (x^2 + y^2 + z^2)$, which represents the invariant length of a vector in four-space (i.e., rotations have no effect):

$$(\bar{x}^0)^2 - \bar{x}^2 - \bar{y}^2 - \bar{z}^2 = (x^0)^2 - x^2 - y^2 - z^2 \quad (R \longrightarrow \bar{R})$$

In view of these properties, consider a similarity transformation of $\boldsymbol{\sigma} \cdot \mathbf{x}$ by an arbitrary 2×2 unitary matrix U , which we'll write as

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

For simplicity, let $\det U = 1$. For unitary U , U^{-1} must be identical to its conjugate transpose:

$$\begin{aligned} U^\dagger &= U^{-1}, \text{ or} \\ \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} &= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

Therefore,

$$U = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}, \quad |a|^2 + |b|^2 = 1 \quad (2.6)$$

Because a and b are arbitrary complex numbers, this leaves three quantities that can be used to represent the three rotations in ordinary three dimensional space.

Now consider the transformation

$$\boldsymbol{\sigma} \cdot \bar{\mathbf{x}} = U(\boldsymbol{\sigma} \cdot \mathbf{x})U^\dagger \quad (2.7)$$

Because of the unitary properties of U , the transformed quantity $\boldsymbol{\sigma} \cdot \bar{\mathbf{x}}$ will also be Hermitian with unit determinant. More importantly, it will also have the same form of $\boldsymbol{\sigma} \cdot \mathbf{x}$; that is,

$$\boldsymbol{\sigma} \cdot \bar{\mathbf{x}} = \begin{bmatrix} \bar{x}^0 + \bar{z} & \bar{x} - i\bar{y} \\ \bar{x} + i\bar{y} & \bar{x}^0 - \bar{z} \end{bmatrix} \quad (2.8)$$

At this point, you should be wondering whether the transformation matrix U can be made to reproduce the system of rotation equations given by (2.1) and (2.2). To find out, let us carry of the transformation (2.7). We get

$$\begin{bmatrix} \bar{x}^0 + \bar{z} & \bar{x} - i\bar{y} \\ \bar{x} + i\bar{y} & \bar{x}^0 - \bar{z} \end{bmatrix} = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \cdot \begin{bmatrix} x^0 + z & x - iy \\ x + iy & x^0 - z \end{bmatrix} \cdot \begin{bmatrix} a^* & -b \\ b^* & a \end{bmatrix}$$

Equating matrix elements and collecting terms, we have the four conditions

$$\begin{aligned}
\bar{x}^0 + \bar{z} &= (aa^* + bb^*)x^0 + (a^*b + ab^*)x + i(a^*b - ab^*)y + (aa^* - bb^*)z \\
\bar{x} - i\bar{y} &= (aa - bb)x - i(aa + bb)y - 2abz \\
\bar{x} + i\bar{y} &= (a^*a^* - b^*b^*)x + i(a^*a^* + b^*b^*)y - 2a^*b^*z \\
\bar{x}^0 - \bar{z} &= (aa^* + bb^*)x^0 - (a^*b + ab^*)x - i(a^*b - ab^*)y - (aa^* - bb^*)z
\end{aligned}$$

From $aa^* + bb^* = 1$, we immediately have $\bar{x}^0 = x^0$. We can still specify up to three identities for a and b . You should have no trouble verifying the following selections for these variables:

Case 1. $a = \cos \frac{1}{2}\theta, b = i \sin \theta$. Then

$$\begin{aligned}
\bar{x} &= x \\
\bar{y} &= y \cos \theta + z \sin \theta \\
\bar{z} &= -y \sin \theta + z \cos \theta
\end{aligned}$$

Case 2. $a = \cos \frac{1}{2}\theta, b = \sin \frac{1}{2}\theta$. Then

$$\begin{aligned}
\bar{x} &= x \cos \theta - z \sin \theta \\
\bar{y} &= y \\
\bar{z} &= x \sin \theta + z \cos \theta
\end{aligned}$$

Case 3. $a = \exp \left[\frac{1}{2}i\theta \right], b = 0$. Then

$$\begin{aligned}
\bar{x} &= x \cos \theta + y \sin \theta \\
\bar{y} &= -x \sin \theta + y \cos \theta \\
\bar{z} &= z
\end{aligned}$$

All of these these expressions are equivalent to

$$\begin{aligned}
U &= \begin{bmatrix} \cos \frac{1}{2}\theta & i \sin \frac{1}{2}\theta \\ i \sin \frac{1}{2}\theta & \cos \frac{1}{2}\theta \end{bmatrix} = \cos \frac{1}{2}\theta + i\sigma_x \sin \frac{1}{2}\theta \sim R_x(\theta) \\
U &= \begin{bmatrix} \cos \frac{1}{2}\theta & \sin \frac{1}{2}\theta \\ -\sin \frac{1}{2}\theta & \cos \frac{1}{2}\theta \end{bmatrix} = \cos \frac{1}{2}\theta + i\sigma_y \sin \frac{1}{2}\theta \sim R_y(\theta) \\
U &= \begin{bmatrix} \exp \left[\frac{1}{2}i\theta \right] & 0 \\ 0 & \exp \left[-\frac{1}{2}i\theta \right] \end{bmatrix} = \cos \frac{1}{2}\theta + i\sigma_z \sin \frac{1}{2}\theta \sim R_z(\theta)
\end{aligned}$$

In similarity to (2.4), these equations can be written with the single elegant expression

$$\begin{aligned}
U &= \exp \left[\frac{1}{2}i\boldsymbol{\sigma} \cdot \boldsymbol{\theta} \right] \\
&= \exp \left[\frac{1}{2}i\boldsymbol{\sigma} \cdot \mathbf{n}\theta \right]
\end{aligned} \tag{2.9}$$

This is the justification for asserting that there is an association between $SO(3)$ and $SU(2)$. The presence of half angles in these quantities was unavoidable, and confirms what we saw in Hamilton's formula. It also sets the stage for much of what is to follow. Note that the correspondence between $SO(3)$ and $SU(2)$ might have been guessed in view of the algebraic similarity of the commutations

$$\begin{aligned}
[J_x, J_y] &= iJ_z \quad \text{and} \\
\left[\frac{\sigma_x}{2}, \frac{\sigma_y}{2} \right] &= \frac{1}{2}i\sigma_z \quad (\text{et cycl.})
\end{aligned}$$

In summary, we say that

$$\begin{aligned}
U &= \exp \left[\frac{1}{2}i\boldsymbol{\sigma} \cdot \boldsymbol{\theta} \right] = \cos \frac{1}{2}\theta + i\boldsymbol{\sigma} \cdot \mathbf{n} \sin \frac{1}{2}\theta \quad \text{corresponds to} \\
R &= \exp [i\mathbf{J} \cdot \mathbf{n}\theta] = \cos \theta + i\boldsymbol{\sigma} \cdot \mathbf{n} \sin \theta
\end{aligned}$$

The correspondence between $SO(3)$ and $SU(2)$ is called an *isomorphism*.

Now, an arbitrary two-component spinor ξ transforms using the same unitary operator U :

$$\bar{\xi} = U\xi$$

It is easy to see that the inner or dot product of a spinor $\xi^\dagger\xi$ with itself transforms according to

$$\begin{aligned}\xi^\dagger\xi &= \begin{bmatrix} \xi_1^* & \xi_2^* \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = |\xi_1|^2 + |\xi_2|^2, \\ \bar{\xi}^\dagger\bar{\xi} &= |\bar{\xi}_1|^2 + |\bar{\xi}_2|^2\end{aligned}$$

However, the outer product transforms like

$$\begin{aligned}\xi\xi^\dagger &= \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \begin{bmatrix} \xi_1^* & \xi_2^* \end{bmatrix} = \begin{bmatrix} |\xi_1|^2 & \xi_1\xi_2^* \\ \xi_1^*\xi_2 & |\xi_2|^2 \end{bmatrix}, \\ \bar{\xi}\bar{\xi}^\dagger &= U(\xi\xi^\dagger)U^\dagger\end{aligned}$$

which is just like the transformation in (2.7). Furthermore, the determinants of these outer products are preserved (i.e., they vanish), which is also characteristic of the quaternion $\sigma \cdot x$. Therefore, we would expect that all spinors transform in the $SU(2)$ symmetry space outlined above for rotations. In the next section, we'll see that Lorentz transformations can be added in as well, and a little later we'll see that spinors transforming under pure Lorentz transformations (no rotations) play a central role in the description of spin-1/2 particles.

The crazy thing about spinors can be traced to the presence of *half angles* in the unitary transformations that they operate under. Ordinarily, when one rotates an object 360° , it goes back to the same thing it started out as. This is common sense, but as you learned in elementary quantum mechanics, common sense can be misleading. Consider the following situation (this is also from *Gravitation*). You are given a cube whose eight corners are tied by rubber bands to the eight respective corners of the room you're sitting in. Everything is neat and symmetric. Now rotate the cube 360° in the xy plane (either direction). The rubber bands are now tangled. Take my word for it that nothing you can do (short of rotating the cube back to its starting point) will untangle the rubber bands. Now rotate the cube *another* 360° in the same direction as before. Alas, the rubber bands are now even more tangled. But wait a minute – by a clever trick, and without touching the cube, you can maneuver the two tangled bunches of rubber bands around each other and untangle the mess, bringing things back to the way they were before. I won't go into it, but the topology of the tangled rubber bands mirrors the world that spinors live in – a 360° rotation doesn't do the job; instead, you need a 720° rotation to make things right again, because half angles are at work here. Spinors are rather like vectors that change sign when rotated 360° , and don't come “home” until two full rotations (720°) have been made.

3. Lorentz Transformations

The group of rotations that we have investigated up to now can be shown to include Lorentz transformations (or Lorentz “boosts”), which can be described as the symmetry group $SO(3,1)$. Recall that two inertial coordinate systems S and \bar{S} moving with constant velocity v with respect to one another in the positive x direction are related by

$$\begin{aligned}\bar{x}^0 &= \gamma(x^0 - \beta x) \\ \bar{x} &= \gamma(x - \beta x^0) \\ \bar{y} &= y \\ \bar{z} &= z\end{aligned}$$

where $\beta = v/c$ and

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

In matrix form, we can write this as

$$\Lambda_x = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that the identity $\gamma^2(1 - \beta^2) = 1$ can be used to make the substitution $\cosh \phi = \gamma$, $\sinh \phi = \gamma\beta$, $\phi = \operatorname{arctanh} \beta$. We can then write

$$\Lambda_x(\phi) = \begin{bmatrix} \cosh \phi & -\sinh \phi & 0 & 0 \\ -\sinh \phi & \cosh \phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.1)$$

It will be helpful at this point to digress to the case of an infinitesimal Lorentz transformation, as it will be of use later. Because the transformation can be written as

$$\bar{x}^\mu = \Lambda^\mu_\alpha(\phi) x^\alpha$$

an infinitesimal transformation in the x -direction can be expressed as

$$\Lambda^\mu_\alpha(\Delta\phi_x) = \delta^\mu_\alpha + \Delta\omega^\mu_\alpha$$

where

$$\Delta\omega^\mu_\alpha = \begin{bmatrix} 0 & -\Delta\phi_x & 0 & 0 \\ -\Delta\phi_x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

An interesting thing occurs when we contract this quantity using the metric tensor $g_{\mu\nu}$; we get, as expected,

$$\Delta\omega_{\nu\alpha} = g_{\mu\nu} \Delta\omega^\mu_\alpha$$

As we will now show, this quantity is antisymmetric with respect to its indices. The line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dx_\mu dx^\mu$ is an invariant, and we know that $d\bar{x}^\mu = \Lambda^\mu_\alpha dx^\alpha$, so

$$d\bar{x}_\mu d\bar{x}^\mu = \Lambda_\mu^\beta \Lambda^\mu_\alpha dx_\beta dx^\alpha$$

(This also shows that $\Lambda_\mu^\beta \Lambda^\mu_\alpha = \delta^\beta_\alpha$.) For an infinitesimal transformation, this requires that

$$\begin{aligned} (\Delta\omega^\beta_\alpha + \Delta\omega_\alpha^\beta) dx_\beta dx^\alpha &= 0 \quad \text{or} \\ (\Delta\omega_{\beta\alpha} + \Delta\omega_{\alpha\beta}) dx^\alpha dx^\beta &= 0, \quad \text{so that} \\ (\Delta\omega_{\beta\alpha} + \Delta\omega_{\alpha\beta}) &= 0 \end{aligned}$$

That is, the contracted form $\Delta\omega_{\alpha\beta}$ is antisymmetric in its two indices. We then have terms like

$$\begin{aligned} \Delta\omega_{01} &= g_{0\mu} \Delta\omega^\mu_1 = \Delta\omega^0_1 = -\Delta\phi_x \quad \text{and} \\ \Delta\omega_{10} &= g_{1\mu} \Delta\omega^\mu_0 = -\Delta\omega^1_0 = \Delta\phi_x \end{aligned}$$

so that

$$\Delta\omega_{\alpha\beta} = \begin{bmatrix} 0 & -\Delta\phi_x & -\Delta\phi_y & -\Delta\phi_z \\ \Delta\phi_x & 0 & 0 & 0 \\ \Delta\phi_y & 0 & 0 & 0 \\ \Delta\phi_z & 0 & 0 & 0 \end{bmatrix}$$

The rotation matrices can be added into this as well. Since the off-diagonal elements of $R_n(\theta)$ are already antisymmetric, contraction with $g_{\mu\nu}$ just changes the sign order. We then have

$$\Delta\omega_{\alpha\beta} = \begin{bmatrix} 0 & -\Delta\phi_x & -\Delta\phi_y & -\Delta\phi_z \\ \Delta\phi_x & 0 & -\Delta\theta_z & \Delta\theta_y \\ \Delta\phi_y & \Delta\theta_z & 0 & -\Delta\theta_x \\ \Delta\phi_z & -\Delta\theta_y & \Delta\theta_x & 0 \end{bmatrix} = \Delta\omega I_{\alpha\beta} \quad (3.2)$$

where $\Delta\omega$ is an infinitesimal “angle” and $I_{\alpha\beta}$ is the associated antisymmetric matrix of ± 1 elements. This matrix will be of considerable use when we investigate the covariance of the Dirac electron equation.

Using a similar approach to what we did with the rotation matrices, we can develop generators for the Lorentz boost transformations. We define the generator $K_x(\phi)$ for the x -direction transformation to be

$$\begin{aligned} K_x(\phi) &= -i \frac{\partial \Lambda_x}{\partial \phi} \Big|_{\phi=0} \\ &= -i \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \quad (3.3)$$

(Obviously, by reversing the velocity direction the -1 terms become $+1$.) Similarly, we have

$$K_y(\phi) = -i \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_z(\phi) = -i \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad (3.4)$$

The commutation relations for these operators are nothing short of amazing. Let us first convert the J rotation matrices to four-dimensional form:

$$J_x(\theta) = -i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad J_y(\theta) = -i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad J_z(\theta) = -i \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is now a simple matter to show that

$$\begin{aligned} [K_x, K_y] &= -iJ_z \\ [J_x, K_y] &= iK_z \\ [J_x, J_y] &= iJ_z \end{aligned} \quad (3.5)$$

Thus, the product of two Lorentz boosts (3.5) results in a rotation! (in electrodynamics, this is responsible for the so-called Thomas precession). Consequently, the boost generators K_n do not form a group by themselves, because they are mixed up with the rotation generators. However, it is easy to show that the combinations defined by $J_{n+} = \frac{1}{2}(J_n + iK_n)$, $J_{n-} = \frac{1}{2}(J_n - iK_n)$ satisfy

$$\begin{aligned} [J_{i+}, J_{j+}] &= i\epsilon_{ijk}J_{k+} \\ [J_{i-}, J_{j-}] &= i\epsilon_{ijk}J_{k-} \\ [J_{i+}, J_{j-}] &= 0 \end{aligned} \quad (3.6)$$

The remarkable thing about these commutation relations is that J_{n+} and J_{n-} each constitute a continuous group and, by (3.6), these two groups are *distinct*. Consequently, the Lorentz group of rotations and boosts must give rise to *two* distinctly different kinds of spinors in the associated space of $SU(2)$; the mathematicians inclined say that $SO(3,1) \sim SU(2) \otimes SU(2)$. The isomorphism between the two groups is therefore double (it’s sometimes referred to as a *double cover*) with respect to the two-dimensional group of unitary transformations. This rather subtle fact has enormous consequences in the physics of spin-1/2 particles since it allows for the existence of two fundamentally different species of spinors.

4. The Lorentz Group and $SU(2)$

The isomorphism between $SO(3)$ and $SU(2)$ is

$$R = \exp i\mathbf{J} \cdot \mathbf{n}\theta \longleftrightarrow U = \exp \frac{1}{2}i\boldsymbol{\sigma} \cdot \mathbf{n}\theta$$

where \longleftrightarrow expresses that the two groups have the indicated correspondence. So what is the correspondence between $SO(3,1)$ and $SU(2)$? From the above expression, we see that the correspondence between the “vectors” J and σ is just

$$\mathbf{J} \longleftrightarrow \frac{1}{2}\boldsymbol{\sigma}$$

Therefore, a likely connection for the combination of rotations and boosts is

$$J_{n-} = \frac{1}{2}(J_n - iK_n) \longleftrightarrow \frac{1}{2}\boldsymbol{\sigma} \quad \text{and} \quad (4.1)$$

$$J_{n+} = \frac{1}{2}(J_n + iK_n) \longleftrightarrow \frac{1}{2}\boldsymbol{\sigma} \quad (4.2)$$

and each of these expressions is expected to correspond to a distinct spinor. Now, the J_n are connected to rotations (θ), while the K_n are associated with Lorentz boosts (ϕ). Therefore, we might expect that $J \longleftrightarrow \frac{1}{2}\boldsymbol{\sigma}$ *always*, while $K \longleftrightarrow \pm \frac{1}{2}i\boldsymbol{\sigma}$. From (4.1) and (4.2), we see that

$$\mathbf{J} \longleftrightarrow \frac{1}{2}\boldsymbol{\sigma}, \quad \mathbf{K} \longleftrightarrow +\frac{1}{2}i\boldsymbol{\sigma} \quad \text{for } J_{n+} = 0$$

$$\mathbf{J} \longleftrightarrow \frac{1}{2}\boldsymbol{\sigma}, \quad \mathbf{K} \longleftrightarrow -\frac{1}{2}i\boldsymbol{\sigma} \quad \text{for } J_{n-} = 0$$

Let’s denote the spinor associated with the first case as φ_R and φ_L for the second. Then their respective transformation must look like

$$\bar{\varphi}_R = \exp\left[\frac{1}{2}i\boldsymbol{\sigma} \cdot \boldsymbol{\theta} + \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\phi}\right] \varphi_R = N\varphi_R \quad (4.3)$$

$$\bar{\varphi}_L = \exp\left[\frac{1}{2}i\boldsymbol{\sigma} \cdot \boldsymbol{\theta} - \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\phi}\right] \varphi_L = M\varphi_L \quad (4.4)$$

(Note the absence of the imaginary i with respect to the boost terms.) In spite of their apparent similarity, the two transformation matrices N and M represent totally different representations of the Lorentz Group; taken together, they constitute a kind of hybrid symmetry known as the $SL(2,C)$ group. The spinors φ_R and φ_L are known as *Weyl spinors*, in honor of the man who first deduced their existence in 1929. In older texts they are sometimes referred to as *dotted* and *undotted* spinors, respectively. We shall see that the designations φ_R and φ_L (standing for *right* and *left*) are very fitting for these quantities.

5. An Aside on Parity

As you are probably aware, the symmetry known as *parity* or *reflection symmetry* involves replacing every instance of x, y, z in an expression with $-x, -y, -z$. Notice that, for ordinary rotations, these reflections have no effect on J , so rotations are parity-conserving. This is not the case for K , however, because a parity operation changes the boost direction and therefore the sign of K changes as well. From (4.3) and (4.4), it is easy to see that φ_R and φ_L change into one another under a parity operation. Consequently, just one of the Weyl spinors is not sufficient to guarantee parity preservation – we need to have both of them working together. By stacking these spinors on top of one another, we can construct a four-component spinor that does preserve parity:

$$\psi = \begin{bmatrix} \varphi_R \\ \varphi_L \end{bmatrix} \quad (5.1)$$

And this is the Dirac spinor! The Dirac spinor applies to massive fermions, all of which are known to obey parity conservation. In view of (4.3) and (4.4), you should now be able to see that Dirac spinor transforms like

$$\begin{bmatrix} \bar{\varphi}_R \\ \bar{\varphi}_L \end{bmatrix} = \begin{bmatrix} \exp\left[\frac{1}{2}i\boldsymbol{\sigma} \cdot \boldsymbol{\theta} + \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\phi}\right] & 0 \\ 0 & \exp\left[\frac{1}{2}i\boldsymbol{\sigma} \cdot \boldsymbol{\theta} - \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\phi}\right] \end{bmatrix} \begin{bmatrix} \varphi_R \\ \varphi_L \end{bmatrix} \quad (5.2)$$

where each term represents a 2×2 matrix.

Later, we will see that for massless particles either of the Weyl spinorial representations holds, in spite of the above assertion that a single Weyl spinor cannot preserve parity. Such particles (neutrinos) were

observed in December 1956. Thus, Weyl's 1929 paper paved the way for one of mankind's most profound discoveries – neutrinos exist and are all described by the Weyl spinor φ_L , which violates parity! Although Pauli was the first to predict their existence (for a totally different reason), he rejected Weyl's paper out of hand solely because of the parity-violation issue. And while Pauli lived to witness the 1956 discovery of neutrinos and the downfall of parity (he died in 1958), Weyl, who passed away in 1955, sadly did not.

6. Dirac's Equation

Now for the Dirac equation, which is the relativistically correct expression for spin-1/2 particles. Consider the individual Weyl spinors transformed by a pure Lorentz boost (no rotations) for a free particle of mass m :

$$\begin{aligned}\varphi_R(\mathbf{p}) &= \exp\left[+\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\phi}\right]\varphi_R(0) = \left[\cosh\frac{1}{2}\phi + \boldsymbol{\sigma}\cdot\mathbf{n}\sinh\frac{1}{2}\phi\right]\varphi_R(0) \\ \varphi_L(\mathbf{p}) &= \exp\left[-\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\phi}\right]\varphi_L(0) = \left[\cosh\frac{1}{2}\phi - \boldsymbol{\sigma}\cdot\mathbf{n}\sinh\frac{1}{2}\phi\right]\varphi_L(0)\end{aligned}$$

where it is assumed that we are transforming from states in which the boost momentum is zero to states with 3-momentum \mathbf{p} . It can easily be shown that

$$\cosh\frac{1}{2}\phi = \sqrt{\frac{\gamma+1}{2}}, \quad (6.1)$$

$$\sinh\frac{1}{2}\phi = \sqrt{\frac{\gamma-1}{2}} \quad (6.2)$$

where $\gamma = (1 - \beta^2)^{-1/2}$. Then

$$\begin{aligned}\varphi_R(\mathbf{p}) &= \left[\sqrt{\frac{\gamma+1}{2}} + \boldsymbol{\sigma}\cdot\mathbf{n}\sqrt{\frac{\gamma-1}{2}}\right]\varphi_R(0) \\ \varphi_L(\mathbf{p}) &= \left[\sqrt{\frac{\gamma+1}{2}} - \boldsymbol{\sigma}\cdot\mathbf{n}\sqrt{\frac{\gamma-1}{2}}\right]\varphi_L(0)\end{aligned}$$

Now multiply the $\gamma - 1$ term by $(\gamma + 1)/(\gamma + 1)$ in both expressions and expand the quantity in brackets using both $E = \gamma mc^2$ and $E^2 = m^2 c^4 + c^2 \mathbf{p} \cdot \mathbf{p}$. You should be able to show without too much difficulty that

$$\varphi_R(\mathbf{p}) = \frac{1}{c\sqrt{2m(E + mc^2)}} [E + mc^2 + c\boldsymbol{\sigma} \cdot \mathbf{p}] \varphi_R(0), \quad (6.3)$$

$$\varphi_L(\mathbf{p}) = \frac{1}{c\sqrt{2m(E + mc^2)}} [E + mc^2 - c\boldsymbol{\sigma} \cdot \mathbf{p}] \varphi_L(0) \quad (6.4)$$

In the state where the momentum is zero, we obviously have $\varphi_R(0) = \varphi_L(0)$. Dividing (6.3) and (6.4) by each other in turns, we then have

$$\varphi_R(\mathbf{p}) = \frac{E + mc^2 + c\boldsymbol{\sigma} \cdot \mathbf{p}}{E + mc^2 - c\boldsymbol{\sigma} \cdot \mathbf{p}} \varphi_L(\mathbf{p}) \quad \text{and} \quad (6.5)$$

$$\varphi_L(\mathbf{p}) = \frac{E + mc^2 - c\boldsymbol{\sigma} \cdot \mathbf{p}}{E + mc^2 + c\boldsymbol{\sigma} \cdot \mathbf{p}} \varphi_R(\mathbf{p}) \quad (6.6)$$

These expressions can be simplified even further. For example, in (6.5) multiply top and bottom by $E + mc^2 + c\boldsymbol{\sigma} \cdot \mathbf{p}$, then use the equivalent energy formula

$$E^2 = m^2 c^4 + c^2 (\boldsymbol{\sigma} \cdot \mathbf{p})^2$$

(remember that \mathbf{p} is not an operator in this expression) and simplify. After some tedious but simple algebra, you should be able to show that

$$\begin{aligned}\varphi_R(\mathbf{p}) &= \frac{E + c\boldsymbol{\sigma} \cdot \mathbf{p}}{mc^2} \varphi_L(\mathbf{p}) \quad \text{and} \\ \varphi_L(\mathbf{p}) &= \frac{E - c\boldsymbol{\sigma} \cdot \mathbf{p}}{mc^2} \varphi_R(\mathbf{p})\end{aligned}$$

We're nearly done. Express this as the homogeneous matrix equation

$$\begin{bmatrix} -mc^2 & E + c\boldsymbol{\sigma} \cdot \mathbf{p} \\ E - c\boldsymbol{\sigma} \cdot \mathbf{p} & -mc^2 \end{bmatrix} \begin{bmatrix} \varphi_R \\ \varphi_L \end{bmatrix} = 0$$

where each component in the matrix is itself a 2×2 matrix. Because $E = cp_0$, we can write this as

$$\begin{bmatrix} -mc & p_0 I + \boldsymbol{\sigma} \cdot \mathbf{p} \\ p_0 I - \boldsymbol{\sigma} \cdot \mathbf{p} & -mc \end{bmatrix} \begin{bmatrix} \varphi_R \\ \varphi_L \end{bmatrix} = 0$$

In four-dimensional form, this goes over to

$$\begin{bmatrix} -mc & 0 & p_0 + p_z & p_x - ip_y \\ 0 & -mc & p_x + ip_y & p_0 - p_z \\ p_0 + p_z & p_x - ip_y & -mc & 0 \\ p_x + ip_y & p_0 - p_z & 0 & -mc \end{bmatrix} \begin{bmatrix} \varphi_{R1} \\ \varphi_{R2} \\ \varphi_{L1} \\ \varphi_{L2} \end{bmatrix} = 0 \quad (6.7)$$

where the column matrix on the right side is my sloppy way of writing the expanded form of the spinor. If we now define the following 4×4 matrices

$$\gamma^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (6.8)$$

$$\gamma^2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \quad \gamma^3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad (6.9)$$

or

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^k = \begin{bmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{bmatrix}, \quad (6.10)$$

then we can write (6.7) in the elegant covariant notation

$$[\gamma^\mu p_\mu - mc] \psi(\mathbf{p}) = 0 \quad (6.11)$$

where $\psi(\mathbf{p})$ is the four-component stacked Weyl spinor [remember that $p^\mu = (E/c, \mathbf{p})$ and $p_\mu = g_{\mu\nu} p^\nu = (E/c, -\mathbf{p})$.] This is the Dirac equation in the momentum representation, and $\psi(\mathbf{p})$ is now called the Dirac spinor. For the coordinate version of this expression, we simply replace p_μ with the momentum operator identity $p_\mu = i\hbar\partial_\mu$ to get

$$[i\hbar\gamma^\mu \partial_\mu - mc] \psi(x) = 0 \quad (6.12)$$

The four matrices above are known as the Dirac gamma matrices in the *Weyl representation*. The Dirac spinor in (5.1) and 6.11) is also in the Weyl representation. The Dirac equation, modified to include the appropriate terms for an electron in the electrostatic field of a proton, provides a fantastically accurate description of the energy levels and characteristics of the hydrogen atom. It's just amazing! In my opinion, the Dirac equation very possibly represents the pinnacle of human intellectual achievement, with quantum electrodynamics (or M-theory?) a close second.

For particles that are not moving, the Dirac equation reduces to

$$\begin{aligned} \gamma^0 p_0 &= mc\psi(x) \quad \text{or} \\ p_0 &= mc\gamma^0\psi(x) \end{aligned} \quad (6.13)$$

While there is nothing wrong with this expression, it would simplify matters somewhat if the matrix γ^0 were diagonal. You probably know that there is an alternate (and equivalent) set of Dirac matrices given by

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma^k = \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix}, \quad (6.14)$$

which is called the standard (or Dirac) representation. To convert from the Weyl form of γ^0 to the other, we use the following similarity transformation:

$$\begin{aligned}\gamma_{\text{Dirac}}^0 &= A (\gamma_{\text{Weyl}}^0) A^{-1} \quad \text{where} \\ A &= A^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\end{aligned}$$

The matrix A also converts the Weyl spinor, the other matrices γ_{Weyl}^k , and the spinor transform matrix (5.2) into the Dirac representation, which become

$$\begin{aligned}\psi_{\text{Dirac}} &= A\psi_{\text{Weyl}} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \varphi_R \\ \varphi_L \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} \varphi_R + \varphi_L \\ \varphi_R - \varphi_L \end{bmatrix}, \\ \gamma_{\text{Dirac}}^k &= \begin{bmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}S_{\text{Dirac}} &= AS_{\text{Weyl}}A^{-1} \\ &= \exp\left(\frac{1}{2}i\boldsymbol{\sigma} \cdot \boldsymbol{\theta}\right) \begin{bmatrix} \cosh \frac{1}{2}\phi & \boldsymbol{\sigma} \cdot \mathbf{n} \sinh \frac{1}{2}\phi \\ \boldsymbol{\sigma} \cdot \mathbf{n} \sinh \frac{1}{2}\phi & \cosh \frac{1}{2}\phi \end{bmatrix}\end{aligned}\quad (6.15)$$

where S_{Weyl} is the transformation matrix in (5.2).

7. Weyl's Neutrino

It was Weyl, in 1929, who noticed something odd about the Dirac equation for massless spin-1/2 particles (at the time, massless spin-1/2 fermions were not known to exist). Going back to (6.11) and putting $m = 0$ and expanding, we get the decoupled set of equations

$$\begin{aligned}[p_0 + \boldsymbol{\sigma} \cdot \mathbf{p}] \varphi_L(\mathbf{p}) &= 0 \quad \text{and} \\ [p_0 - \boldsymbol{\sigma} \cdot \mathbf{p}] \varphi_R(\mathbf{p}) &= 0\end{aligned}$$

When $m = 0$, the energy equation reads $E = c|\mathbf{p}|$, or $p_0 = |\mathbf{p}|$. Let us define the helicity of a massless particle as the dimensionless quantity

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{p}} = \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|\mathbf{p}|}\quad (7.1)$$

Clearly, this is a measure of the spin component with regard to the direction of motion; a positive helicity indicates that the spin is in the direction of motion (in the right-hand sense), and negative helicity when it is opposite to the motion. Thus,

$$[\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}] \varphi_L(\mathbf{p}) = -\varphi_L(\mathbf{p}) \quad \text{and} \quad (7.2)$$

$$[\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}] \varphi_R(\mathbf{p}) = +\varphi_R(\mathbf{p}) \quad (7.3)$$

As I mentioned earlier, these spinors violate conservation of parity. For this reason, Pauli considered them to be unphysical and made a number of rather harsh statements regarding Weyl's 1929 paper. The discovery of the neutrino in 1956, however, completely vindicated Weyl. It turns out that Nature respects parity with regard to all the fundamental forces with the exception of the weak interaction, which involves neutrinos. It also turns out that the neutrino is strictly described by the left-handed Weyl spinor φ_L . There are no right-handed neutrinos in Nature at all – God decided in the beginning that Nature should be left-handed!

8. An Aside on the Neutrino

By the mid-1950s, particle physicists had acquired some experience with three recently-discovered unstable particles called pi-mesons or pions (π^- , π^+ , π^0) and their decay characteristics. In what surely ranks

as the quickest experiment of its kind (it took several days), in 1957 Lederman observed the following decay process:

$$\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$$

(there's a similar decay in which $\pi^+ \rightarrow \bar{\mu}^+ + \nu_\mu$) where the plus- and minus-pions decay into muons (μ^-), antimuons ($\bar{\mu}^+$), muon antineutrinos ($\bar{\nu}_\mu$) and muon neutrinos (ν_μ). Because neutrinos are uncharged and do not feel the strong force, they are practically ghost particles – they rarely interact with anything and hence are extremely difficult to detect. However, muons are charged and can easily be observed in the laboratory. For decay processes in which the pions are initially at rest, the daughter particles must come out back to back in order to conserve momentum. By following the spins and directions of the outgoing muons, Lederman was able to deduce the following rule:

1. ALL NEUTRINOS ARE LEFT-HANDED (described by φ_L)
2. ALL ANTINEUTRINOS ARE RIGHT-HANDED (described by φ_R)

(Lederman received the Nobel Prize for his experimental verification of parity violation in 1988.) Neutrinos must be massless and travel at the speed of light if these laws are to hold. Otherwise, a Lorentz transformation could be used to effectively make a neutrino travel in a direction opposite to its line of motion, and the above rules would not hold; in that case, neutrinos and their antimatter partners would both exhibit left- and right-handedness. In recent years, there has been speculation that neutrinos have a small but non-zero mass, and that they can convert into different types (there are three kinds of neutrino). This belies the fact that to date no right-handed neutrinos have ever been observed, but the jury is still out on this.

9. Alternative Derivation of Dirac's Equation

In addition to his non-relativistic wave equation, which was based upon $E = p^2/2m$, Schrödinger derived a relativistically correct form using the cartesian energy relation

$$E^2 = m^2c^4 + c^2p^2 \tag{9.1}$$

Using the usual quantum transcription rules

$$\begin{aligned} E &\rightarrow i\hbar c \partial/\partial x^0, \\ c\hat{p}_j &\rightarrow -i\hbar c \vec{\nabla}_j \end{aligned}$$

he obtained, for a free particle,

$$-\hbar^2 c^2 \frac{\partial^2 \Psi}{(\partial x^0)^2} = m^2 c^4 \Psi - \hbar^2 c^2 \nabla^2 \Psi$$

Unfortunately, this equation didn't work for the hydrogen atom (the calculated energy spectrum is too wide compared to observation). The above equation, which is called the Klein-Gordan equation, works fine for spinless particles but fails for fermions like electrons. Dirac also came across this equation, and decided that it should reduce somehow to an expression that was linear in the time derivative, so he tried a "square root" operation for (9.1) and proposed

$$\begin{aligned} E &= \beta mc^2 + c\alpha_j p_j \quad \text{or} \\ i\hbar \frac{\partial \psi}{\partial t} &= \beta mc^2 \psi - i\hbar c \alpha_j \cdot \vec{\nabla}_j \psi \end{aligned} \tag{9.2}$$

where β and α_j are coefficients. Because ordinary numbers cannot satisfy this expression, Dirac assumed that these coefficients are 4×4 matrices having the commutation properties

$$\begin{aligned} \alpha_j \beta + \beta \alpha_j &= 0 \\ \alpha_i \alpha_j + \alpha_j \alpha_i &= 0 \quad (i \neq j) \\ \alpha_j^2 &= \beta^2 = 1 \end{aligned}$$

These properties are satisfied by

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (9.3)$$

$$\alpha_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad (9.4)$$

The α_j quantities are Hermitian (like β) and can be expressed in terms of the Pauli matrices as

$$\alpha_j = \begin{bmatrix} 0 & \sigma_j \\ \sigma_i & 0 \end{bmatrix} \quad (9.5)$$

where each entry in the matrix is itself a 2×2 matrix. Similarly, we have

$$\beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (9.6)$$

These are called the Dirac alpha and beta matrices. If we now multiply (9.2) through on the left by β and divide by c , we get

$$i\hbar\beta\frac{\partial\psi}{\partial x^0} = mc\psi - i\hbar\beta\alpha_j\vec{\nabla}_j\psi$$

If we now let $\gamma^0 = \beta$ and $\gamma^j = \beta\alpha_j$; this becomes

$$[i\hbar\gamma^\mu\partial_\mu - mc]\psi = 0 \quad (9.7)$$

which is the Dirac equation again, this time in the Dirac or standard representation.

While we're at it, I may as well show you how the Dirac equation automatically includes fermion spin. The Dirac Hamiltonian operator is

$$\hat{H} = \beta mc^2 + c\alpha_j\hat{p}_j$$

where $\hat{p}_j = -i\hbar\partial/\partial x^j$. Now, an operator \hat{Q} is said to be a constant of the motion if its time derivative, defined as

$$i\hbar\frac{d\hat{Q}}{dt} = \hat{Q}\hat{H} - \hat{H}\hat{Q},$$

is identically zero. The time derivative of the angular momentum operator \hat{L}_n , defined as

$$\hat{L}_n = \epsilon_{njk} x_j \hat{p}_k$$

is then

$$\begin{aligned} i\hbar\frac{d\hat{L}_n}{dt} &= \hat{L}_n\hat{H} - \hat{H}\hat{L}_n \\ &= \epsilon_{njk} x_j \hat{p}_k (\beta mc^2 + c\alpha_m \hat{p}_m) - (\beta mc^2 + c\alpha_m \hat{p}_m) \epsilon_{njk} x_j \hat{p}_k \end{aligned} \quad (9.8)$$

Since the βmc^2 term contains no \hat{p} , it commutes with everything; therefore, we can just drop that term and write

$$i\hbar\frac{d\hat{L}_n}{dt} = c\alpha_m \epsilon_{njk} (x_j \hat{p}_k \hat{p}_m - \hat{p}_m x_j \hat{p}_k)$$

Now, from the commutation condition

$$[x_j, \hat{p}_m] = x_j \hat{p}_m - \hat{p}_m x_j = i\hbar \delta_{jm}$$

we have $\hat{p}_m x_j = x_j \hat{p}_m - i\hbar \delta_{jm}$, so that Eq. (9.8) becomes

$$\begin{aligned} i\hbar \frac{d\hat{L}_n}{dt} &= c\alpha_m \epsilon_{njk} (x_j \hat{p}_k \hat{p}_m - x_j \hat{p}_m \hat{p}_k) + i\hbar c \alpha_j \epsilon_{njk} \hat{p}_k \\ &= i\hbar c \alpha_j \epsilon_{njk} \hat{p}_k \end{aligned} \quad (9.9)$$

Since this result is non-zero, we know that none of the angular momentum components \hat{L}_n is a constant of motion, at least for a Dirac particle. But Dirac figured that nature must conserve something resembling angular momentum, and he considered adding something to the \hat{L}_n that would satisfy the desired conservation criterion. He then wrote down the following three 4×4 matrices (there seems to be no end of these matrices!):

$$\hat{S}_n = \frac{1}{2}\hbar \begin{bmatrix} \sigma_n & 0 \\ 0 & \sigma_n \end{bmatrix}$$

and proceeded to compute $i\hbar d\hat{S}_n/dt$. In matrix notation, this is

$$i\hbar \frac{d\hat{S}_n}{dt} = \frac{1}{2}\hbar c \begin{bmatrix} \sigma_n & 0 \\ 0 & \sigma_n \end{bmatrix} \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix} \hat{p}_k - \frac{1}{2}\hbar c \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix} \begin{bmatrix} \sigma_n & 0 \\ 0 & \sigma_n \end{bmatrix} \hat{p}_k$$

or

$$i\hbar \frac{d\hat{S}_n}{dt} = \frac{1}{2}\hbar c \begin{bmatrix} 0 & \sigma_n \sigma_k \\ \sigma_n \sigma_k & 0 \end{bmatrix} \hat{p}_k - \frac{1}{2}\hbar c \begin{bmatrix} 0 & \sigma_k \sigma_n \\ \sigma_k \sigma_n & 0 \end{bmatrix} \hat{p}_k$$

Combining these two matrices, we have

$$\begin{aligned} i\hbar \frac{d\hat{S}_n}{dt} &= \frac{1}{2}\hbar c \begin{bmatrix} 0 & \sigma_n \sigma_k - \sigma_k \sigma_n \\ \sigma_n \sigma_k - \sigma_k \sigma_n & 0 \end{bmatrix} \hat{p}_k \\ &= \frac{1}{2}\hbar c \begin{bmatrix} 0 & -2i \epsilon_{njk} \sigma_j \\ -2i \epsilon_{njk} \sigma_j & 0 \end{bmatrix} \hat{p}_k \end{aligned} \quad (9.10)$$

Using the commutation properties for the Pauli matrices, (9.10) reduces, finally, to

$$\begin{aligned} i\hbar \frac{d\hat{S}_n}{dt} &= -i\hbar c \begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix} \epsilon_{njk} \hat{p}_k \\ &= -i\hbar c \alpha_j \epsilon_{njk} \hat{p}_k \end{aligned}$$

But this is just (9.9) with a minus sign. Therefore, we have the conservation condition

$$i\hbar \frac{d(\hat{L}_n + \hat{S}_n)}{dt} = 0$$

Dirac called the \hat{S}_n the spin matrices. This last result demonstrates that, for spin-1/2 particles at least, the quantity that is conserved is the sum of the angular momentum and the spin.

10. Covariance of the Dirac Equation

When I first saw the Dirac equation, my first thought was that its obviously covariant form would allow it to be easily recast in curved spacetime. I had a lot to learn, but first I had to be convinced that the γ -matrices do not transform as four-vectors, for if they did, then the Dirac spinor would have to be just a four-component scalar, which is not the case. Instead, the γ -matrices are considered fixed, and the spinors transform under Lorentz transformations according to (6.15), as you already know. But there is a slightly more elegant way to express the unitary transformation matrix that generalizes all possible directions in the boost parameter. If the Dirac equation is truly covariant, and if the γ -matrices are not four-vectors, then just how does the Dirac spinor transform in the complete Lorentz transformation of arbitrary boost directions? Well, most texts give the answer as

$$\begin{aligned} \bar{\psi}(\bar{x}) &= S \psi(x) \quad \text{and} \\ \psi(x) &= S^{-1} \bar{\psi}(\bar{x}) \quad \text{where} \\ S &= \exp\left(-\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu}\right) \end{aligned}$$

and where $\sigma^{\mu\nu}$ and $\omega_{\mu\nu}$ are both antisymmetric 4×4 matrices. Where the hell do these matrices come from? For rotations and boosts, we have the Dirac spinor transforming like

$$S_{\text{Dirac}} = \exp\left(\frac{1}{2}i\boldsymbol{\sigma} \cdot \boldsymbol{\theta}\right) \begin{bmatrix} \cosh \frac{1}{2}\phi & \boldsymbol{\sigma} \cdot \mathbf{n} \sinh \frac{1}{2}\phi \\ \boldsymbol{\sigma} \cdot \mathbf{n} \sinh \frac{1}{2}\phi & \cosh \frac{1}{2}\phi \end{bmatrix}$$

so the transformation matrix must somehow be equal to S . To derive it, let's start by considering the Lorentz covariance of the Dirac equation in what we will call the unprimed system O :

$$[i\hbar\gamma^\mu\partial_\mu - mc]\psi(x) = 0 \quad (10.1)$$

Assume that there is another observer in the primed system O' whose coordinate system is described by the infinitesimal Lorentz transformation

$$\begin{aligned} x' &= a^\mu{}_\alpha x^\alpha \quad \text{where} \\ a^\mu{}_\alpha &= \delta^\mu{}_\alpha + \Delta\omega^\mu{}_\alpha \end{aligned}$$

where $\Delta\omega^\mu{}_\alpha$ was defined earlier. In the primed system, the Dirac equation is

$$i\hbar\gamma^\mu\partial_{\mu'}\psi'(x') = mc\psi'(x')$$

(Remember that the γ -matrices are fixed.) We then have only two transformations to deal with

$$\begin{aligned} \partial_\mu &= a^\alpha{}_\mu \partial_{\alpha'} \quad \text{and} \\ \psi(x) &= S^{-1}\psi'(x') \end{aligned}$$

Plugging these expressions into (10.1), we get

$$i\hbar\gamma^\mu S^{-1}a^\alpha{}_\mu \partial_{\alpha'}\psi'(x') = mcS^{-1}\psi'(x')$$

(I moved the S^{-1} just to the right of γ^μ because it's considered a constant matrix.) Left multiplying by the matrix S gives

$$i\hbar S\gamma^\mu S^{-1}a^\alpha{}_\mu \partial_{\alpha'}\psi'(x') = mc\psi'(x') \quad (10.2)$$

This expression will have exactly the same form as that for O if we can show that

$$S\gamma^\mu S^{-1}a^\alpha{}_\mu = \gamma^\alpha$$

Multiplying on the right by $a^\lambda{}_\alpha$ turns this into

$$\begin{aligned} S\gamma^\lambda S^{-1} &= \gamma^\alpha a^\lambda{}_\alpha \quad \text{or, equivalently,} \\ S^{-1}\gamma^\lambda S &= \gamma^\alpha a^\lambda{}_\alpha \end{aligned}$$

(Remember that $a^\alpha{}_\mu a^\mu{}_\beta = \delta_\beta{}^\alpha$.) Since the transformation is infinitesimal, we let S vary only slightly from unity and adopt the forms

$$\begin{aligned} S &= 1 - k\Delta\omega_{\mu\nu}\sigma^{\mu\nu}, \\ S^{-1} &= 1 + k\Delta\omega_{\mu\nu}\sigma^{\mu\nu} \end{aligned}$$

where k is some convenient constant to be determined and $\sigma^{\mu\nu}$ is an antisymmetric matrix of unknown makeup for the time being (it must be completely antisymmetric, otherwise $\Delta\omega_{\mu\nu}$ would wipe out any symmetric terms). Plugging this into (10.2), we have

$$\begin{aligned} \gamma^\alpha a^\lambda{}_\alpha &= \gamma^\alpha (\delta^\lambda{}_\alpha + \Delta\omega^\lambda{}_\alpha) = \gamma^\lambda + \gamma^\alpha \Delta\omega^\lambda{}_\alpha \\ &= (1 - k\Delta\omega_{\mu\nu}\sigma^{\mu\nu})\gamma^\lambda (1 + k\Delta\omega_{\mu\nu}\sigma^{\mu\nu}) \end{aligned}$$

or

$$\gamma^\lambda + \gamma^\alpha \Delta\omega^\lambda{}_\alpha = \gamma^\lambda - k\Delta\omega_{\mu\nu}\sigma^{\mu\nu}\gamma^\lambda + k\gamma^\lambda \Delta\omega_{\mu\nu}\sigma^{\mu\nu}$$

At this point we assume that $\Delta\omega_{\mu\nu}$ and γ^λ commute (you'll see why shortly); we then arrive at

$$\gamma^\nu \Delta\omega^\lambda{}_\nu = k\Delta\omega_{\mu\nu}(\gamma^\lambda\sigma^{\mu\nu} - \sigma^{\mu\nu}\gamma^\lambda)$$

We would now like to get rid of $\Delta\omega_{\mu\nu}$. Notice that $\gamma^\nu \Delta\omega^\lambda{}_\nu = \gamma^\nu g^{\mu\lambda} \Delta\omega_{\mu\nu}$ and that

$$\begin{aligned}\gamma^\nu g^{\mu\lambda} \Delta\omega_{\mu\nu} &= \frac{1}{2}\gamma^\nu g^{\mu\lambda} \Delta\omega_{\mu\nu} + \frac{1}{2}\gamma^\nu g^{\mu\lambda} \Delta\omega_{\mu\nu} \\ &= \frac{1}{2}\gamma^\nu g^{\mu\lambda} \Delta\omega_{\mu\nu} - \frac{1}{2}\gamma^\mu g^{\nu\lambda} \Delta\omega_{\mu\nu} \\ &= \frac{1}{2}\Delta\omega_{\mu\nu} [\gamma^\nu g^{\mu\lambda} - \gamma^\mu g^{\nu\lambda}]\end{aligned}$$

(Watch your index juggling!) We then have

$$\begin{aligned}\frac{1}{2}\Delta\omega_{\mu\nu} [\gamma^\nu g^{\mu\lambda} - \gamma^\mu g^{\nu\lambda}] &= k\Delta\omega_{\mu\nu}(\gamma^\lambda\sigma^{\mu\nu} - \sigma^{\mu\nu}\gamma^\lambda) \quad \text{or} \\ \frac{1}{2}[\gamma^\nu g^{\mu\lambda} - \gamma^\mu g^{\nu\lambda}] &= k(\gamma^\lambda\sigma^{\mu\nu} - \sigma^{\mu\nu}\gamma^\lambda)\end{aligned}\tag{10.3}$$

since $\Delta\omega_{\mu\nu}$ is non-zero. We've rid ourselves of $\Delta\omega_{\mu\nu}$, but still have no idea what $\sigma^{\mu\nu}$ is. Since the right-hand side of (10.3) is a commutator involving the γ^ν , we suspect that $\sigma^{\mu\nu}$ might consist of combinations of the upper-index γ -matrices, and this is indeed the case. By writing

$$\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$$

you can show that this satisfies (10.3) provided $k = -i/4$.

Now, as you might have already suspected, the matrix $\Delta\omega_{\mu\nu}$ is the same one we introduced back in Section 3 (after all, Lorentz boosts and rotations must be brought into the analysis *somewhere*). This assumption made, we have now derived the infinitesimal version of the transformation matrix. For finite transformations, we write the $\Delta\omega_{\mu\nu} = \Delta\omega I_{\mu\nu}$ as we did earlier. A finite angle represents a sequential series of transformations involving $\Delta\omega$. If we set $\omega = n\Delta\omega$, then

$$\begin{aligned}S &\rightarrow \lim_{n \rightarrow \infty} \left[1 - \frac{i}{4} I_{\mu\nu} \sigma^{\mu\nu} \frac{\omega}{n} \right]^n \quad \text{or} \\ S &= \exp\left(-\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu}\right)\end{aligned}$$

where $\omega_{\mu\nu} = \omega I_{\mu\nu}$. This is the form for S you'll most likely see in the literature.

To test all this, let's say we're transforming from a system at rest to a system moving with velocity (boost) β in the x -direction. Then

$$\begin{aligned}S &= \exp\left(-\frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}\right) = \exp\left(\frac{1}{2} \omega_{01} \sigma^{01}\right) \quad \text{or} \\ S &= \cosh \frac{1}{2} \phi - \sinh \frac{1}{2} \phi \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

which is the answer you'll see in the textbooks. (It is also reminiscent of Hamilton's rotation formula of Section 1, making me wonder if S is the relativistic variant of that formula.)

11. The Dirac Spinor

So what does a Dirac spinor actually look like? To start, let's expand (6.13) for a particle at rest, using $p_0 = i\hbar\partial_0$ and the standard form of the β matrix:

$$\begin{aligned}i\hbar\partial_0\psi(x) &= mc\gamma^0\psi(x) \quad \text{or} \\ \partial_t \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} &= -imc^2/\hbar \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}\end{aligned}$$

The solution to this system of equations is just

$$\begin{aligned}
\psi_1(x) &= u_1(0) \exp(-iEt/\hbar) \\
\psi_2(x) &= u_2(0) \exp(-iEt/\hbar) \\
\psi_3(x) &= v_1(0) \exp(+iEt/\hbar) \\
\psi_4(x) &= v_2(0) \exp(+iEt/\hbar)
\end{aligned} \tag{11.1}$$

where $E = |mc^2|$ and

$$u_1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad u_2(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_1(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_2(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This is the solution to Dirac's equation for a spin-1/2 particle at rest. The quantities $u_{1,2}$ and $v_{1,2}$ are base spinors. To get the solution for particles that are moving, we use the matrix (6.15), which Lorentz-transforms these spinors to a moving frame with momentum \mathbf{p} . First note, however, that

$$\begin{aligned}
\cosh \frac{1}{2}\phi &= \sqrt{\frac{\gamma+1}{2}} = \sqrt{\frac{E+mc^2}{2mc^2}}, \\
\sinh \frac{1}{2}\phi &= \sqrt{\frac{\gamma-1}{2}} = \sqrt{\frac{E-mc^2}{2mc^2}}, \\
\tanh \frac{1}{2}\phi &= \frac{cp}{E+mc^2}
\end{aligned}$$

We then have, for $\theta = 0$,

$$\begin{aligned}
\begin{bmatrix} \cosh \frac{1}{2}\phi & \boldsymbol{\sigma} \cdot \mathbf{n} \sinh \frac{1}{2}\phi \\ \boldsymbol{\sigma} \cdot \mathbf{n} \sinh \frac{1}{2}\phi & \cosh \frac{1}{2}\phi \end{bmatrix} &= \cosh \frac{1}{2}\phi \begin{bmatrix} 1 & \boldsymbol{\sigma} \cdot \mathbf{n} \tanh \frac{1}{2}\phi \\ \boldsymbol{\sigma} \cdot \mathbf{n} \tanh \frac{1}{2}\phi & 1 \end{bmatrix} \\
&= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{bmatrix} 1 & 0 & \frac{p_z}{E+mc^2} & \frac{p_x-ip_y}{E+mc^2} \\ 0 & 1 & \frac{p_x+ip_y}{E+mc^2} & \frac{-p_z}{E+mc^2} \\ \frac{p_z}{E+mc^2} & \frac{p_x-ip_y}{E+mc^2} & 1 & 0 \\ \frac{p_x+ip_y}{E+mc^2} & \frac{-p_z}{E+mc^2} & 0 & 1 \end{bmatrix}
\end{aligned}$$

A straightforward application of this matrix to the base spinors then gives us, for the moving frame,

$$\begin{aligned}
u_1(p) &= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{E+mc^2} \\ \frac{p_x+ip_y}{E+mc^2} \end{bmatrix}, & u_2(p) &= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{bmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+mc^2} \\ \frac{-p_z}{E+mc^2} \end{bmatrix}, \\
v_1(p) &= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{bmatrix} \frac{p_z}{E+mc^2} \\ \frac{p_x+ip_y}{E+mc^2} \\ 1 \\ 0 \end{bmatrix}, & v_2(p) &= \sqrt{\frac{E+mc^2}{2mc^2}} \begin{bmatrix} \frac{p_x-ip_y}{E+mc^2} \\ \frac{-p_z}{E+mc^2} \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

In view of the fact that the Dirac equation has solutions for both positive and negative energy (11.1), the spinors $u_{1,2}(p)$ are called positive-energy solutions for *matter*, while the $v_{1,2}(p)$ are the negative-energy solutions for *antimatter*. When he first derived his famous equation in 1928, Dirac was initially puzzled by the negative-energy solutions because they implied that the associated particles would have positive charge (he thought that they might in fact be related to protons). In the face of withering skepticism by his colleagues, Dirac proposed that the negative-energy spinors actually described a new form of matter that he called *antimatter*. It is a tribute to Dirac's intellect and professional courage that he was able to sort out the ambiguities in this interpretation and stick to his convictions. In 1932, the Caltech experimental

physicist Carl Anderson discovered the positively-charged anti-electron (now called the *positron*) in one of his experiments.

I cannot adequately convey the admiration I feel for Dirac in making this discovery, nor my gratitude to God for having lifted the scales from mankind’s eyes and giving us this fantastic gift. Stephen Hawking has stated that if Dirac had somehow been able to patent his discovery, the royalties he would have received from the world’s electronics manufacturers alone would have made him a multibillionaire. As it was, in 1933 Dirac settled for a few thousand dollars and the Nobel Prize in Physics!

As we leave the Dirac spinor, keep in mind that the above analysis holds only for *free* spin1/2 particles. Adding interaction effects is straightforward, but the analysis gets much messier. You will be happy to know, however, that the solution to the Dirac equation for the hydrogen atom can be obtained in closed form, and its predictions match experiment to within a small fraction of a percent.

12. Fast and Slow Electrons

The Weyl representation is tailor-made for the description of electrons moving at relativistic velocities. At high speeds, the electron mass term in the Dirac equation can be considered negligible, and we can write

$$(\gamma^\mu p_\mu - mc)\psi \rightarrow \gamma^\mu p_\mu \psi = 0$$

But this is just a restatement of the zero-mass problem that we addressed in Section 7. As we have seen, a moving electron has only two degrees of freedom – it can spin in the direction of motion (positive helicity) or it can spin opposite to the line of motion (negative helicity).

For slow-moving electrons, the three-momentum p_i can be neglected; we are then left with

$$(\gamma^0 p_0 - mc)\psi = 0 \tag{12.1}$$

When this is the case, we choose to abandon the Weyl representation in favor of the Dirac set of gamma matrices. Then (12.1) becomes

$$\left[\frac{E}{c} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - mcI \right] \begin{bmatrix} \phi \\ \chi \end{bmatrix} = 0$$

(The Dirac spinor is written in this way according to convention; we could have just as well used the ψ_{Dirac} form from Section 6.) Using $E = mc^2$, this becomes simply

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi \\ \chi \end{bmatrix} = 0$$

which leaves $\chi = 0$. This means that for slowly-moving electrons the lower two-component spinor χ in the Dirac equation is “small” compared with its upper counterpart ϕ . One can take advantage of this fact when deriving the non-relativistic approximation of the Dirac equation, which of course reduces to the two-component spinor version of Schrödinger’s equation.

13. Final Thoughts

I’ll make this final part quick. The gamma matrices have many amazing properties, and they figure prominently in all aspects of modern quantum physics – we’ve only barely scratched the surface. And yet, they’re just constant matrices. From the identity $g^{\mu\nu}(x) = (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)/2$, you might have suspected that there are versions of the gamma matrices that are coordinate-dependent, and there are. From this perspective, the gamma matrices behave a little more like vector or tensor quantities (observe how many times we raised and lowered indices on the γ -matrices and other matrix quantities using the metric tensor, in spite of the fact that none of these matrices is a true four-vector). Years ago, while playing with Weyl’s 1918 theory, I naively transformed the Dirac matrices into polar form using

$$\begin{aligned} \gamma'_\mu &= \frac{\partial x^\alpha}{\partial x^{\mu'}} \gamma_\alpha, \\ x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned} \tag{13.1}$$

and got

$$\begin{aligned} \gamma'_0 &= \gamma_0 \\ \gamma'_1 &= \begin{bmatrix} 0 & 0 & \cos \theta & \sin \theta e^{-i\phi} \\ 0 & 0 & \sin \theta e^{i\phi} & -\cos \theta \\ -\cos \theta & -\sin \theta e^{-i\phi} & 0 & 0 \\ -\sin \theta e^{i\phi} & \cos \theta & 0 & 0 \end{bmatrix} \\ \gamma'_2 &= r \begin{bmatrix} 0 & 0 & -\sin \theta & \cos \theta e^{-i\phi} \\ 0 & 0 & \cos \theta e^{i\phi} & \sin \theta \\ \sin \theta & -\cos \theta e^{-i\phi} & 0 & 0 \\ -\cos \theta e^{i\phi} & -\sin \theta & 0 & 0 \end{bmatrix} \\ \gamma'_3 &= r \sin \theta \begin{bmatrix} 0 & 0 & 0 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 0 \\ 0 & ie^{-i\phi} & 0 & 0 \\ -ie^{i\phi} & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

All of which exhibit the same commutation properties as the γ -matrices along with the polar anticommutation relations

$$\gamma'_\mu \gamma'_\nu + \gamma'_\nu \gamma'_\mu = 2g_{\mu\nu}(r, \theta)$$

Anyway, I solved the Dirac electron equation in the spherical field of a proton using these matrices and got the same result given in my textbooks, so I suppose the approach has some merit (however, Pauli has shown that in flat space, all gamma matrices are equivalent to the Dirac matrices up to a unitary transformation). [The transformation coefficients in (13.1) provide an example of what is known as a *vierbein* in mathematical physics, although vierbeins are normally used to transform flat-space quantities into their curved-space counterparts.]

More interesting, however, is the observation that the covariant derivatives of the gamma matrices obey an algebra that is consistent with the property

$$\begin{aligned} \gamma'_{\mu|\nu} &= -\gamma'_{\nu|\mu} \quad \text{where} \\ \gamma'_{\mu|\nu} &= \gamma'_{\mu\nu} - \gamma'_\lambda \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} \end{aligned}$$

where the quantity in brackets is the Christoffel symbol of the second kind. (Vectors displaying this property are known as *Killing vectors*. Since the γ'_μ are not truly vectors, I don't know if this property has any real significance.) In Weyl's spacetime gauge theory, the covariant derivative of the metric tensor is non-zero and involves the Weyl field ϕ_μ . It would be interesting to know how Weyl would have approached the spherical and curved-space forms of the Dirac matrices and if the Weyl field plays any role in this type of analysis. Years ago, I used a revised form of Weyl's theory to see if the gyromagnetic ratio of the electron differed from integer 2 using curved-space variants of the spherical Dirac matrices presented above (but expressed as the 4-vector quantities $\bar{\psi}\gamma'_\mu\psi$). I had hoped to get something like 2.002 (and maybe you know the reason why). The result? It was too complex, and I couldn't do it. Maybe you'll have better luck, but then again it may just be a colossal waste of time.

References

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