RIEMANNIAN VECTORS IN A WEYL SPACE

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Summary

Although Einstein was initially impressed with Hermann Weyl's 1918 metric gauge theory of the combined electromagnetic-gravitational field, he famously reversed his praise and rejected it on physical grounds – the line element ds, as a proper-time measure of physical processes, was not gauge invariant. But regardless of the legitimacy of Einstein's argument, Weyl's theory suffers from another defect, which is its inability to accommodate vector fields that are invariant with respect to parallel transport or differentiation. In this paper we examine this defect in the theory and propose a revision that preserves the length of certain types of vectors. The revised space is seen to be mathematically consistent, although it still does not refute Einstein's objection. Using derived properties of constant-length vectors, simple derivations of the Klein-Gordon and Dirac equations are presented using the revised Weyl formalism, which indicate that the Weyl ϕ-field may indeed be related to the electromagnetic four-potential.

1. Notation

Both the metric tensor $g_{\mu\nu}$ and the coefficient of connection $\Gamma^\alpha_{\mu\nu}$ are symmetric in their lower indices. In Riemannian space, the connection is identified with the Christoffel symbol of the second rank

$$\Gamma^\alpha_{\mu\nu} = -\begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix}$$

consistent with the vanishing of the covariant derivative of the metric tensor (Ricci's theorem). Except when denoted otherwise, ordinary partial differentiation is indicated by a single subscripted bar (e.g., $g_{\mu\nu|\alpha}$), while covariant differentiation is indicated by a double subscripted bar ($g_{\mu\nu||\alpha}$, etc.).

Covariant differentiation of a covariant(contravariant) tensor introduces a positive(negative) connection term for each index.

2. Introduction

In the years immediately following Einstein's 1915 announcement of the theory of general relativity, numerous physicists attempted to generalize the theory in order to incorporate electrodynamics as a purely geometrical construct. By far the most interesting of these early attempts to unite gravitation and electromagnetism was published in 1918 by the German mathematical physicist Hermann Weyl, then chair of the mathematics department at the Eidgenössische Technische Hochschule (Swiss Federal Technical Institute) in Zürich.

Einstein's gravity theory is couched in the language of Riemannian geometry, which counts among its few arbitrary axioms the demand that vector length or magnitude be invariant under the process known as parallel transport. That is, if a vector is transported parallel to itself from one point in a Riemannian manifold to another, its orientation or direction may change but its magnitude does not. By relaxing this single requirement, Weyl discovered an intuitively straightforward way to generalize Riemannian geometry that paved the way for his 1918 theory of the unified gravitational-electromagnetic field.

Weyl proposed that physics should be invariant with respect to a rescaling of the metric tensor ($g_{\mu\nu} \rightarrow \lambda g_{\mu\nu}$), where $\lambda(x)$ is an arbitrary gauge factor. Thus, all metric quantities are gauge-transformed, including the metric tensor, the metric determinant $\sqrt{-g}$, tetrads, the coefficients of connection, and the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. Although initially impressed with Weyl's idea, Einstein soon rejected the theory on physical grounds. Einstein noted that the line element could be viewed as the proper-time measure of the ticking of a clock associated with certain physical processes (such as the spacing of atomic spectral lines) and, under a regauging of ds, these processes would become dependent upon their prehistories. Weyl was unable to effectively counter Einstein's criticism, and by 1921 the theory had been largely discounted.

Since that time, Weyl's original theory has been reconsidered by numerous researchers in an attempt to establish its relevance in modern physics, especially field theory, and there has been some speculation that Einstein's criticism might be surmountable. However, completely apart from Einstein's objection, Weyl's theory suffers from another, purely mathematical, inconsistency that would have doomed the theory from the beginning, anyway. This inconsistency, which involves the theory's inability to allow for parallel transport of vectors having constant magnitudes (which we will henceforth call *Riemannian vectors*), can be traced to Weyl's connection term. A revised form of the Weyl connection, free of this inconsistency, is proposed in the following.

3. Parallel Vector Transport

Weyl himself was instrumental in defining both the process of parallel transport and the definition of the connection quantity in terms of a transported vector

$$d\xi^{\alpha} = \Gamma^{\alpha}_{\mu\nu}\xi^{\mu}dx^{\nu} \tag{3.1}$$

where $\Gamma^{\alpha}_{\mu\nu}(x)$ is the connection. In order to allow for variable vector length, Weyl had to assume that a similar expression held for the change in vector magnitude, and he wrote

$$\begin{aligned} L^2 &= g_{\mu\nu}dx^{\mu}dx^{\nu}, \\ dL &= a\phi_{\alpha}\, dx^{\alpha} L \end{aligned} \tag{3.2}$$

where a is a suitable constant and $\phi_{\alpha}(x)$ is a new vector field that Weyl subsequently identified as the electromagnetic four-potential.

However, there are vector quantities whose magnitudes are pure numbers, and it is difficult to imagine how the lengths of these "constant-length" vectors might vary under parallel transport or differentiation (note that this change in length is distinctly different from that resulting from a gauge transformation). As an example, consider the simplest Riemannian vector, the unit vector $U^{\alpha} = dx^{\alpha}/ds$, whose magnitude is unity:

$$1 = g_{\mu\nu}U^{\mu}U^{\nu}$$

Parallel transferring both sides and relabeling indices, we have

$$\begin{aligned} 0 &= g_{\mu\nu|\alpha}U^{\mu}U^{\nu}U^{\alpha} + g_{\mu\nu}U^{\mu}dU^{\nu} + g_{\mu\nu}dU^{\mu}U^{\nu} \\ &= g_{\mu\nu||\alpha}U^{\mu}U^{\nu}U^{\alpha} \end{aligned} \tag{3.3}$$

where we have used (3.1). In Riemannian space, the covariant derivative of the metric tensor vanishes identically, in accordance with Ricci's theorem. Clearly, in Weyl's theory a non-zero metric covariant derivative is the key to the variation of vector magnitude under parallel transport. If this derivative vanishes, then the connection can be expressed solely in terms of the metric tensor and its first derivatives (i.e., the Christoffel symbols).

Using Weyl's ansatz (3.2), it is a simple matter to show that the metric covariant derivative is

$$g_{\mu\nu||\alpha} = 2ag_{\mu\nu}\phi_{\alpha} \tag{3.4}$$

It is similarly easy to show that this expression leads to a unique identification of the connection coefficient in what is now called a *Weyl space*:

$$\Gamma^{\alpha}_{\mu\nu} = -\begin{Bmatrix}\alpha \\ \mu\nu\end{Bmatrix} + a\delta^{\alpha}_{\mu}\phi_{\nu} + a\delta^{\alpha}_{\nu}\phi_{\mu} - ag_{\mu\nu}g^{\alpha\beta}\phi_{\beta} \tag{3.5}$$

Using these expressions, Weyl was able to develop a mathematically consistent description of both gravitation and electrodynamics which, by employing a modified Einstein-Maxwell action principle, led to equations of motion. It was quickly realized by Pauli and others that the gravitational equations of motion in Weyl's theory were equivalent to those of Einstein's theory under very plausible conditions.

4. Riemannian Vectors

In a Weyl space with a non-vanishing vector field ϕ_{α}, the length of an arbitrary vector is not only allowed to vary, its variance is actually *mandated* – no vector is "constant." For example, from (3.2) parallel transport of the unit vector leaves a null result only for the trivial case $\phi_{\alpha} = 0$. Even in Weyl's time, physicists pointed out that there are vectors whose magnitudes are pure numbers, and that these quantities should not vary under parallel transfer or differentiation. Perhaps the most obvious example is the momentum four-vector

$$g_{\mu\nu}p^\mu p^\nu = m^2c^2$$

whose length is a constant that logically should not change from point to point.

In order to accommodate Riemannian vectors in a Weyl space, it is obvious that the Weyl connection, which was obtained via his definition of the metric covariant derivative, has to be modified.

5. Revision of the Weyl Connection

Consider again the expression for the assumed invariance of the unit vector $U^\mu = dx^\mu/ds$,

$$g_{\mu\nu||\alpha}U^\mu U^\nu U^\alpha = 0$$

It is clear from this equation that the metric covariant derivative either vanishes identically or satisfies the peculiar cyclic symmetry condition

$$g_{\mu\nu||\alpha} + g_{\alpha\mu||\nu} + g_{\nu\alpha||\mu} = 0 \tag{5.1}$$

It is easy to show that the only definition of the connection consistent with this condition is

$$\Gamma^\alpha_{\mu\nu} = -\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix} + a\delta^\alpha_\mu\phi_\nu + a\delta^\alpha_\nu\phi_\mu - 2ag_{\mu\nu}g^{\alpha\beta}\phi_\beta \tag{5.2}$$

In view of the close similarity of this expression with Weyl's, we will call the space defined by (5.2) a *generalized Weyl space*. It will be noted that the revised connection, unlike the one in Weyl's original theory, is not gauge invariant, although the contracted connection $\Gamma^\alpha_{\mu\alpha}$ is fully gauge invariant.

With this definition, along with the symmetry condition in (5.1), it can be straightforwardly shown that the metric covariant derivative is now

$$g_{\mu\nu||\beta} = 2ag_{\mu\nu}\phi_\beta - ag_{\beta\mu}\phi_\nu - ag_{\nu\beta}\phi_\mu \tag{5.3}$$

$$g^{\mu\nu}{}_{||\alpha} = -2ag^{\mu\nu}\phi_\alpha + a\delta^\mu_\alpha g^{\lambda\nu}\phi_\lambda + a\delta^\nu_\alpha g^{\mu\lambda}\phi_\lambda \tag{5.4}$$

With the help of these expressions, it is easy to show that the following identities also hold in this Weyl space:

$$\begin{aligned} g^{\mu\nu}{}_{||\nu} &= a(n-1)g^{\mu\nu}\phi_\nu \qquad (5.6)\\ \sqrt{-g}_{||\nu} &= a(n-1)\sqrt{-g}\,\phi_\nu \\ \phi_\alpha &= \frac{1}{2a(n-1)}g^{\mu\nu}g_{\mu\nu||\alpha} \end{aligned}$$

where n is the dimension of spacetime (later we will be working solely with $n = 4$).

By contraction of (5.3) with $g^{\alpha\beta}$, it is easy to see that the connection in (5.2) can also be written as

$$\Gamma^\alpha_{\mu\nu} = -\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix} - g^{\alpha\beta}g_{\mu\nu||\beta} \tag{5.7}$$

This expression was actually first suggested by Schrödinger, who in 1950 attempted to define the most general type of connection possible (symmetric and otherwise). He proposed that such a connection could be expressed by

$$\Gamma^\alpha_{\mu\nu} = -\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix} - g^{\alpha\beta}T_{\mu\nu||\beta}$$

where Schrödinger's unspecified T-quantity has exactly the same symmetry properties as $g_{\mu\nu||\beta}$. This observation, along with the fact that the revised connection can accommodate Riemannian vectors under parallel transport, gives considerable support to the proposition that (5.2) is the true connection.

Now that we have the necessary tools, let us now look at how Riemannian vectors behave in a generalized Weyl space.

6. Properties of Riemannian Vectors

Given any contravariant Riemannian vector $\xi^\mu(x)$, its constant magnitude or length can be expressed as

$$L^2 = g_{\mu\nu}\xi^\mu\xi^\nu$$

Covariant differentiation of both sides with respect to x^α followed by multiplication by ξ^α gives

$$0 = g_{\mu\nu||\alpha}\xi^\mu\xi^\nu\xi^\alpha + 2g_{\mu\nu}\xi^\nu\xi^\mu{}_{||\alpha}\xi^\alpha$$

By (5.1), the first term vanishes and we are left with

$$\xi_\mu\,\xi^\mu{}_{||\alpha}\xi^\alpha = 0$$

If ξ^μ is the unit vector dx^μ/ds, then we know that this is just the contracted equation of the geodesics, $U^\mu{}_{||\alpha}U^\alpha = 0$. Consequently, we will take as the first general property of Riemannian vectors the identity

$$\xi^\mu{}_{||\alpha}\xi^\alpha = 0 \tag{6.1}$$

Also, using the fact that $\xi_\mu\,\xi^\mu$ is a constant, (6.1) can be written as

$$\xi_{\mu||\alpha}\,\xi^\mu\xi^\alpha = 0$$

indicating that $\xi_{\mu||\alpha}$ is either identically zero or antisymmetric in the indices:

$$\xi_{\mu||\alpha} = -\xi_{\alpha||\mu} \tag{6.2}$$

Since we will have no use for vectors that cannot be covariantly differentiated, we will take this as the second general property of an arbitrary Riemannian vector.

In Riemannian space, any vector satisfying (6.2) is called a *Killing vector*. Its admission into a metric indicates that the metric has a hidden or implicit symmetry. In the Weyl space we are considering, however, this is not the case. To see this, recall that for a Killing vector χ^μ to exist, the metric of a infinitesimally-transformed coordinate system $\overline{x}$ must be identical to the transformed metric *in that same system*. Consequently, we demand that

$$\begin{aligned}\overline{x}^\mu &= x^\mu + \epsilon\chi^\mu \\ \overline{g}_{\mu\nu}(\overline{x}) &= g_{\mu\nu}(\overline{x})\end{aligned}$$

Using the identities

$$\begin{aligned}\overline{g}_{\mu\nu}(\overline{x}) &= g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial \overline{x}^\mu}\frac{\partial x^\beta}{\partial \overline{x}^\nu} \\ g_{\alpha\beta}(x) &= g_{\alpha\beta}(\overline{x}) - \epsilon g_{\alpha\beta|\lambda}\chi^\lambda\end{aligned}$$

it is easy to show that the *Killing equation* is

$$2g_{\alpha\beta||\lambda}\chi^\lambda + \chi_{\alpha||\beta} + \chi_{\beta||\alpha} = 0 \tag{6.3}$$

which reduces to the usual form (6.2) when space is Riemannian. Note, however, that contraction of (6.3) with $\chi^\alpha\chi^\beta$ gives

$$\chi_{\alpha||\beta}\,\chi^\alpha\chi^\beta = 0$$

while contraction with $g^{\alpha\beta}$ yields

$$2a(n-1)\chi^\lambda\phi_\lambda + g^{\alpha\beta}\chi_{\alpha||\beta} = 0$$

This shows that the covariant derivative of a Killing vector in a generalized Weyl space is *not* antisymmetric with respect to its lower indices.

Let us now see how the Weyl field ϕ_μ itself behaves under parallel transport. We have

$$L^2 = g^{\mu\nu}\phi_\mu\phi_\nu, \quad 2L\,dL = g^{\mu\nu}{}_{||\alpha}\phi_\mu\phi_\nu dx^\alpha$$

However, with the use of (5.4) we see that dL is identically zero. Since length invariance holds also for covariant differentiation, we have determined that the Weyl vector ϕ_μ is a Riemannian vector and should be subject to the conditions

$$\begin{aligned} \phi^\mu{}_{||\alpha}\phi^\alpha &= 0 \, , \\ \phi_{\mu||\alpha} + \phi_{\alpha||\mu} &= 0 \end{aligned}$$

This is an odd result if we expect ϕ_μ to have any relevance to the electromagnetic four-potential A_μ. However, Galehouse has proposed that, in his consideration of quantum geodesics in five-dimensional space,

$$\frac{dx^\mu}{ds} = g^{\mu\nu}A_\nu$$

indicating that in some respects the potential behaves as the unit tangent vector, which is a Riemannian vector. If legitimate, this observation would support the proposition that ϕ_μ is also a Riemannian vector.

As a demonstration of this otherwise empty formalism, let us derive the conservation law for energy-momentum. The matter tensor $T^{\mu\nu}$ for a moving collection of incoherent, non-interacting particles of density $\rho(x)$ is defined by

$$T^{\mu\nu} = \rho\, U^\mu U^\nu$$

The divergence of this quantity vanishes, in accordance with general relativity:

$$T^{\mu\nu}{}_{||\nu} = \rho\, U^\mu{}_{||\nu} U^\nu + U^\mu(\rho\, U^\nu)_{||\nu} = 0$$

The first term on the right represents the equations of the geodesics, which vanish, leaving the conservation law

$$(\rho\, U^\nu)_{||\nu} = 0$$

It should be noted that this result cannot be derived from the original Weyl theory.

7. The Dirac Gamma Matrices as Riemannian Vectors

A very special kind of "vector" is the Dirac gamma matrix, which is used to define the metric in spinor space:

$$\begin{aligned} 2g^{\mu\nu}(x) &= \gamma^\mu(x)\gamma^\nu(x) + \gamma^\nu(x)\gamma^\mu(x) \, , \\ \gamma_\mu\gamma^\mu &= n \, , \quad (\gamma^0)^2 = g^{00} \, , \quad (\gamma^i)^2 = g^{ii} \end{aligned} \tag{7.1}$$

Of course, the gamma matrices are not vectors in the normal sense (although the quantity $\overline{\psi}(x)\gamma^\mu\psi(x)$, where $\overline{\psi}$ is the adjoint of the Dirac spinor ψ, is a true contravariant vector), but for subsequent purposes we will treat them as vectors.

But why would we want to consider these matrices as Riemannian vectors in the first place? To answer this, consider the Heisenberg equation of motion, which relates the time translation of some operator $\widehat{O}$ with the Hamiltonian H in elementary quantum mechanics:

$$i\hbar\frac{d\widehat{O}}{dt} = [\widehat{O}, H]$$

Since γ^μ is a Dirac matrix, we consider the time derivative of the position operator $\widehat{x}^i$ with the Dirac Hamiltonian

$$\begin{aligned} H &= mc^2\gamma^0 + c\alpha^j p_j \\ &= mc^2\gamma^0 - i\hbar c\,\alpha^j\nabla_j \end{aligned}$$

where the α^j are the three hermitian Dirac alpha matrices. We then have

$$\begin{aligned} i\hbar\frac{dx^i}{dt} &= i\hbar c\delta^i{}_j\alpha^j \text{ , or} \\ \alpha^i &= \frac{dx^i}{cdt} \simeq \frac{dx^i}{ds} \end{aligned}$$

In view of this and the definition of the gamma matrices

$$\gamma^i = \gamma^0\alpha^i$$

we have

$$\gamma^\mu \sim \gamma^0\frac{dx^\mu}{ds} \tag{7.2}$$

We avoid the equality sign here, because obviously the unit vector does not behave as a 4×4 matrix. But since all the quantities in (7.2) are of magnitude unity, the asserted relationship between the gamma matrices and the unit tangent vectors is at least plausible.

Now that we have rather more confidence in the gamma matrices behaving as Riemannian vectors, let us investigate some of their properties. By covariant differentiation of the covariant form

$$2g_{\mu\nu} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu$$

we have

$$2g_{\mu\nu||\alpha} = \gamma_{\mu||\alpha}\gamma_\nu + \gamma_\mu\gamma_{\nu||\alpha} + \gamma_{\nu||\alpha}\gamma_\mu + \gamma_\nu\gamma_{\mu||\alpha}$$

It is now easy to see that, given the symmetrization condition $g_{\mu\nu||\alpha} + g_{\alpha\mu||\nu} + g_{\nu\alpha||\mu} \equiv \left\{g_{\mu\nu||\alpha}\right\} = 0$, we have

$$\left\{\left(\gamma_{\mu||\alpha} + \gamma_{\alpha||\mu}\right)\gamma_\nu + \gamma_\nu\left(\gamma_{\mu||\alpha} + \gamma_{\alpha||\mu}\right)\right\} = 0$$

This can only be true if

$$\gamma_{\mu||\alpha} + \gamma_{\alpha||\mu} = 0$$

Thus, the indices in the differentiated gamma matrices are antisymmetric, consistent with one of the properties of a Riemannian vector. This same result could have been obtained by an alternative route. The "square root" of the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$can be written as

$$\begin{aligned} A\,ds &= \gamma_\mu dx^\mu \text{ , or} \\ A &= \gamma_\mu\frac{dx^\mu}{ds} \end{aligned}$$

where A is a constant idempotent matrix of trace zero (A cannot be the identity matrix, because this would conflict with the properties of the gamma matrices). Differentiation with respect to ds then gives

$$\begin{aligned} 0 &= \gamma_{\mu|\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + \gamma_\mu\frac{d^2x^\mu}{ds^2} \\ &= \left(\gamma_{\mu|\nu} + \gamma_\lambda\Gamma^\lambda_{\mu\nu}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \\ &= \gamma_{\mu||\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \end{aligned}$$

which again shows $\gamma_{\mu||\nu}$ to be antisymmetric.

Similarly, covariant differentiation of (7.1) gives

$$2g^{\mu\nu}_{\ \ ||\nu} = \gamma^\mu{}_{||\nu}\gamma^\nu + \gamma^\mu\gamma^\nu{}_{||\nu} + \gamma^\nu{}_{||\nu}\gamma^\mu + \gamma^\nu\gamma^\mu{}_{||\nu}$$

Noting that

$$\begin{aligned} g^{\mu\nu}_{\ \ ||\nu} &= a(n-1)g^{\mu\nu}\phi_\nu \text{ and} \\ \gamma^\nu{}_{||\nu} &= (g^{\mu\nu}\gamma_\mu)_{||\nu} \\ &= a(n-1)g^{\mu\nu}\phi_\mu\gamma_\nu \\ &= a(n-1)\gamma^\nu\phi_\nu \end{aligned} \tag{7.3}$$

we have

$$2a(n-1)g^{\mu\nu}\phi_\nu = \gamma^\mu{}_{||\nu}\gamma^\nu + a(n-1)\gamma^\mu\gamma^\nu\phi_\nu + a(n-1)\gamma^\nu\gamma^\mu\phi_\nu + \gamma^\nu\gamma^\mu{}_{||\nu}$$

This reduces to

$$\gamma^\mu{}_{||\nu}\gamma^\nu + \gamma^\nu\gamma^\mu{}_{||\nu} = 0$$

which is remarkably similar to (6.1).

Lastly, for completeness we shall derive a few more properties of Riemannian vectors, several of which equally well to the gamma "vectors." Double covariant differentiation of a Riemannian vector introduces the Riemann-Christoffel tensor via

$$\xi_{\alpha||\mu||\nu} - \xi_{\alpha||\nu||\mu} = -\xi_\lambda R^\lambda{}_{\alpha\mu\nu}$$

Cyclic permutation of the lower indices in the Riemann-Christoffel tensor results in the *Bianchi identities*

$$R^\lambda{}_{\alpha\mu\nu} + R^\lambda{}_{\nu\alpha\mu} + R^\lambda{}_{\mu\nu\alpha} = 0$$

Using these identities and the antisymmetry of $\xi_{\alpha||\mu}$, it is easy to show that this vector also obeys a Bianchi-like cyclic symmetry:

$$\xi_{\alpha||\mu||\nu} + \xi_{\nu||\alpha||\mu} + \xi_{\mu||\nu||\alpha} = 0 \tag{7.4}$$

From these expressions, we also get

$$\xi_{\alpha||\mu||\nu} = \xi_\lambda R^\lambda{}_{\nu\alpha\mu}$$

Therefore, a Riemannian vector can be covariantly differentiated twice only when the Riemann-Christoffel tensor is non-zero.

Consider now the expansion of $\xi_{\alpha||\mu||\nu}$:

$$\xi_{\alpha||\mu||\nu} = \xi_{\alpha||\mu|\nu} + \xi_{\alpha||\lambda}\Gamma^\lambda_{\mu\nu} + \xi_{\lambda||\mu}\Gamma^\lambda_{\alpha\nu}$$

Symmetrization gives

$$\left\{\xi_{\alpha||\mu||\nu}\right\} = \left\{\xi_{\alpha||\mu|\nu}\right\} + \left\{\xi_{\alpha||\lambda}\Gamma^\lambda_{\mu\nu}\right\} + \left\{\xi_{\lambda||\mu}\Gamma^\lambda_{\alpha\nu}\right\} = 0$$

Because $\xi_{\alpha||\lambda}$ is antisymmetric in its indices while $\Gamma^\lambda_{\mu\nu}$ is symmetric, the last two terms cancel and we have

$$\left\{\xi_{\alpha||\mu|\nu}\right\} = 0$$

From elementary tensor theory, any antisymmetric tensor of rank two that satisfies this condition is derivable from a potential quantity, so that

$$\xi_{\alpha||\mu} = k_{\alpha|\mu} - k_{\mu|\alpha}$$

The most visible example of this is the Maxwell electrodynamic stress tensor,

$$F_{\mu\nu} = A_{\mu|\nu} - A_{\nu|\mu}$$

where A_μ is the four-potential.

This decomposition of Riemannian vectors can be extended in the case of the Dirac matrices. Because $g_{\mu\nu||\alpha}$ can be decomposed into terms involving the metric and the Weyl field ϕ_μ, it should be possible to do the same for $\gamma_{\mu||\nu}$. Using (5.3), let us expand in terms of the gamma matrices:

$$\begin{aligned}
2g_{\mu\nu||\alpha} &= 4ag_{\mu\nu}\phi_\alpha - 2ag_{\alpha\mu}\phi_\nu - 2ag_{\nu\alpha}\phi_\mu \\
&= \gamma_{\mu||\alpha}\gamma_\nu + \gamma_\mu\gamma_{\nu||\alpha} + \gamma_{\nu||\alpha}\gamma_\mu + \gamma_\nu\gamma_{\mu||\alpha} \\
&= a\gamma_\mu\gamma_\nu\phi_\alpha + a\gamma_\mu\gamma_\nu\phi_\alpha + a\gamma_\nu\gamma_\mu\phi_\alpha + a\gamma_\nu\gamma_\mu\phi_\alpha \\
&\quad -a\gamma_\alpha\gamma_\mu\phi_\nu - a\gamma_\mu\gamma_\alpha\phi_\nu - a\gamma_\nu\gamma_\alpha\phi_\mu - a\gamma_\alpha\gamma_\nu\phi_\mu \\
&= a(\gamma_\mu\phi_\alpha - \gamma_\alpha\phi_\mu)\gamma_\nu + a(\gamma_\nu\phi_\alpha - \gamma_\alpha\phi_\nu)\gamma_\mu \\
&\quad +a\gamma_\nu(\gamma_\mu\phi_\alpha - \gamma_\alpha\phi_\mu) + a\gamma_\mu(\gamma_\nu\phi_\alpha - \gamma_\alpha\phi_\nu)
\end{aligned}$$

Equating terms in the second and last lines (either beginning or ending with γ_μ or γ_ν), we see that a sufficient (if not necessary) solution is

$$\gamma_{\mu||\alpha} = a\left(\gamma_\mu \phi_\alpha - \gamma_\alpha \phi_\mu\right)$$

This result indicates that the γ_μ can be covariantly differentiated only in a Weyl manifold. From this expression and (7.4), it can easily be shown that the gamma matrices also satisfy the peculiar identity

$$\gamma_\mu F_{\alpha\beta} + \gamma_\beta F_{\mu\alpha} + \gamma_\alpha F_{\beta\mu} = 0$$

where $F_{\alpha\beta} = \phi_{\alpha||\beta} - \phi_{\beta||\alpha}$. By raising and lowering indices, it can also be shown that

$$\begin{aligned} \gamma^\mu{}_{||\alpha} &= a\left(\delta^\mu{}_\alpha \gamma^\lambda \phi_\lambda - \gamma^\mu \phi_\alpha\right) \\ \gamma^\mu{}_{||\mu} &= a(n-1)\gamma^\mu \phi_\mu \end{aligned}$$

results that are fully consistent with previous expressions.

8. Derivation of the Klein-Gordon and Dirac Equations in Weyl Space

In Riemannian space, the Laplacian of a scalar function $\varphi(x)$ is expressed by

$$\begin{aligned} \nabla^2 \varphi &= \frac{1}{\sqrt{-g}}(\sqrt{-g} g^{\mu\nu} \varphi_{|\mu})_{||\nu} \\ &= \frac{1}{\sqrt{-g}} D_\mu D_\nu \left(\sqrt{-g} g^{\mu\nu} \varphi\right) \end{aligned} \tag{8.1}$$

where D is the covariant derivative operator. When the spacetime dimension exceeds three, the notation is replaced by the d'Alembertian operator, $\Box^2$. In a Weyl space, the covariant derivative operators also act on the metric terms, giving

$$\begin{aligned} \sqrt{-g}_{||\mu} &= a(n-1)\sqrt{-g}\, \phi_\mu, \\ g^{\mu\nu}{}_{||\mu} &= a(n-1) g^{\mu\nu} \phi_\mu \end{aligned}$$

Expanding (8.1), we have

$$\begin{aligned} \Box^2 \varphi &= 4a^2(n-1)^2 \sqrt{-g} g^{\mu\nu} \phi_\mu \phi_\nu \varphi + 2a(n-1)\sqrt{-g} g^{\mu\nu} \phi_{\mu||\nu} \varphi \\ &\quad + 4a(n-1)\sqrt{-g} g^{\mu\nu} \phi_\mu \varphi_{|\nu} + \sqrt{-g} g^{\mu\nu} \varphi_{|\mu||\nu} \end{aligned} \tag{8.2}$$

Now, the Klein-Gordon equation of quantum mechanics describes a spin-zero boson of mass m and is given by the covariant expression

$$\Box^2 \varphi = -\frac{m^2 c^2}{\hbar^2} \tag{8.3}$$

Following the fashion standard for problems of this sort, we assume a plane-wave solution of the form

$$\varphi = \exp\left[\int \left(k_\mu - \phi_\mu\right)\, dx^\mu\right]$$

where $k_\mu(x)$ is arbitrary for the time being (the integral is necessary because the arguments are functions of the coordinates). Inserting this into (8.2), we indeed obtain a solution provided

$$\begin{aligned} a &= \frac{1}{2(n-1)} \\ g^{\mu\nu} k_{\mu||\nu} &= 0 \\ g^{\mu\nu} k_\mu k_\nu &= -\frac{m^2 c^2}{\hbar^2} \end{aligned} \tag{8.4}$$

We thus see that the vector k_μ has to be a Riemannian vector. We can bring this into more recognizable form by the replacements

$$\begin{aligned} k_\mu &= -\frac{i}{\hbar} p_\mu \\ \phi_\mu &= \frac{iq}{\hbar c} A_\mu \end{aligned}$$

where q is the particle charge. Equation (8.4) now becomes

$$g^{\mu\nu} p_\mu p_\nu = m^2 c^2$$

which we recognize as the magnitude of the relativistic four-momentum:

$$p_\mu = (\frac{E}{c}, p^i)$$

The solution φ is now

$$\varphi = \exp\left[-\frac{i}{\hbar} \int \left(p_\mu - \frac{q}{c} A_\mu \right) dx^\mu \right]$$

which describes a spin-zero particle in an electromagnetic field with the "minimal" momentum correction provided by A_μ.

A similar treatment can be assumed for the derivation of the Dirac equation, which explains spin-1/2 particles (fermions). The expression for the divergence in curvilinear tensor format is

$$\frac{1}{\sqrt{-g}} \left(\sqrt{-g} \xi^\mu \right)_{||\mu}$$

Consideration of the form of the d'Alembertian operator in (8.3) indicates that this is the "square root" of the d'Alembertian, or

$$\Box\psi = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} \gamma^\mu \psi \right)_{||\mu} = \frac{imc}{\hbar} \psi$$

where the "scalar" ψ is now a 4-component spinor. Expanding, we get the Dirac equation for a fermion in an electromagnetic field:

$$\gamma^\mu \left(\partial_\mu - \frac{iq}{\hbar c} \right) \psi = \frac{imc}{\hbar} \psi$$

9. More on the Dirac Gamma Matrices

The gamma matrices have many amazing properties, and they figure prominently in all aspects of modern quantum physics. Here we have focused on the coordinate-dependent versions of these matrices in curved space which, as Weyl showed in 1929, can only be reached via tetrad transformations, which transform Lorentz vectors to their coordinate (curved space) counterparts:

$$\gamma_\mu(x) = e^a_{\ \mu}(x)\, \gamma_a$$

where the γ_a matrices are assumed to be in the standard (Dirac) representation. In Lorentz space, the flat-space γ_a have vanishing covariant derivatives, so the antiysymmetry of $\gamma_{a||\mu}$ is academic. But by transforming the γ_a into spherical coordinates, the antisymmetry condition can easily be demonstrated. To this end, let us perform the transformation using

$$\begin{aligned} \gamma_\mu &= \frac{\partial x^\alpha}{\partial x^\mu} \gamma_\alpha, \text{ with} \\ x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

(We will have to assume that the zero-index matrix γ_0 does not change.) Carrying out the calculations, we then have

$$\gamma_1 = \begin{bmatrix} 0 & 0 & \cos\theta & \sin\theta e^{-i\phi} \\ 0 & 0 & \sin\theta e^{i\phi} & -\cos\theta \\ -\cos\theta & -\sin\theta e^{-i\phi} & 0 & 0 \\ -\sin\theta e^{i\phi} & \cos\theta & 0 & 0 \end{bmatrix}$$

$$\gamma_2 = r \begin{bmatrix} 0 & 0 & -\sin\theta & \cos\theta e^{-i\phi} \\ 0 & 0 & \cos\theta e^{i\phi} & \sin\theta \\ \sin\theta & -\cos\theta e^{-i\phi} & 0 & 0 \\ -\cos\theta e^{i\phi} & -\sin\theta & 0 & 0 \end{bmatrix}$$

$$\gamma_3 = r\sin\theta \begin{bmatrix} 0 & 0 & 0 & -ie^{-i\phi} \\ 0 & 0 & ie^{i\phi} & 0 \\ 0 & ie^{-i\phi} & 0 & 0 \\ -ie^{i\phi} & 0 & 0 & 0 \end{bmatrix}$$

All of which exhibit the same anticommutation properties as the γ_a matrices along with the polar anticommutation relations

$$2g_{\mu\nu}(r,\theta) = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu$$

which, it is easy to demonstrate, are all consistent with $\boldsymbol{\gamma}_{\mu||\nu} = -\gamma_{\nu||\mu}$.

10. Final Thoughts

Although Weyl's 1918 theory was quickly dispatched by Einstein, it has the peculiar habit of persistently showing up in modern physics journals. And while its continuing popularity might be attributed to its sheer mathematical beauty, it would also seem that the theory has persisted at least in part because of the suspicion that Weyl had discovered something of fundamental importance. Although Einstein's objections to the theory seem to carry less weight today than they did in 1918, the gauge-invariant (or conformal) aspects of the theory are still beset with problems involving physical interpretation and geometrical relevance. If the line element ds, as Einstein had asserted, is indeed rescaled at every spacetime point in a Weyl manifold, all time-dependent phenomena would depend upon their prehistories, and the theory need not be further considered, regardless of its presumed beauty.

Here we have tackled a much simpler problem, the consistency of Weyl's theory with regard to constant-length vectors. As we have shown, the theory is easily amended, hopefully in a manner that would have met with Weyl's approval. Still, it is barely possible that the revised theory can be used to shed additional light on Einstein's argument. For example, Weyl proposed that the magnitude of a vector ξ^μ would change according to

$$dL = a\phi_\alpha L dx^\alpha$$

Integration of this expression around a closed curve leads to

$$\oint \frac{dL}{L} = a \oint \phi_\alpha \, dx^\alpha$$

But, as Einstein observed, for vector length to return into itself, Weyl's ϕ-field would have to satisfy curl $\phi = 0$; the electromagnetic stress tensor $F_{\mu\nu}$ then vanishes, and Weyl's geometry reverts to the Riemannian case. However, in the generalized theory we have

$$\begin{aligned} 2L\,dL &= g_{\mu\nu||\alpha}\xi^\mu \xi^\nu dx^\alpha \\ &= 2a(g_{\mu\nu}\phi_\alpha - g_{\alpha\nu}\phi_\mu)\,\xi^\mu\xi^\nu dx^\alpha \text{ or} \\ \frac{dL}{L} &= a\left[\phi_\alpha - \frac{\xi_\alpha \xi^\lambda \phi_\lambda}{\xi_\nu \xi^\nu}\right] dx^\alpha \end{aligned} \tag{9.1}$$

We now see that the situation is more complicated, as it involves the vector itself.* But it is at least possible that vector length could be preserved in (9.1) without placing any requirements on the curl of ϕ_μ.

Could we not just demand that $\oint dL/L = 0$ and be done with it? This would only require that

$$\oint dL/L = 2\pi i N$$

where N is an integer. London investigated this very possibility, and with it found that the quantized atomic radii of the simple Bohr atom could be derived from Weyl's theory. But this requires that the Weyl field ϕ_μ be pure imaginary (and the connection terms complex), results that only present more complications involving physical relevancy. In spite of these problems with Weyl's theory, its innate and indisputable mathematical beauty seems to point to a truth that remains ever elusive, an "old man's toy" that continues to tease and inspire.

References

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**Exercise.* By relabeling indices in (9.1), show that

$$L\, dL = \frac{1}{2}\, a\, e_{\rho\lambda\mu\beta}\, e^{\rho\lambda}{}_{\alpha\nu}\, \phi^\beta \xi^\mu \xi^\nu dx^\alpha$$

where the e-quantity is the completely-antisymmetric *Levi-Civita tensor* (see Adler et al., p. 101). This shows a little more clearly that $dL = 0$ if ξ^μ is any vector quantity proportional to ϕ^μ or dx^μ.